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Destroying the Ramsey property by the removal of edges

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Abstract

For any two graphs G and H, the Ramsey number $R(G, H)$ is the minimum number of vertices required in a complete graph to guarantee that every red/blue coloring of the edges of that complete graph contains either a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Hence, the removal of a single vertex in the complete graph destroys this property. Rather than remove a vertex (along with all of its incident edges), we consider the problem of selecting a vertex and removing edges incident with it. Our goal is to determine, for various pairs of graphs G and H, the exact number of edges that must be removed in this way in order to destroy the Ramsey property. We give precise evaluations of this number in the cases where G is a tree and H is a complete graph and in the cases where G and H are both stars. Partial results are obtained in other cases in which G and H are trees, not both of which are stars.

1 Introduction

If G and H are any two graphs, the Ramsey number $R(G, H)$ is defined to be the least natural number p such that every red/blue coloring of the edges of the complete graph K_p of order p contains a red subgraph isomorphic to G or a blue subgraph isomorphic to H. In a recent note, Cowen [7] built on the work of Golomb [10] by showing that whenever the Ramsey number $R(K_m, K_n) = p$, there exists a red/blue coloring of the edges in $K_p - e$ (a complete graph of order p with a single edge removed) that lacks a red K_m and a blue K_n. In other words, the Ramsey property is destroyed by removing a single edge. It can be observed that this result easily

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extends to multicolor Ramsey numbers, but we will not need the multicolor version here.

The statement $R(G, H) = p$ has two implications. First, every red/blue coloring of the edges in K_p results in a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Second, there exists some red/blue coloring of the edges in K_{p-1} that lacks a red G and a blue H as subgraphs. Hence, for any pair of graphs, the removal of a vertex (and all of its incident edges) destroys the Ramsey property. Our focus will be on determining the exact number of edges incident with a given vertex that must be removed in order to destroy the Ramsey property. For $k \in \mathbb{N}$, define the *k-deleted Ramsey number* $D_k(G, H)$ to be the least $p \in \mathbb{N}$ such that every red/blue coloring of the edges in $K_p - E(K_{1,k})$ contains a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Here, $E(K_{1,k})$ is the edge set of the complete bipartite graph $K_{1,k}$ in which the partite sets have cardinalities 1 and k. The graph $K_{1,k}$ is a tree that is usually referred to as a star. By convention, we set

$$D_0(G, H) := R(G, H).$$

In 1973, Chvátal and Harary [6] defined the number $R(G, \mathcal{F}, c)$, where G is a graph, $\mathcal{F}$ is a family of graphs, and c is a positive integer, to be the greatest integer n such that every coloring of G using c colors results in at least n monochromatic occurrences of members of $\mathcal{F}$. When $\mathcal{F} = \{F\}$, we write $R(G, F, c)$ in place of $R(G, \mathcal{F}, c)$. In this regard, our symbol $D_k(F, F)$ is the least positive integer p such that

$$R(K_{p-1} - E(K_{1,k}), F, 2) = 0 \quad \text{and} \quad R(K_p - E(K_{1,k}), F, 2) > 0.$$

If we note that every red/blue coloring of $K_{p+1} - E(K_{1,k})$ contains a red/blue coloring of K_p (remove the vertex in the partite set with cardinality 1 in $K_{1,k}$), then it follows that

$$D_k(G, H) \leq R(G, H) + 1.$$

On the other hand, if every red/blue coloring of $K_p - E(K_{1,k})$ results in a red G or a blue H, then so does every red/blue coloring of K_p. Hence,

$$R(G, H) \leq D_k(G, H).$$

Since the various Ramsey numbers only take on values from $\mathbb{N}$, it follows that $D_k(G, H)$ is either equal to $R(G, H)$ or $R(G, H) + 1$. Following this observation, Cowen's result [7] is equivalent to

$$D_1(K_m, K_n) = R(K_m, K_n) + 1.$$

For all $k' \leq k$, we find that

$$D_{k'}(G, H) \leq D_k(G, H),$$

since every red/blue coloring of $K_p - E(K_{1,k'})$ contains a red G or a blue H whenever every red blue coloring of $K_p - E(K_{1,k})$ contains a red G or a blue H. Thus, for any pair of graphs, G and H, define the *deleted edge number* $de(G, H)$ to be the least $k \in \mathbb{N}$ such that

$$D_{k-1}(G, H) < D_k(G, H).$$

That is, $de(G, H)$ is the least k such that

$$D_{k-1}(G, H) = R(G, H) \quad \text{and} \quad D_k(G, H) = R(G, H) + 1.$$

Cowen's result [7] is then equivalent to $de(K_m, K_n) = 1$ for all $m, n \in N$. Since removing a vertex and all of its incident edges destroys the Ramsey property, it follows that

$$1 \le de(G, H) \le R(G, H) - 1.$$

While the definition of $de(G, H)$ was motivated by Cowen's paper [7], it may be viewed as a variation of the size Ramsey number $\hat{r}(G, H)$ introduced by Erdős, Faudree, Rousseau, and Schelp [9] and the upper and lower size Ramsey numbers, $u(G, H)$ and $\ell(G, H)$, respectively, considered by Erdős and Faudree [8]. Here, $\hat{r}(G, H)$ is defined to be the minimum number of edges in a graph F such that every red/blue coloring of the edges of F results in a red copy of G or a blue copy of H. The main result of Cowen's note [7] actually follows from the evaluation of $\hat{r}(K_m, K_n)$ given in Theorem 1 of [9].

The upper and lower size Ramsey numbers discussed in [8] are related to $\hat{r}(G, H)$, but focus on subgraphs of $K_{R(G,H)}$. The upper size Ramsey number $u(G, H)$ is defined to be the minimum number such that if a subgraph L of $K_{R(G,H)}$ has at least $u(G, H)$ edges, then every red/blue coloring of the edges of L contains a red G or a blue H. The lower size Ramsey number is the minimum number of edges in any subgraph L of $K_{R(G,H)}$ such that every red/blue coloring of the edges of L contains a red G or a blue H. Since $de(G, H)$ is determined by removing edges from $K_{R(G,H)}$ in a specified way, it follows that

$$\ell(G, H) \le \binom{r}{2} - de(G, H) + 1 \le u(G, H).$$

This inequality implies that

$$\frac{r^2 - r + 2}{2} - u(G, H) \le de(G, H) \le \frac{r^2 - r + 2}{2} - \ell(G, H).$$

As we shall discover, $de(G, H)$ serves as a measure of connectivity for the pair of graphs G and H. Therefore, our primary focus in this paper will be on the computation of $de(G, H)$ when G and H are at the extremes of connectivity. Specifically, we consider the cases where G and H are complete graphs or trees. Section 2 offers an exact evaluation of $de(T, K_n)$, where T is any tree. In Section 3, we focus on the evaluation of $de(T, T')$, where T and T' are trees. Here, we are able to provide exact evaluations when T and T' are stars, and partial results in other cases. We conclude the paper by listing open problems and some potential applications of deleted edge numbers.

2 Trees Versus Complete Graphs

In this section, we focus on the evaluation of $de(G, H)$, where G is a tree and H is a complete graph. It is well known that if T_m is a tree of order m, then

$$R(T_m, K_n) = (m-1)(n-1) + 1.$$

The fact that this number is a lower bound for $R(T_m, K_n)$ was proved by Chvátal and Harary [5], and that it is an upper bound was shown by Chvátal [4]. Since $T_2 = K_2$, we already know that $de(T_2, K_n) = 1$. The remaining tree versus complete graph cases are addressed by the following theorem.

Theorem 2.1. *Let T_m be a tree of order $m \geq 2$ and assume that $n \geq 3$. Then $de(T_m, K_n) = m - 1$.*

Proof. This result will follow from showing that

$$D_{m-2}(T_m, K_n) = R(T_m, K_n) = (m-1)(n-1) + 1 \tag{1}$$

and

$$D_{m-1}(T_m, K_n) = R(T_m, K_n) + 1 = (m-1)(n-1) + 2. \tag{2}$$

Equation (1) will follow from proving that

$$D_{m-2}(T_m, K_n) \leq (m-1)(n-1) + 1.$$

We will proceed by (strong) induction on $m+n$. Our initial case was handled above:

$$D_0(T_2, K_3) = R(T_2, K_3) = 3.$$

Assume that

$$D_{m'-2}(T_{m'}, K_{n'}) \leq (m'-1)(n'-1) + 1$$

for all $m' + n' < m + n$ and consider a red/blue coloring of the edges of

$$K_{(m-1)(n-1)+1} - E(K_{1,m-2}).$$

Denote the vertices of the missing star by a and $b_1, b_2, \ldots, b_{m-2}$, where a is the center (i.e., the vertex of degree $m-2$). Denote by T' the tree formed by removing a single leaf from T_m and let x be the vertex in T' that was adjacent to the removed leaf. First, we apply the inductive hypothesis to the $K_{(m-1)(n-2)+1}$ formed by removing the vertices $a, b_1, b_2, \ldots, b_{m-2}$. Then there exists a red T_m or a blue K_{n-1}. In the former case, we are done, so assume there is a blue K_{n-1} and observe that it does not include any of the vertices $a, b_1, b_2, \ldots, b_{m-2}$. Removing the vertices in this complete subgraph, we obtain a red/blue coloring of

$$K_{(m-2)(n-1)+1} - E(K_{1,m-2}).$$

By the inductive hypothesis, there exists a red T' or a blue K_n. Assume the former case. Hence, the original coloring contains a red T' and a blue K_{n-1} that are disjoint. Since the K_{n-1} does not contain any of the vertices $a, b_1, b_2, \ldots, b_{m-2}$, all edges connecting x to the vertices in the K_{n-1} are included. If any such edge is red, we obtain a red T_m. Otherwise, they are all blue, and we obtain a blue K_n. This completes the proof of Equation (1).

Equation (2) will follow from proving that

$$D_{m-1}(T_m, K_n) > (m-1)(n-1) + 1.$$

That is, we must provide a red/blue coloring of

$$K_{(m-1)(n-1)+1} - E(K_{1,m-1})$$

that lacks a red T_m and a blue K_n. Begin with $n-1$ copies of red K_{m-1}-subgraphs, interconnected by blue edges. The resulting $K_{(m-1)(n-1)}$ lacks a red T_m since the largest connected red subgraph has order $m-1$ and it lacks a blue K_n since every complete blue subgraph contains at most one vertex from each red K_{m-1}. Select one of the red K_{m-1}-subgraphs and denote its vertices by $b_1, b_2, \ldots, b_{m-1}$. Add in vertex a and connect it via blue edges to all vertices in the $K_{(m-1)(n-1)}$, except for $b_1, b_2, \ldots, b_{m-1}$ (these will be the missing edges). The result is a red/blue coloring of

$$K_{(m-1)(n-1)+1} - E(K_{1,m-1})$$

that lacks a red T_m and a blue K_m. Equation (2) follows, completing the proof of the theorem. □

3 Trees Versus Trees

Now we turn our attention to trees versus trees, starting with the case of stars versus stars. In [14], it was proved that if $m, n \geq 1$, then

$$R(K_{1,m}, K_{1,n}) = \begin{cases} m+n-1 & \text{if } m \text{ and } n \text{ are both even} \\ m+n & \text{if } m \text{ or } n \text{ are odd,} \end{cases}$$

and in [2], this result was generalized to more than two stars. We will prove that whenever $m, n \geq 2$ are both even, $D_1(K_{1,m}, K_{1,n}) \geq m+n$, from which it follows that $de(K_{1,m}, K_{1,n}) = 1$. This result was first observed by Erdős and Faudree [8], but for the sake of completion, we give a proof.

Theorem 3.1. *If $m, n \geq 2$ are both even, then $de(K_{1,m}, K_{1,n}) = 1$.*

Proof. To prove this result, we must show that $D_1(K_{1,m}, K_{1,n}) \geq m + n$, which can be achieved by producing a red/blue coloring of $K_{m+n-1} - e$ that lacks a red $K_{1,m}$ and a blue $K_{1,n}$. We start with the observation that since $m+n-2$ is even, the complete graph K_{m+n-2} has a 1-factorization (see Theorem 9.1 of Harary's book [13]). Now, we observe that the subgraph spanned by the red edges contains a matching of size at least $\frac{m-2}{2}$. If we switch the edges in such a matching from red to blue, we produce a red/blue coloring of K_{m+n-2} that contains $m-2$ vertices with red degree $m-2$ and blue degree $n-1$ and $n-1$ vertices with red degree $m-1$ and blue degree $n-2$. Denote the set containing the first collection of vertices by A and the set containing the second collection of vertices by B. Finally, we introduce a new vertex, connecting it via red edges to the vertices in A and via blue edges to all but one vertex in B. No vertex in the resulting $K_{m+n-1} - e$ has a red degree exceeding $m-1$ or a blue degree exceeding $n-1$. □

It was observed by Erdős, Faudree, Rousseau, and Schelp (see Theorem 2 of [9]) that every red/blue coloring of the edges of $K_{1,m+n-1}$ produces a red $K_{1,m}$ or a blue $K_{1,n}$. This leads to the following theorem.

Theorem 3.2. *If $m \geq 1$ or $n \geq 1$ is odd, then $de(K_{1,m}, K_{1,n}) = m + n - 1$.*

Proof. It is known that when m or n are odd,

$$R(K_{1,m}, K_{1,n}) = m + n.$$

Consider a red/blue coloring of K_{m+n} and note that the removal of $m+n-2$ edges incident with a fixed vertex leaves at least one vertex having degree $m+n-1$. Hence, all edges incident with a fixed vertex must be removed in order to destroy the Ramsey property. □

Before we consider deleted edge numbers involving non-star trees we state the following special case of Lemma 2.3 given by Guo and Volkmann in [11]. Note that for any vertex x in a graph G, we denote by $deg_G(x)$ the degree of x in G. The minimum and maximum degrees in G are then defined by the following:

$$\delta(G) := \min\{deg_G(x) \mid x \in V(G)\} \quad \text{and} \quad \Delta(G) := \max\{deg_G(x) \mid x \in V(G)\}.$$

Lemma 3.3. *Let G be a connected graph satisfying $\delta(G) \geq n \geq 2$ and having order $|G| \geq n+2$. If T is any non-star tree of order $n+2$, then G contains a subgraph isomorphic to T.*

Although we are unable to determine $de(T, K_{1,n})$ for all non-star trees T, the following theorem provides an upper bound for the corresponding 1-deleted Ramsey numbers.

Theorem 3.4. *Let T_m be a non-star tree of order $m \geq 4$ and $n \geq 1$. Then*

$$D_1(T_m, K_{1,n}) \leq m + n - 1.$$

Proof. Consider a red/blue coloring of the edges of $K_{m+n-1} - e$. Denote the subgraph spanned by the red edges by G_R and the subgraph spanned by the blue edges by G_B. If the coloring lacks a blue $K_{1,n}$, then $\Delta(G_B) \leq n - 1$. Since two vertices in $K_{m+n-1} - e$ have degree $m + n - 3$ (call them a and b), it follows that

$$\delta(G_R) \geq m + n - 3 - (n - 1) = m - 2.$$

Before we can apply Lemma 3.3, it is necessary to argue that G_R contains a connected component having order at least m. Let G be the largest connected component of G_R. Then G must contain some vertex x other than a or b, as including only a and/or b would result in G being an empty graph. Such an x has degree

$$deg_G(x) \geq m + n - 2 - (n - 1) = m - 1$$

in G, forcing G to have order at least m. Thus, we are able to apply Lemma 3.3, from which it follows that G contains a subgraph isomorphic to T_m. $\square$

In order to consider the implications of Theorem 3.4 on the deleted edge number, observe that $de(G, H) \geq k$ if and only if

$$D_{k-1}(G, H) \leq R(G, H),$$

for graphs G and H. Hence, if T_m is a non-star tree of order $m \geq 4$ that satisfies

$$R(T_m, K_{1,n}) = m + n - 1,$$

then $de(T_m, K_{1,n}) \geq 2$. In 1974, Burr [1] proved that if T_m is any tree of order m for which $(m - 1)|(n - 1)$, then

$$R(T_m, K_{1,n}) = m + n - 1.$$

Burr's result, combined with Theorem 3.4, implies the following corollary.

Corollary 3.5. *Let T_m be a non-star tree of order $m \geq 4$. Then for all $n \geq 2$ such that $(m - 1)|(n - 1)$, $de(T_m, K_{1,n}) \geq 2$.*

The following theorem is motivated by Theorem 3.2 in [11]. Using the notation introduced by Guo and Volkmann in [11], we denote by T_m^* a tree of order m with $\Delta(T_m^*) = m - 2$. Such a tree is necessarily a broom, with one vertex of degree $m - 2$, one vertex of degree 2, and all other vertices having degree 1.

Theorem 3.6. *Let T_m^* and T_n^* be trees of orders $m \geq 4$ and $n \geq 4$ satisfying*

$$\Delta(T_m^*) = m - 2 \quad \textit{and} \quad \Delta(T_n^*) = n - 2.$$

Then $D_1(T_m^*, T_n^*) \leq m + n - 3$.

Proof. Consider a red/blue coloring of $K_{m+n-3} - e$ that lacks a blue T_n^*. Let G_R and G_B denote the graphs spanned by the red and blue edges, respectively. If $\Delta(G_B) \leq n-3$, then $\Delta(G_R) \geq m-2$, and by Lemma 3.3, G_R contains a subgraph isomorphic to T_m^*. If $\Delta(G_B) \geq n-2$, choose a vertex v having blue degree $deg_{G_B}(v) = n-2$. Pick a vertex set $A \subseteq N(v, G_B)$ with cardinality $|A| = n-2$, where $N(v, G_B)$ is the set of all vertices in G_B that are adjacent to v (i.e., the neighbors of v in G_B). Also, define the set

$$B := V(K_{m+n-3} - e) - (A \cup \{v\}),$$

which necessarily has cardinality $|B| = m-2$. Since the original red/blue coloring of $K_{m+n-3} - e$ lacks a blue T_n^*, it follows that all edges interconnecting A and B must be red. We claim that the subgraph spanned by these red edges contains a red T_m^*, despite the possibility that one of them is the missing edge. Without loss of generality suppose that ab is the missing edge with $a \in A$ and $b \in B$. Since $n \geq 4$, A contains some other vertex $x \neq a$, which is adjacent via red edges to all vertices in B. Let x be the vertex in the red T_m^* having degree $m-2$. Since $m \geq 4$, B contains some other vertex $y \neq b$, which forms the vertex of degree 2 in T_m^* by including the red edge ay. Hence, we have shown that every red/blue coloring of $K_{m+n-3} - e$ contains a red T_m^* or a blue T_n^*, completing the proof of the theorem. □

As with Theorem 3.4 and Corollary 3.5, we can compare the previous theorem to the exact Ramsey numbers determined in Theorem 3.2 in [11]. Specifically, when $(m-1)|(n-3)$ or $(n-1)|(m-3)$, it is known that

$$R(T_m^*, T_n^*) = m + n - 3,$$

resulting in the following corollary.

Corollary 3.7. *If $(m-1)|(n-3)$ or if $(n-1)|(m-3)$ where $m, n \geq 4$, then $de(T_m^*, T_n^*) \geq 2$.*

4 Conclusion

Most of the proofs given in this paper mirror the analogous proofs for Ramsey numbers, but we find that we can "take up the slack" in the original proofs. The limitations for continuing to evaluate deleted edge numbers for other pairs of graphs is only restricted to those whose Ramsey numbers are known. In this regard, future research into deleted edge numbers may involve cycles, books, wheels, complete graphs missing a single edge, and other graphs not studied here. We conclude by listing some additional problems for future inquiry.

1. While we know the exact values of very few of the classical Ramsey numbers $R(K_m, K_n)$, we are hopeful that the observation

$$D_1(K_m, K_n) = R(K_m, K_n) + 1$$

may offer some assistance in improving the known bounds for some m and n. For example, if the best known upper bound is $R(K_m, K_n) \leq p$, then proving

$$D_1(K_m, K_n) \leq p$$

would imply the improved lower bound

$$R(K_m, K_n) \leq D_1(K_m, K_n) - 1 \leq p - 1.$$

Radziszowski's dynamic survey [15] offers a comprehensive listing of the best known bounds for classical Ramsey numbers and would be a good starting point for initiating such applications.

2. When G and H are trees, we found examples at the opposite ends of the interval

$$1 \leq de(G, H) \leq R(G, H) - 1.$$

Are there trees G and H for which the deleted edge number is not one of the extremes (i.e., $1 < de(G, H) < R(G, H) - 1$)? Guo and Volkmann's results [11] involving non-star trees and Burr and Robert's results involving paths [2, 3] seem like good starting points in seeking out such an example.

3. In a sense, the deleted edge number seems to offer a measure of connectivity for the given pair of graphs. For pairs of complete graphs, the deleted edge number is 1, but when T_m is a tree of order m, we found that

$$de(T_m, K_n) = m - 1.$$

Is it true that if G is a connected graph of order m, then

$$1 \leq de(G, K_n) \leq m - 1?$$

If so, the deleted edge number can serve as a measure of connectivity for G.

4. Variations of classical Ramsey numbers such as Gallai-Ramsey numbers (e.g., see [12]) lead one to define a rainbow triangle-free deleted edge number. In this regard, one can study the number of edges necessary to destroy the Ramsey property for the t-colored Gallai-Ramsey number $gr^t(G)$, defined to be the least natural number p such that every rainbow triangle-free t-coloring of K_p contains a monochromatic copy of G.

5. An analogue of Cowen's result [7] holds for r-uniform hypergraphs. While fewer exact Ramsey numbers are known in the hypergraph setting, the deleted edge number for r-uniform hypergraphs can certainly be considered in such cases.

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