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Pattern restricted quasi-Stirling permutations

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Abstract

We define a variation of Stirling permutations, called quasi-Stirling permutations, to be permutations on the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ that avoid the patterns 1212 and 2121. Their study is motivated by a known relationship between Stirling permutations and increasing ordered rooted labeled trees. We construct a bijection between quasi-Stirling permutations and the set of ordered rooted labeled trees and investigate pattern avoidance for these permutations.

1 Introduction

Stirling permutations were introduced in 1978 by Gessel and Stanley [8] in the course of their investigation of Stirling polynomials, which themselves are defined in terms of the Stirling numbers of the first and second kinds. The unsigned Stirling numbers

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of the first kinds, denoted $c(n, m)$, give the number of permutations of length n that are comprised of exactly m cycles. The Stirling numbers of the second kind, denoted $S(n, m)$, give the number of set partitions of the set $\{1, 2, \ldots, n\}$ into exactly m parts. The Stirling polynomials are defined in terms of a parameter k as

$$f_k(x) = S(x + k, x) \quad \text{and} \quad g_k(x) = c(x, x - k).$$

These polynomials were used to study the set of so-called Stirling permutations. This set, typically denoted by $\mathcal{Q}_n$, consists of permutations on the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ that avoid the pattern 212. That is, if $\pi = \pi_1\pi_2 \ldots \pi_{2n} \in \mathcal{Q}_n$ and $i < j < k$ with $\pi_i = \pi_k$, then we must have $\pi_j > \pi_i$. In [8], Gessel and Stanley enumerated these permutations and found that $|\mathcal{Q}_n| = (2n-1)!!$. They subsequently refined this enumeration by the number of descents and found that the resulting numbers $B_{n,i}$ satisfy Eulerian-like polynomials in $f_k(n)$ and $g_k(n)$. In particular, if $B_{n,i}$ denotes the number of Stirling permutations of length $2n$ with exactly i descents, then

$$\sum_{n=0}^{\infty} f_k(n)x^n = \left(\sum_{i=1}^{k} B_{k,i}x^i\right) / (1-x)^{2k+1}$$

and

$$\sum_{n=0}^{\infty} g_k(n)x^n = \left(\sum_{i=k+1}^{2k} B_{2k-i+1,i}x^i\right) / (1-x)^{2k+1}.$$

Stirling permutations have received a considerable amount of attention since their conception. They have been found to be in bijection with several interesting combinatorial objects [9, 10, 15, 4] (including ordered rooted labeled trees, which we discuss further in this paper); several permutation statistics on them have been studied, including descents, ascents, plateaus, and left-to-right maxima and minima [8, 12, 2, 18]; and several applications of these permutations have been discovered, particularly to the study of other combinatorial objects (see for example, [9, 17, 14, 15]). In addition, pattern avoidance on Stirling permutations was explored in [13, 18, 3].

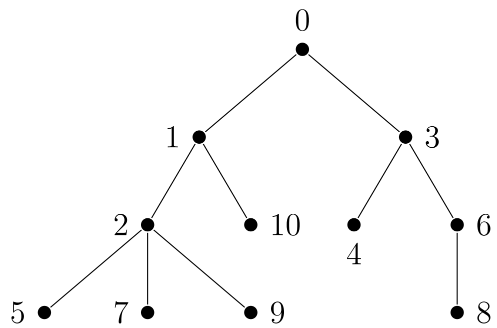


Figure 1: Consider the increasing ordered rooted tree pictured here. Traversing counterclockwise, we obtain the Stirling permutation $\pi = 125577992(10)(10)134468863$.

In this paper, the authors define quasi-Stirling permutations. This definition is motivated by a characterization of Stirling permutations via a bijection with increasing ordered rooted trees with labels in $[n]$ (see for example [9]). Given an increasing ordered rooted tree on $[n]$ as seen in Figure 1, one can obtain a Stirling permutation by performing a depth-first walk, i.e., by traversing the tree counterclockwise and recording the vertex below each edge traversed. Since each edge must be traversed twice, one obtains a permutation on the multiset $[n, n]$. Since this tree is increasing (and so no descendant of a given vertex is less than the vertex itself), this permutation will avoid the pattern 212. This map is invertible and thus this defines a bijection.

In this paper, we extend this bijection to all ordered rooted labeled trees. In Section 2, we introduce the necessary background. In Section 3, we describe the bijection between rooted labeled ordered trees and quasi-Stirling permutations. In Section 4, we present the bulk of our enumerative results. Namely, this section includes the enumeration of quasi-Stirling permutations that avoid all sets of size at least 2 containing patterns of length 3. We also enumerate these permutations with respect to plateaus. Finally, in Section 5, we present a few open questions and directions for future research.

2 Background

2.1 Permutation statistics and pattern avoidance

We denote the set of all permutations on the multiset $[n, n] = \{1, 1, 2, 2, \ldots, n, n\}$ by $\mathcal{S}_n^2$. For a permutation $\pi \in \mathcal{S}_n^2$ and for $i < j$, we will denote by $\pi_{[i,j]}$ the segment $\pi_i\pi_{i+1}\ldots\pi_j$. For a given segment $\pi_{[i,j]}$, we denote by $\text{Set}(\pi_{[i,j]})$ the subset of $[n]$ given by:

$$\text{Set}(\pi_{[i,j]}) = \{\pi_i, \pi_{i+1}, \ldots, \pi_j\}$$

without multiplicities. For example, $\text{Set}(144288) = \{1, 2, 4, 8\}$. For $\pi \in \mathcal{S}_n^2$ and $i < j$, we say that the segment $\pi_{[i,j]}$ is **weakly increasing** if $\pi_i \leq \pi_{i+1} \leq \cdots \leq \pi_j$ and similarly, we say that the segment is **weakly decreasing** if $\pi_i \geq \pi_{i+1} \geq \cdots \geq \pi_j$.

For a permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in \mathcal{S}_n^2$, we say i is a **descent** of π if $\pi_i > \pi_{i+1}$, and we denote the number of descents of π by $\text{des}(\pi)$. Similarly, we say i is an **ascent** of π if $\pi_i < \pi_{i+1}$, and we denote the number of ascents of π by $\text{asc}(\pi)$. Finally, we say that i is a **plateau** of π if $\pi_i = \pi_{i+1}$ and denote the number of plateaus by $\text{pl}(\pi)$. For example, if $\pi = 77611632554423 \in S_{7,7}$, then $\text{des}(\pi) = 6$, $\text{asc}(\pi) = 3$, and $\text{pl}(\pi) = 4$. Notice that for any $\pi \in \mathcal{S}_n^2$, we must have $\text{des}(\pi) + \text{asc}(\pi) + \text{pl}(\pi) = 2n - 1$.

Pattern avoidance is a well-studied concept for traditional permutations (see [19] for the seminal paper on the topic and [11] for a survey) as well as for generalizations of permutations and other combinatorial objects. For sequences σ and τ, we say σ and τ are in the same relative order if (1) $\sigma_i = \sigma_j$ if and only if $\tau_i = \tau_j$, and (2) $\sigma_i < \sigma_j$ if and only if $\tau_i < \tau_j$. We say that a permutation $\pi = \pi_1\pi_2\ldots\pi_{2n} \in \mathcal{S}_n^2$ **contains** the pattern $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ if there is some $i_1 < i_2 < \cdots < i_k$ so that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is in the same relative order as σ. If π does not contain σ, we say that π

avoids σ. We denote by $\mathcal{S}_n^2(\sigma_1, \sigma_2, \ldots, \sigma_m)$ the set of permutations in $\mathcal{S}_n^2$ that avoid all of the patterns $\sigma_1, \sigma_2, \ldots, \sigma_m$.

The set $\mathcal{Q}_n$ of **Stirling permutations** consists of the permutations in $\mathcal{S}_n^2$ that avoid the pattern 212. That is, $\mathcal{Q}_n := S_n^2(212)$ and thus there is no $i < j < k$ so that $\pi_i = \pi_k$ with $\pi_j < \pi_i$. For example, 135532214664 is a Stirling permutation since there is no subsequence in the same relative order as 212. However, the permutation 4225541331 is not a Stirling permutation since it contains the subsequence 424, which is an occurrence of the pattern 212. The permutation 4225541331 does avoid the pattern 1212 and also avoids the pattern 123.

We define the set of **quasi-Stirling permutations** to be the set $\overline{\mathcal{Q}}_n := \mathcal{S}_n^2(1212, 2121)$, i.e. those permutations in $\mathcal{S}_n^2$ that avoid both 1212 and 2121. For example, 4225541331 is a quasi-Stirling permutation, but 4221554331 is not since the subsequence 4141 is an occurrence of the pattern 2121. We will denote the set of quasi-Stirling permutations that additionally avoid the patterns in the set $\Lambda = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ by $\overline{\mathcal{Q}}_n(\sigma_1, \sigma_2, \ldots, \sigma_m)$ or just $\overline{\mathcal{Q}}_n(\Lambda)$. We use the notation $\overline{q}_n(\sigma_1, \sigma_2, \ldots, \sigma_m)$ or $\overline{q}_n(\Lambda)$ to denote the size of the set $\overline{\mathcal{Q}}_n(\Lambda)$, i.e.

$$\overline{q}_n(\Lambda) = |\overline{\mathcal{Q}}_n(\Lambda)|.$$

Finally, we say that two patterns σ and τ are **$\overline{\mathcal{Q}}$-Wilf-equivalent** if $\overline{q}_n(\sigma) = \overline{q}_n(\tau)$ for all $n \geq 1$ and we say that two sets $\Lambda_1 = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ and $\Lambda_2 = \{\tau_1, \tau_2, \ldots, \tau_m\}$ are $\overline{\mathcal{Q}}$-Wilf-equivalent if $\overline{q}_n(\Lambda_1) = \overline{q}_n(\Lambda_2)$ for all $n \geq 1$.

2.2 Ordered rooted trees

We define a **rooted labeled tree** on $[n] = \{1, 2, \ldots, n\}$ to be a tree on $n+1$ nodes, one labeled 0 and n of which are labeled with the numbers from $[n]$. The vertex labeled 0 we take to be the **root**. We draw these trees as in Figure 2 with the root at the top. In the remainder of the paper, we suppress the label of the root, since it is always the label 0.

A vertex j is a **descendant** of vertex i if $i \neq j$ and i appears in the unique path from the root to j. In this case, we say that i is an **ancestor** of j. If i and j are adjacent in the forest, then we say that i is the **parent** of j and that j is a **child** of i. For example, in Figure 2, vertices 2 and 11 are ancestors of 7, while vertex 8 is a child of 1 and a descendant of 9.

We say that a rooted labeled tree is **ordered** if the left-to-right order in which the children of each vertex appear matters. In other words, different embeddings of the tree into the plane determine distinct ordered rooted labeled trees. (See [5] for a more detailed construction.) We can see that the two trees in Figure 2 are distinct as ordered rooted labeled trees since the children of several vertices appear in different orders in the given embedding. We let $\mathcal{T}_n$ denote the set of ordered rooted labeled trees $n+1$ vertices with the root labeled 0 and the other nodes labeled with elements from the set $[n]$.

It is well-known (see [5, 21]) that the number of unlabelled ordered trees on $n+1$ vertices is given by the n-th Catalan number. Therefore, by labelling the root with 0

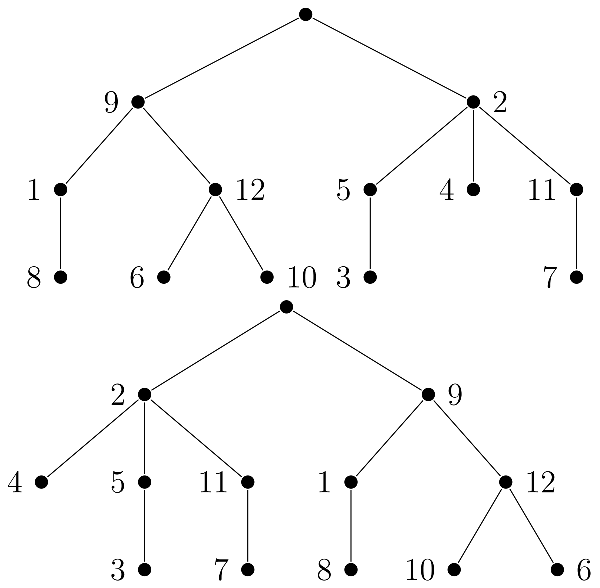


Figure 2: Two examples of rooted labeled trees on [12]. These trees are different as ordered rooted labeled trees.

and all remaining vertices with elements from $[n]$, we see that there are $n! \cdot C(n)$ elements of the set $\mathcal{T}_n$.

3 Quasi-Stirling permutations and rooted trees

The authors discovered quasi-Stirling permutations in the process of generalizing the map between increasing rooted labeled trees and Stirling permutations described in [21]. Such permutations arose naturally when investigating what happens if the domain of the map is extended to include all ordered rooted trees on $[n]$ (i.e. removing the condition that they must be increasing). We describe the resulting map here.

Denote by $\varphi : \mathcal{T}_n \to \mathcal{S}_n^2$ the map which sends an ordered rooted labeled tree to a permutation of the multiset $[n, n]$ obtained by traversing the tree counterclockwise and recording each vertex twice. More precisely, we perform a depth-first walk, and every time an edge between parent i and child j is traversed, we record j. See Example 3.1 for a detailed example of this map. After the example, we prove that this map actually is a bijection between ordered rooted labeled trees and the set of quasi-Stirling permutations, $\overline{\mathcal{Q}}_n$, and describe its inverse (demonstrated by Example 3.2).

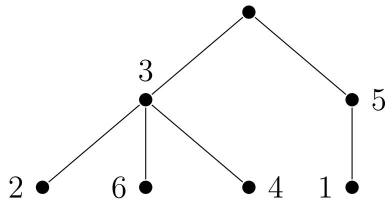
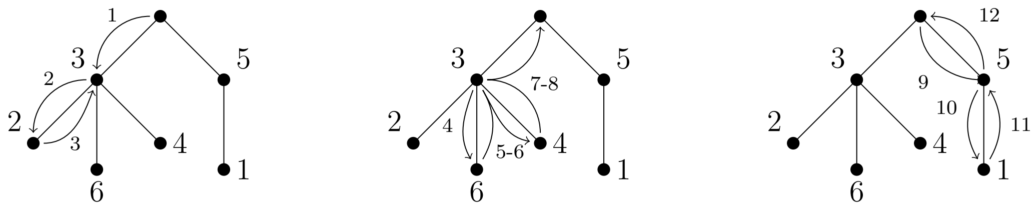


Figure 3: The rooted labeled ordered tree T associated to the permutation $\pi = \varphi(T) = 32266445115 \in \mathcal{S}_{6,6}$. Since this permutation avoids 1212 and 2121, it is indeed a quasi-Stirling permutation.



(a) Steps 1–3 (b) Steps 4–8 (c) Steps 9–12

Figure 4: Step-by-step depiction of the tree-traversal in computing $\varphi(T)$ from Example 3.1. In this case, we get $\varphi(T) = 322664435115$.

Example 3.1. Consider the rooted labeled ordered tree T in Figure 3. We construct the quasi-Stirling permutation $\pi = \varphi(T)$ using a step-by-step process, at step i obtaining the first i elements, denoted $\pi_{[1,i]}$, of the permutation $\pi \in \mathcal{S}_n^2$. A visual depiction of this step-by-step process is given in Figure 4.

We begin by traversing to the left-most child node of the root, the vertex labeled 3 and record $\pi_1 = 3$. We continue our traversal to its left-most child 2 and record its value giving us $\pi_{[1,2]} = 32$. The node labeled 2 has no children, so we traverse back up the tree to 3. We record the node associated to the edge we are traversing giving us a second 2. Node 3 still has children that have not yet been visited, so we traverse the edge to the left-most unvisited child node 6, and record its value. Thus $\pi_{[1,4]} = 3226$. We continue in this way until all nodes have been visited, giving $\varphi(T) = \pi := \pi_{[1,12]} = 322664435115$.

The next theorem asserts that this map is indeed a bijection that sends rooted ordered trees to quasi-Stirling permutations. We do this by first showing that the image of φ is contained in $\overline{\mathcal{Q}}_n$ and then by constructing an inverse to φ.

Theorem 3.1. *The map φ described above is a bijection between $\mathcal{T}_n$ and $\overline{\mathcal{Q}}_n$.*

Proof. Given any ordered rooted labeled tree $T \in \mathcal{T}_n$, $\varphi(T)$ is clearly an element of $\mathcal{S}_n^2$. To show that $\varphi(T) \in \overline{\mathcal{Q}}_n$, we must show that this permutation avoids 1212 and 2121. If j is a descendant of i, i must be read first while traversing the tree and j must be read twice before i is read again. If i and j are not related (that is, neither one is a descendant of the other) and i appears to the left of j in the tree, then i must be read twice before j is ever read while performing this traversal of the tree.

Thus the pattern *abab* never appears, and in particular 1212 and 2121 never appear, so π avoids these patterns.

It remains to show that the map is invertible on the set $\overline{\mathcal{Q}}_n$ of quasi-Stirling permutations. Let $a, b \in [n]$. Consider that an occurrence of a plateau aa indicates a leaf with label a; an occurrence of ab where $a \neq b$ and a is appearing for the first time indicates that a is the parent of b; an occurrence of ab where $a \neq b$ and a and b are both appearing for the second time indicates that a is a child of b; and an occurrence of ab where $a \neq b$, a is appearing for the second time, and b is appearing for the first time indicates that a and b share a parent and a lies directly to the left of b in the embedding of the tree.

Taken altogether, we can construct the tree $\varphi^{-1}(\pi)$ for any permutation $\pi \in \overline{\mathcal{Q}}_n$. First, we want to partition π into m blocks where the first element of each block is equal to the last. Thus

$$\pi = [a_1 b_1 \cdots c_1 a_1][a_2 b_2 \cdots c_2 a_2] \cdots [a_m b_m \cdots c_m a_m].$$

Since π avoids 1212 and 2121, each block has either exactly 0 or 2 copies of each element in $[n]$. The root of $\varphi^{-1}(\pi)$ will have m child nodes labeled from left-to-right $a_1, a_2, \ldots, a_m$. Continue this process recursively for each block to obtain the subtree associated to each child of the root. A block of the form aa produces a leaf. □

In Example 3.2, we demonstrate the inverse of φ.

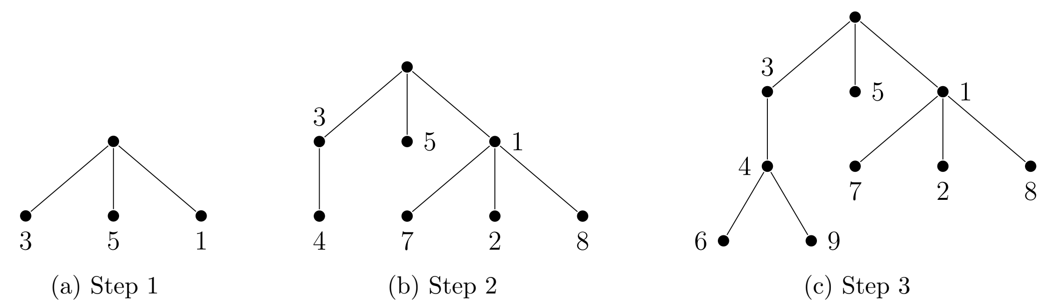


Figure 5: Step-by-step depiction of the tree-traversal in computing $\varphi(T)$ from Example 3.2.

Example 3.2. Given a quasi-Stirling permutation, $\pi = 346699435517722881$, we will construct the tree $T = \varphi^{-1}(\pi)$. A visual depiction of these three steps can be found in Figure 5.

1. Partition our permutation π into subpermutations such that the first element of each subpermutation equals the last element of the subpermutation.

$$\pi = [34669943][55][17722881]$$

In this example 3, 5, and 1 are the child nodes of the root of $T = \varphi^{-1}(\pi)$.

2. Next, partition each permutation of the children of our root, using the same rule in step one: first element is equal to the last element. Since the consecutive 5's indicate a plateau, we know that 5 is a leaf in T and that branch of the tree is complete.

$$\pi = [3[466994]3][55][1[77][22][88]1]$$

   Therefore 4 is the only child node of 3, but node 1 has three children: $7, 2$, and 8. Also note that $7, 2$, and 8 are leafs, so that subtree is now complete.

3. Now we can partition our last permutation, giving us the child nodes of 4, which are 6 and 9.

$$\pi = [3[4[66][99]4]3][55][17722881]$$

   Since the consecutive 6's and 9's indicate plateaus, we know our process is complete, and a quick traversal of the tree shows that we have the correct tree for our permutation.

The final result can be seen in Figure 5(c).

Finally, we can use Theorem 3.1 to enumerate quasi-Stirling permutations. Since it unlabeled ordered rooted trees on $[n]$ are enumerated by the Catalan numbers, we obtain an immediate enumeration of the quasi-Stirling permutations.

Corollary 3.1. *For any $n \geq 1$, $|\overline{\mathcal{Q}}_n| = n! \cdot C_n$, where C_n is the nth Catalan number.*

It is clear from this result that quasi-Stirling permutations are in fact in bijection with several other combinatorial objects as well.

4 Enumeration results

In this section, we consider the set of quasi-Stirling permutations that avoid a given set of patterns of length three. Using two trivial symmetries we can reduce the number of cases we need to consider. The **reverse** of the permutation $\pi = \pi_1\pi_2 \dots \pi_{2n} \in \mathcal{S}_n^2$ is the permutation $\pi^r = \pi_{2n}\pi_{2n-1} \dots \pi_2\pi_1$ and the **complement** of the permutation $\pi \in \mathcal{S}_n^2$ is the permutation π^c where $\pi_i^c = n+1-\pi_i$. For example, if $\pi = 25513443661277$, then $\pi^r = 77216634431552$ and $\pi^c = 63375445227611$. Clearly, for any $\pi \in \mathcal{S}_n^2$, π avoids σ if and only if π^r avoids σ^r and π^c avoids σ^c. It is also clear that the set of quasi-Stirling permutations is closed under both of these operations. Thus, in this section, for each possible proper nontrivial subset $\Lambda \subseteq \mathcal{S}_3$, we need only consider subsets Λ which cannot be obtained via complements or reverses of another subset on our list.

In the table in Figure 6, we list a representative (of the set of subsets obtained by the reverse and complement symmetries) for sets of size two through five of patterns in $\mathcal{S}_3$. This table provides a summary of the results on pattern avoidance in quasi-Stirling permutations that the authors have obtained.

In certain cases, we will need to make use of the following lemma.

Representative Λ	Enumeration of $\overline{\mathcal{Q}}_n(\Lambda)$	Theorem
$\{123, 321\}$	0, for $n \geq 5$	Theorem 4.1
$\{132, 312\}$	$4 \cdot 3^{n-2}$, for $n \geq 2$	Theorem 4.2
$\{132, 231\}$		Theorem 4.3
$\{312, 321\}$		Theorem 4.4
$\{132, 213\}$	g.f.: $\mathcal{A}(x) = \dfrac{(1-x)^2}{x^3 - 3x + 1}$	Theorem 4.5
$\{132, 321\}$	$2n^2 - 3n + 2$	Theorem 4.6
$\{123, 132, 321\}$	0, for $n \geq 5$	Theorem 4.7
$\{132, 213, 321\}$	$2n$, for $n \geq 2$	Theorem 4.8
$\{123, 213, 312\}$		Theorem 4.9
$\{132, 213, 312\}$		Theorem 4.10
$\{123, 132, 213\}$	$\frac{1}{4}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1} - 2(-1)^{n+1}\right)$	Theorem 4.11
$\{123, 132, 312\}$	$\frac{1}{2}(n^2 + 3n - 2)$	Theorem 4.12
$\{123, 132, 213, 321\}$	0, for $n \geq 5$	Theorem 4.13
$\{123, 132, 231, 321\}$		
$\{123, 132, 312, 321\}$		
$\{123, 132, 213, 231\}$	4, for $n \geq 2$	Theorem 4.14
$\{123, 132, 231, 312\}$	3, for $n \geq 3$	Theorem 4.15
$\{132, 213, 231, 312\}$	2, for $n \geq 3$	Theorem 4.16
$\{123, 132, 213, 231, 321\}$	0, for $n \geq 5$	Theorem 4.17
$\{123, 132, 213, 231, 312\}$	1, for $n \geq 3$	Theorem 4.18

Figure 6: A representative of each Wilf-equivalence class is listed and the set of quasi-Stirling permutations that avoid that set of patterns is enumerated.

Lemma 4.1. *For $n \geq 5$, no permutation $\pi \in \mathcal{S}_n^2$ avoids both* 123 *and* 321.

Clearly this lemma holds, just as it does in $\mathcal{S}_n$. Indeed, given any subsequence of length five containing five distinct numbers, there must be an occurrence of either 123 or 321.

4.1 Avoiding one pattern in $\mathcal{S}_3$

In the case of single patterns, we can deduce that the patterns 123 and 321 are trivially $\overline{\mathcal{Q}}$-Wilf-equivalent since $123 = (321)^r$. Similarly, 213, 312, 132, and 231 are all $\overline{\mathcal{Q}}$-Wilf-equivalent since $213 = (312)^r = (231)^c = (132)^{rc}$. It remains an open question to enumerate quasi-Stirling permutations that avoid a single pattern of length three. In the table in Figure 7, we compute $\overline{q}_n(132)$ and $\overline{q}_n(123)$ for $n \in [5]$.

n	1	2	3	4	5
$\overline{q}_n(132)$	1	4	19	102	590
$\overline{q}_n(123)$	1	4	19	96	510

Figure 7: The values $\overline{q}_n(132)$ and $\overline{q}_n(123)$ for $n \in [5]$.

4.2 Avoiding two patterns in $\mathcal{S}_3$

Theorem 4.1. *For all* $n \geq 5$, $\overline{q}_n(123, 321) = 0$.

Proof. This follows immediately from Lemma 4.1. □

Theorem 4.2. *For all* $n \geq 2$, $\overline{q}_n(132, 312) = 4 \cdot 3^{n-2}$.

Proof. Let $\Lambda = \{132, 312\}$ and $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$. Notice that we must have $\pi_{2n} = n$ or $\pi_{2n} = 1$. Indeed, if this were not the case, then for any $i, j < 2n$ such that $\pi_i = 1$ and $\pi_j = n$, we would have that either $i < j$, in which case $\pi_i\pi_j\pi_{2n}$ is an occurrence of 132, or that $i > j$, in which case $\pi_j\pi_i\pi_{2n}$ is an occurrence of 312. Thus we have two cases:

(1) $\pi = \alpha 1 \beta 1$, or

(2) $\pi = \alpha n \beta n$.

In Case (1), β is a weakly increasing segment since π avoids 132 and for each $a \in \alpha$ and $b \in \beta$, we must have that $a < b$ since π avoids 312. In Case (2), β is a weakly decreasing segment since π avoids 312 and for each $a \in \alpha$ and $b \in \beta$, we must have that $a > b$ since π avoids 132. Therefore, for $n \geq 2$,

$$\overline{q}_n(\Lambda) = 2\sum_{i=0}^{n-1} \overline{q}_i(\Lambda).$$

Solving this recurrence with the initial conditions $q_0(\Lambda) = 1$ and $q_1(\Lambda) = 1$, we obtain the result. □

Theorem 4.3. *For all* $n \geq 2$, $\overline{q}_n(132, 231) = 4 \cdot 3^{n-2}$.

Proof. Let $\Lambda = \{132, 231\}$. For a given quasi-Stirling permutation $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ with $n > 2$, π must satisfy one of the following:

(1) $\pi = nn\alpha$,

(2) $\pi = \alpha nn$, or

(3) $\pi = n\alpha n$.

Indeed, if we have some $i < j < k$ so that $\pi_i \neq n$, $\pi_j = n$, and $\pi_k \neq n$, then $\pi_i\pi_j\pi_k$ would be an occurrence of 132 or 231. In each case, α is any permutation in $\overline{\mathcal{Q}}_{n-1}(\Lambda)$ since n cannot be part of a 132 or a 231 pattern. We therefore have that for $n > 2$,

$$\overline{q}_n(\Lambda) = 3\overline{q}_{n-1}(\Lambda).$$

Since there are four quasi-Stirling permutations in $\overline{\mathcal{Q}}_2(\Lambda)$, the result follows. □

Theorem 4.4. *For all* $n \geq 2$, $\overline{q}_n(312, 321) = 4 \cdot 3^{n-2}$.

Proof. Let $\Lambda = \{312, 321\}$ and $n > 2$. Write $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. Since π avoids 312 and 321, we must have that $|\operatorname{Set}(\beta) \cup \operatorname{Set}(\gamma)| \leq 1$. Thus,

(1) $\pi = \alpha nn$,

(2) $\pi = \alpha nbbn$ for $b \in [n-1]$,

(3) $\pi = \alpha nnbb$, for $b \in [n-1]$, or

(4) $\pi = \alpha_1 b \alpha_2 nnb$ for $b \in [n-1]$.

Let $F_n = \{\pi \in \overline{\mathcal{Q}}_n(\Lambda) \mid \pi_{2n-1} = \pi_{2n}\}$ and $G_n = \{\pi \in \overline{\mathcal{Q}}_n(\Lambda) \mid \pi_{2n-1} \neq \pi_{2n}\}$. Denote the sizes of these sets by $f_n = |F_n|$ and $g_n = |G_n|$. Then clearly, F_n and G_n are disjoint sets so that $F_n \cup G_n = \overline{\mathcal{Q}}_n(\Lambda)$. Thus, $\overline{q}_n(\Lambda) = f_n + g_n$. Notice that Cases (1) and (3) above are enumerated by f_n and Cases (2) and (4) are enumerated by g_n. In Case (1), α can be any permutation in $\overline{\mathcal{Q}}_{n-1}(\Lambda)$ and in Case (3), αbb can be any permutation in F_{n-1}. In Case (2), αbb is any permutation in F_{n-1} and in Case (4), $\alpha_1 b \alpha_2 b$ is any permutation in $\overline{\mathcal{Q}}_{n-1}(\Lambda)$. By observing these cases, we can see that $f_n = \overline{q}_{n-1}(\Lambda) + f_{n-1}$ and that $g_n = \overline{q}_{n-1}(\Lambda) + f_{n-1}$. In particular, $f_n = g_n$.

Taken together we see that

$$\overline{q}_n(\Lambda) = f_n + g_n = 2(\overline{q}_{n-1}(\Lambda) + f_{n-1}) = 3\overline{q}_{n-1}(\Lambda).$$

This recurrence together with the initial condition $\overline{q}_2(\Lambda) = 4$ gives us the result. □

For a set $\Lambda \subseteq \mathcal{S}_3$, define

$$Q_\Lambda(x) = \sum_{n \geq 1} |\overline{\mathcal{Q}}_n(\Lambda)| x^n$$

to be the generating function for $\overline{\mathcal{Q}}_n(\Lambda)$. We will use this notation in the next theorem.

Theorem 4.5. *Let* $\Lambda = \{132, 213\}$. *Then*

$$Q_\Lambda(x) = \frac{(1-x)^2}{1 - 3x + x^3}.$$

Proof. Let $\Lambda = \{132, 213\}$ and $n \geq 2$. Write $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. Since π avoids 213, α and β are weakly increasing. If $a \in \operatorname{Set}(\alpha)$ and $b \in \operatorname{Set}(\beta)$, then either $a > b$ in which case abn is an occurrence of 213, or $a < b$ in which case anb is an occurrence of 132. We cannot have that $a = b$ since $anbn$ would then be an occurrence of 2121. Thus, we must have that either $\operatorname{Set}(\alpha) = \varnothing$ or $\operatorname{Set}(\beta) = \varnothing$. Thus there are four possibilities:

(1) $\pi = nn\gamma$,

(2) $\pi = n\beta n\gamma$ where $\operatorname{Set}(\beta) \neq \varnothing$,

(3) $\pi = \alpha nn\gamma$ where $\operatorname{Set}(\alpha) \neq \varnothing$ and $\operatorname{Set}(\alpha) \cap \operatorname{Set}(\gamma) = \varnothing$, or

(4) $\pi = \alpha nn\gamma$ where $\text{Set}(\alpha) \cap \text{Set}(\gamma) \neq \varnothing$.

Notice that for any $a \in \alpha$, $b \in \beta$, and $c \in \gamma$, we must have that $a \geq c$ and $b > c$ since π avoids 132. Let

$$F_n = \left\{\pi \in \overline{\mathcal{Q}}_n(\Lambda) \mid \pi = nn\gamma \text{ (Case (1))}\right\},$$
$$G_n = \left\{\pi \in \overline{\mathcal{Q}}_n(\Lambda) \mid \pi = n\beta n\gamma, \text{Set}(\beta) \neq \varnothing \text{ (Case (2))}\right\}, \textit{and}$$
$$H_n = \left\{\pi \in \overline{\mathcal{Q}}_n(\Lambda) \mid \pi = \alpha nn\gamma, \text{Set}(\alpha) \neq \varnothing \text{ (Case (3) and Case (4))}\right\}.$$

We define $f_n = |F_n|$, $g_n = |G_n|$, and $h_n = |H_n|$. Since F_n, G_n, and H_n are disjoint sets with $F_n \cup G_n \cup H_n = \overline{\mathcal{Q}}_n(\Lambda)$, we have $\overline{q}_n(\Lambda) = f_n + g_n + h_n$. Let us consider each case.

- For all $n \geq 2$, we have that γ is any permutation in $\overline{\mathcal{Q}}_{n-1}(\Lambda)$. Thus we have that $f_n = \overline{q}_{n-1}(\Lambda)$.
- For all $n \geq 2$, we have that $\pi = n(i+1)(i+1)\ldots(n-1)(n-1)n\gamma$ where $\gamma \in \overline{\mathcal{Q}}_i(\Lambda)$. Thus we have that
$$g_n = \sum_{k=0}^{n-2} \overline{q}_k(\Lambda).$$
- Finally, for all $n \geq 2$, α is weakly increasing and $a \geq c$ for all $a \in \alpha$ and $c \in \gamma$. Thus we must have that α ends in $(n-1)$. If $\text{Set}(\alpha) \cap \text{Set}(\gamma) = \varnothing$, then α ends in $(n-1)(n-1)$ and thus $\alpha\gamma$ is a permutation in F_n or H_n. If $\text{Set}(\alpha) \cap \text{Set}(\gamma) \neq \varnothing$, then we have $|\text{Set}(\alpha) \cap \text{Set}(\gamma)| = 1$ since π avoids 132. Given that $\text{Set}(\alpha) \cap \text{Set}(\gamma = \{a\}$, since π avoids 132, we must have that $a = \min(\alpha)$ and $a = \max(\gamma)$. writing $\pi = \alpha_1 a \alpha_2 nn\gamma_1 a\gamma_2$, we must have $\text{Set}(\alpha_1) = \varnothing$ and $\text{Set}(\gamma_1) = \varnothing$. Further, α_2 is weakly increasing and γ_2 is any permutation in $\overline{\mathcal{Q}}_{a-1}(\Lambda)$. Therefore
$$h_n = f_{n-1} + h_{n-1} + \sum_{k=0}^{n-2} \overline{q}_k(\Lambda) = f_{n-1} + h_{n-1} + \overline{q}_{n-2}(\Lambda) + g_{n-1} = \overline{q}_{n-2}(\Lambda) + \overline{q}_{n-1}(\Lambda).$$

Thus we obtain the recurrence

$$\overline{q}_n(\Lambda) = \overline{q}_{n-1}(\Lambda) + \sum_{k=0}^{n-2} \overline{q}_k(\Lambda) + \overline{q}_{n-2}(\Lambda) + \overline{q}_{n-1}(\Lambda) = \overline{q}_{n-1}(\Lambda) + \overline{q}_{n-2}(\Lambda) + \sum_{k=0}^{n-1} \overline{q}_k(\Lambda).$$

Solving for the generating function $Q_\Lambda(x)$, where $Q_\Lambda(x) = \sum_{n=0}^{\infty} \overline{q}_n(\Lambda)x^n$, we then obtain the result from this recurrence. □

Theorem 4.6. *For $n \geq 2$, $\overline{q}_n(132, 321) = 2n^2 - 3n + 2$.*

Proof. Let $\Lambda = \{132, 321\}$ and $n > 2$. Write $\pi \in \overline{\mathcal{Q}}_n(123, 312)$ as $\pi = \alpha n\beta n\gamma$. Since π avoids 132, we must have that for any $a \in \alpha$, $b \in \beta$, and $c \in \gamma$, $a > b$, $b > c$, and $a \geq c$. Since π avoids 321, it must be the case that at least one of $\text{Set}(\beta)$ and $\text{Set}(\gamma)$ is empty. Thus there are four possibilities:

(1) $\pi = \alpha nn$,

(2) $\pi = \alpha n\beta n$ where $\mathrm{Set}(\beta) \neq \varnothing$,

(3) $\pi = \alpha nn\gamma$ where $\mathrm{Set}(\gamma) \neq \varnothing$ and $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma) = \varnothing$, or

(4) $\pi = \alpha nn\gamma$ where $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma) \neq \varnothing$.

In Case (1), α is any permutation in $\overline{\mathcal{Q}}_n(\Lambda)$. In Case (2), since π avoid 321, β must be weakly increasing and since $\mathrm{Set}(\beta) \neq \varnothing$, α is also weakly increasing. Thus there are $n-1$ permutations in this case. In Case (3), we similarly have that α and γ are weakly increasing and there are $n-1$ permutations in this case.

Finally, consider Case (4). Since π avoids 132, we must have that $|\,\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma)| = 1$, $a = \min(\mathrm{Set}(\alpha))$, and $a = \max(\mathrm{Set}(\gamma))$. Let us write $\pi = \alpha_1 a\alpha_2 nn\gamma_1 a\gamma_2$. Then α_1, α_2, and γ_1 are weakly increasing and $\mathrm{Set}(\gamma_2) = \varnothing$. Further, since π avoids 321, at least one of $\mathrm{Set}(\alpha_1)$ and $\mathrm{Set}(\gamma_1)$ are empty. There is one way for both $\mathrm{Set}(\alpha_1)$ and $\mathrm{Set}(\gamma_1)$ to be empty, there is $(n-2)$ permutations of the form $\alpha_1 a\alpha_2 nna$ with $\mathrm{Set}(\alpha_1) \neq \varnothing$, and there is $(n-2)$ permutations of the form $a\alpha_2 nn\gamma_1 a$ with $\mathrm{Set}(\gamma_1) \neq \varnothing$.

Then we obtain the recurrence:

$$\overline{q}_n(\Lambda) = \overline{q}_{n-1}(\Lambda) + (n-1) + (n-1) + 1 + (n-2) + (n-2) = \overline{q}_{n-1}(\Lambda) + 4n - 5.$$

Solving this recurrence, we obtain the result. □

4.3 Avoiding three patterns in $\mathcal{S}_3$

Theorem 4.7. *For all $n \geq 5$, $\overline{q}_n(123, 132, 321) = 0$.*

Proof. This follows immediately from Lemma 4.1. □

Theorem 4.8. *For all $n \geq 2$, $\overline{q}_n(132, 213, 321) = 2n$.*

Proof. Let $\Lambda = \{132, 213, 321\}$. For any $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$, write π as $\pi = \alpha n\beta n\gamma$. Since π avoids 213, α must be weakly increasing. Since π avoids 321, both β and γ are weakly increasing as well. In addition, for any $a \in \mathrm{Set}(\alpha)$, $b \in \mathrm{Set}(\beta)$, and $c \in \mathrm{Set}(\gamma)$, $a > b$, $b > c$, and $a \geq c$ since π avoids 132. For $n \geq 3$, we must also have $a > c$ since π avoids 213 and 132. Notice that since π avoids 213 and $a > b$ for any $a \in \mathrm{Set}(\alpha)$ and $b \in \mathrm{Set}(\beta)$, we must actually have that either $\mathrm{Set}(\alpha) = \varnothing$ or $\mathrm{Set}(\beta) = \varnothing$.

Thus either $\{\mathrm{Set}(\alpha), \mathrm{Set}(\gamma)\}$ or $\{\mathrm{Set}(\beta), \mathrm{Set}(\gamma)\}$ is a partition of $[n-1]$ into two (possibly empty) disjoint sets of consecutive numbers. There are exactly n ways to do this, thus there are $2n$ permutations in $\overline{\mathcal{Q}}_n(\Lambda)$. □

Example 4.1. For an example of the process in the proof of the previous theorem, suppose the partition of [8] into two subsets is given by

$$\{\{1, 2, 3\}, \{4, 5, 6, 7, 8\}\}.$$

Then either $\mathrm{Set}(\alpha) = \{4, 5, 6, 7, 8\}$ and $\mathrm{Set}(\gamma) = \{1, 2, 3\}$ in which case

$$\pi = 445566778899112233,$$

or $\mathrm{Set}(\beta) = \{4, 5, 6, 7, 8\}$ and $\mathrm{Set}(\gamma) = \{1, 2, 3\}$ in which case

$$\pi = 994455667788112233.$$

Theorem 4.9. *For all $n \geq 2$, $\overline{q}_n(123, 213, 312) = 2n$.*

Proof. Let $\Lambda = \{123, 213, 312\}$ and $n \geq 2$. Suppose $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ with $\pi = \alpha n \beta n \gamma$. Since π avoids 312, we must have that γ is weakly decreasing. Further, since π avoids 123 and 213, we must have that $|\,\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)| \leq 1$. Thus, there are four possibilities (all of which include γ as a weakly decreasing segment):

(1) $\pi = aann\gamma$ where $a \in [n-1]$,

(2) $\pi = nbbn\gamma$ where $b \in [n-1]$,

(3) $\pi = nn\gamma$, or

(4) $\pi = (n-1)nn(n-1)\gamma$.

Since γ is always determined, there are clearly $2n$ total permutations. □

Theorem 4.10. *For all $n \geq 2$, $\overline{q}_n(132, 213, 312) = 2n$.*

Proof. This proof is very similar to the proof of Theorem 4.8. Let $\Lambda = \{132, 213, 312\}$ and $n \geq 2$. Write $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$.

Since π avoids 213, α must be weakly increasing, and since π avoids 312, both β and γ are weakly decreasing. In addition, for any $a \in \mathrm{Set}(\alpha)$, $b \in \mathrm{Set}(\beta)$, and $c \in \mathrm{Set}(\gamma)$, $a > b$, $b > c$, and $a \geq c$ since π avoids 132. For $n \geq 3$, either $\mathrm{Set}(\beta) \neq \varnothing$, in which $a > c$ since π avoids 132 and 213, or $\mathrm{Set}(\beta) = \varnothing$ in which we must also have $a > c$ since π avoids 213 and 132. Notice that since π avoids 213 and $a > b$ for any $a \in \mathrm{Set}(\alpha)$ and $b \in \mathrm{Set}(\beta)$, we must actually have that either $\mathrm{Set}(\alpha) = \varnothing$ or $\mathrm{Set}(\beta) = \varnothing$.

Thus either $\{\mathrm{Set}(\alpha), \mathrm{Set}(\gamma)\}$ or $\{\mathrm{Set}(\beta), \mathrm{Set}(\gamma)\}$ is a partition of $[n-1]$ into two (possibly empty) disjoint sets of consecutive numbers. There are exactly n ways to do this, thus there are $2n$ permutations.

□

Example 4.2. For an example of the process in the proof of the previous theorem, suppose the partition of [8] into two subsets is given by

$$\{\{1, 2, 3\}, \{4, 5, 6, 7, 8\}\}.$$

Then either $\mathrm{Set}(\alpha) = \{4, 5, 6, 7, 8\}$ and $\mathrm{Set}(\gamma) = \{1, 2, 3\}$, in which case we obtain the permutation $\pi = 445566778899332211$, or $\mathrm{Set}(\beta) = \{4, 5, 6, 7, 8\}$ and $\mathrm{Set}(\gamma) = \{1, 2, 3\}$, in which case $\pi = 988776655449332211$.

Theorem 4.11. *For all $n \geq 1$,*

$$\overline{q}_n(123, 132, 213) = \frac{(1+\sqrt{2})^{n+1}}{4} + \frac{(1-\sqrt{2})^{n+1}}{4} - \frac{(-1)^{n+1}}{2}.$$

Proof. Let $\Lambda = \{123, 132, 213\}$ and $n > 3$. Write $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. Since π avoids 123 and 213, $|\operatorname{Set}(\alpha) \cup \operatorname{Set}(\beta)| \leq 1$. Since π avoids 132, $\operatorname{Set}(\alpha) \cup \operatorname{Set}(\beta) = \varnothing$ or $\{n-1\}$. There are four cases:

(1) $\pi = nn\gamma$,

(2) $\pi = n(n-1)(n-1)n\gamma$,

(3) $\pi = (n-1)(n-1)nn\gamma$, or

(4) $\pi = (n-1)nn\gamma_1(n-1)\gamma_2$.

Notice that in Case (4), since π avoids 123 and 213, we have $|\operatorname{Set}(\gamma_1)| \leq 1$. Since π also avoids 132, $\operatorname{Set}(\gamma_1) = \varnothing$ or $\{n-2\}$.

Thus, there are $\overline{q}_{n-1}(\Lambda)$ permutations of Type (1), $\overline{q}_{n-2}(\Lambda)$ permutations of Type (2), $\overline{q}_{n-2}(\Lambda)$ permutations of Type (3), and $\overline{q}_{n-2}(\Lambda) + \overline{q}_{n-3}(\Lambda)$ permutations of Type (4). Thus, we obtain the recurrence

$$\overline{q}_n(\Lambda) = \overline{q}_{n-1}(\Lambda) + 3\overline{q}_{n-2}(\Lambda) + \overline{q}_{n-3}(\Lambda).$$

By solving this recurrence given the initial conditions that $\overline{q}_0(\Lambda) = 1, \overline{q}_1(\Lambda) = 1$, and $\overline{q}_2(\Lambda) = 4$, we obtain the result. □

Theorem 4.12. *For all $n \geq 1$, $\overline{q}_n(123, 132, 312) = \frac{1}{2}(n^2 + 3n - 2)$.*

Proof. Let $\Lambda = \{123, 132, 312\}$ and $n \geq 2$. Write $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. Then α, β, and γ are all weakly decreasing since π avoids 123 and 312. In addition, since π avoids 132, we must have that for any $a \in \operatorname{Set}(\alpha)$, $b \in \operatorname{Set}(\beta)$, and $c \in \operatorname{Set}(\gamma)$, $a > b$, $b > c$, and $a \geq c$. We can only have $a = c$ if $\operatorname{Set}(\beta) = \varnothing$. This leaves two possibilities:

(1) $\operatorname{Set}(\alpha) \cap \operatorname{Set}(\gamma) = \varnothing$, or

(2) $\operatorname{Set}(\alpha) \cap \operatorname{Set}(\gamma) \neq \varnothing$ and $\operatorname{Set}(\beta) = \varnothing$.

In Case (1), $\{\operatorname{Set}(\alpha), \operatorname{Set}(\beta), \operatorname{Set}(\gamma)\}$ is a partition of $[n-1]$ into three (possibly empty) disjoint subsets of consecutive numbers. There are clearly $\binom{n+1}{2}$ such partitions.

In Case (2), we must have that $|\operatorname{Set}(\alpha) \cap \operatorname{Set}(\gamma)| = 1$ since π avoids both 123 and 312. Since both α and γ are weakly decreasing and $a \geq c$ for all $a \in \operatorname{Set}(\alpha)$ and $c \in \operatorname{Set}(\gamma)$, we thus have $\alpha' anna \gamma'$, and in fact $\alpha a a \gamma$ is exactly the weakly decreasing permutation in $\mathcal{S}_{n-1,n-1}$. Since $a \in [n-1]$ and α' and γ' are determined (namely $\alpha = (n-1)(n-1)\ldots(a+1)(a+1)$ and $\beta = (a-1)(a-1)\ldots 2211$), there are $n-1$ such permutations in Case (2). Taking Cases (1) and (2) together, we obtain the result. □

4.4 Avoiding four patterns in $\mathcal{S}_3$

Theorem 4.13. *For all $n \geq 5$, we have that $\overline{q}_n(123, 132, 213, 321)$, $\overline{q}_n(123, 132, 231, 321)$, and $\overline{q}_n(123, 132, 312, 321)$ are all 0.*

Proof. This follows immediately from Lemma 4.1. □

Theorem 4.14. *For all $n \geq 2$, $\overline{q}_n(123, 132, 213, 231) = 4$.*

Proof. Let $\Lambda = \{123, 132, 213, 231\}$ and suppose $n > 2$. Then for any $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$, we must have that $\pi = nn\alpha$ with $\alpha \in \overline{\mathcal{Q}}_{n-1}$. Indeed, if there were some $i < j < k$ so that $\pi_i \neq n$ and either $\pi_j = n$ or $\pi_k = n$, where π_i, π_j, and π_k are distinct, then $\pi_i\pi_j\pi_k$ would be an occurrence of a forbidden pattern. Thus $\overline{q}_n(\Lambda) = \overline{q}_{n-1}(\Lambda)$. Since $\overline{\mathcal{Q}}_2(\Lambda) = \{1122, 1221, 2211, 2112\}$, we must have that $\overline{q}_n(\Lambda) = 4$ for all $n > 2$. □

Theorem 4.15. *For all $n \geq 3$, $\overline{q}_n(123, 132, 231, 312) = 3$.*

Proof. Let $\Lambda = \{123, 132, 231, 312\}$ and suppose $n > 2$. Let $\pi \in \overline{\mathcal{Q}}_n(\Lambda)$. Since π avoids 132 and 231, n must lie either at the beginning or end of the permutation. If $\pi_1 = n$, then the values in $\{1, 1, 2, 2, \ldots, n-1, n-1\}$ must appear in weakly decreasing order since π avoids 312. Similarly, if $\pi_{2n} = n$, then the values in $\{1, 1, 2, 2, \ldots, n-1, n-1\}$ must appear in weakly decreasing order since π avoids 123. Thus the only elements of $\overline{\mathcal{Q}}_n(\Lambda)$ are $nn(n-1)(n-1)\cdots 2211$, $n(n-1)(n-1)\cdots 2211n$, and $(n-1)(n-1)\cdots 2211nn$. □

Theorem 4.16. *For all $n \geq 3$, $\overline{q}_n(132, 213, 231, 312) = 2$.*

Proof. For all $n \geq 3$, the only patterns of length 3 that are allowed in a permutation $\pi \in \overline{\mathcal{Q}}_n(132, 213, 231, 312)$ are 123 and 321. Thus only the weakly increasing permutation $1122\ldots nn$ and the weakly decreasing permutation $nn\ldots 2211$ are in $\overline{\mathcal{Q}}_n(132, 213, 231, 312)$. □

4.5 Avoiding five patterns in $\mathcal{S}_3$

Theorem 4.17. *For all $n \geq 5$, $\overline{q}_n(123, 132, 213, 231, 321) = 0$.*

Proof. This follows immediately from Lemma 4.1. □

Theorem 4.18. *For all $n \geq 3$, $\overline{q}_n(123, 132, 213, 231, 312) = 1$.*

Proof. Let $n \geq 3$, and suppose $\pi \in \overline{\mathcal{Q}}_n(123, 132, 213, 231, 312)$. Since the only pattern of length 3 that is allowed to appear is the decreasing pattern, every length 3 subsequence of π is weakly decreasing, thus π is the weakly decreasing permutation. □

4.6 Number of plateaus

We finish this section by enumerating the set of quasi-Stirling permutations with a given number of plateaus, i.e. *consecutive* occurrences of the pattern 11. We note that the bijection φ, as defined in the previous section, sends leaves of an ordered rooted labeled tree to plateaus in the associated quasi-Stirling permutation. In other words, if the tree T has a leaf labeled i, then there is some $j \in [2n-1]$ so that for $\pi = \varphi(T)$, $\pi_j = \pi_{j+1} = i$. It is a well-known result that the number of ordered rooted unlabeled trees on $[n]$ with k leaves is given by the Narayana numbers,

$$\frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}.$$

See for example [21] and [20, A001263]. From this, one can immediately obtain the following proposition.

Proposition 4.1. *The number of quasi-Stirling permutations on* $[n, n]$ *with exactly* k *plateaus is equal to*

$$\frac{n!}{k}\binom{n-1}{k-1}\binom{n}{k-1}.$$

5 Open questions

Avoid other patterns. In this paper, we do not enumerate the number of quasi-Stirling permutations that avoid a single pattern of length 3. Though we made some progress, we were unable to produce a recurrence or closed form for these permutations. Thus it remains open to enumerate $\overline{\mathcal{Q}}_n(321)$ and $\overline{\mathcal{Q}}_n(312)$. Additionally, it is open to enumerate the set of quasi-Stirling permutations that avoid other classes of patterns, like consecutive patterns, vincular or bivincular patterns, mesh patterns, etc. (See [11] for example.) It may also be interesting to consider the number of quasi-Stirling permutations avoiding certain patterns in $\mathcal{S}_n^2$, like the pattern 12321, which would correspond to the number of rooted ordered labeled trees that avoid the pattern 123 (as defined in [1] for unordered trees).

Statistics. In this paper, we enumerate quasi-Stirling permutations by the number of plateaus. It is an open question to enumerate these permutations by the number of descents, by the number of left-to-right maxima, or any other statistic. Based on numerical evidence, we conjecture that the number of quasi-Stirling permutations on $[n, n]$ with exactly $n-1$ descents is $(n+1)^{n-1}$. Since this number coincides with the number of rooted *unordered* forests on n nodes, this may indicate one can use the bijection φ to rooted ordered trees in some way to prove this conjecture.

Generalizations. In several contexts, Stirling permutations have been generalized to permutations of any multiset that avoid 212. A similar generalization could be done for quasi-Stirling permutations. It may be interesting to investigate such permutations.

Pattern avoidance in forests. Pattern-avoidance is closely related to the study of pattern avoidance in forests. For example, the number of inversions (i.e., an occurrence of 21) in a rooted labeled ordered tree is equal to the number of occurrences of the pattern 2112 in quasi-Stirling permutations. The authors know of no study of pattern avoidance of ordered trees, but much has been studied for inversions of unordered trees (see for example, [16, 6]) and other statistics and pattern avoidance in unordered trees (see for example [7, 1]).

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