AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **74(2)** (2019), Pages 344–363

The r-matching sequencibility of complete multi-k-partite k-graphs

Adam Mammoliti

School of Mathematics and Statistics
UNSW
Sydney, NSW 2052
Australia
adam.mammoliti@outlook.com.au

Abstract

Alspach [*Bull. Inst. Combin. Appl.* 52 (2008), 7–20] defined the maximal matching sequencibility of a graph G, denoted $ms(G)$, to be the largest integer s for which there is an ordering of the edges of G such that every s consecutive edges form a matching. In this paper, we consider the natural analogue for hypergraphs of this and related results and determine $ms(\lambda\mathcal{K}_{n_1,\dots,n_k})$ where $\lambda\mathcal{K}_{n_1,\dots,n_k}$ denotes the multi-k-partite k-graph with edge multiplicity λ and parts of sizes $n_1,\dots,n_k$, respectively. It turns out that these invariants may be given surprisingly precise and somewhat elegant descriptions, in a much more general setting.

1 Introduction

Alspach [1] defined the (maximal) *matching sequencibility* of a graph G, denoted $ms(G)$, to be the maximum integer s such that there exist an ordering of the edges of G so that each s consecutive edges form a matching. Alspach [1] determined the value of $ms(K_n)$, as follows.

Theorem 1.1. *For an integer $n \geq 3$,*

$$ms(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor .$$

Katona [4] implicitly considered the *cyclic matching sequencibility* $cms(G)$ of a graph G which is the natural analogue of the matching sequencibility for G when cyclic orderings are allowed. Brualdi, Kiernan, Meyer and Schroeder [3] defined this invariant explicitly and proved the cyclic analogue of Theorem 1.1, below, thus strengthening a weaker result by Katona [4].

ISSN: 2202-3518

Theorem 1.2 (Brualdi et al. [3])**.** *For an integer $n \geq 4$,*

$$cms(K_n) = \left\lfloor \frac{n-2}{2} \right\rfloor .$$

Let $K_{n,m}$ be the complete bipartite graph with parts of cardinality n and m. Brualdi et al. [3] also found the matching and cyclic matching sequencibility of complete bipartite graphs, as follows.

Theorem 1.3. *For integers n and m with $2 \leq n \leq m$,*

$$ms(K_{n,m}) = cms(K_{n,m}) = \begin{cases} n & \text{if } n < m\,; \\ n-1 & \text{if } n = m\,. \end{cases}$$

The aim of this paper is to generalise Theorem 1.3 considerably with respect to a more general notion of matching sequencibility and a more general notion of graphs. It turns out that the resulting invariants may be given surprisingly precise and somewhat elegant descriptions; see Theorem 1.4 below. We will consider the following generalisation of matching sequencibility given in [6]. For a graph G, $ms_r(G)$ denotes the analogue of $ms(G)$ where consecutive edges are required to form a graph with maximal vertex degree at most r. Similarly, $cms_r(G)$ is defined in analogy to $ms_r(G)$ where we allow cyclic orderings of G's edges. A hypergraph $\mathcal{H}$ is a pair (V, E) where V is a set and E is a multiset of subsets of V. The complete k-partite k-graph with parts of cardinalities $n_1, \dots, n_k$, denoted $\mathcal{K}_{n_1,\dots,n_k}$, is the hypergraph whose vertex set is the union of disjoint sets $N_1, \dots, N_k$ of cardinalities $n_1, \dots, n_k$, respectively, and whose edge set is the family of every k-set containing exactly one member of $N_1, \dots, N_k$, respectively. For a hypergraph $\mathcal{H} = (V, E)$, we let $ms_r(\mathcal{H})$ and $cms_r(\mathcal{H})$ denote the natural analogues of $ms_r(G)$ and $cms_r(G)$ for hypergraphs, respectively. Furthermore, for any positive integer λ, let $\lambda\mathcal{H}$ be the hypergraph $\mathcal{H}' = (V, E')$ where E' contains λ distinct copies of e for each $e \in E$. For $r \geq \Delta(\mathcal{H})$, the maximal vertex degree of $\mathcal{H}$, these invariants trivially equal $|E(\mathcal{H})|$. We will extend the above definitions of $ms_r(\mathcal{H})$ and $cms_r(\mathcal{H})$ for $r < \Delta(\mathcal{H})$ to non-trivial definitions of these invariants for $r \geq 1$. However, the details are technical and will be given later, in Subsection 2.1.

The main result of this paper is the following theorem, which succeeds, perhaps surprisingly, to precisely describe the values of $ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k})$ and $cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k})$.

Theorem 1.4. *Let $1 \leq n_1 = n_2 = \cdots = n_u < n_{u+1} \leq \cdots \leq n_k$ and $r = r_1\lambda\prod_{i=2}^{k} n_i + r_2$, for non-negative integers r_1, r_2 with $0 \leq r_2 \leq \lambda\prod_{i=2}^{k} n_i - 1$. Then*

$$ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = \begin{cases} rn_1 & \text{if } n_1^{u-1} \mid r_2 \text{ or (1), below, holds}\,; \\ rn_1 - 1 & \text{otherwise}\,, \end{cases}$$

and

$$cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = \begin{cases} rn_1 & \text{if } n_1^{u-1} \mid r_2; \\ rn_1 - 1 & \text{otherwise}\,, \end{cases}$$

where

$$\left(\left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor + 1\right) \left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor \leq \lambda \prod_{i=u+1}^{k} n_i \leq \left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor \left(\left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor + 1\right). \quad (1)$$

Theorem 1.4 includes Theorem 1.3 as a special case, which is more evident from Theorem 1.4 when $r = 1$, given below.

Corollary 1.5. *Let $n_1 \leq n_2 \leq \cdots \leq n_k$. Then*

$$ms(\lambda\mathcal{K}_{n_1,\ldots,n_k}) = cms(\lambda\mathcal{K}_{n_1,\ldots,n_k}) = \begin{cases} rn_1 & \text{if } n_1 < n_2\,; \\ rn_1 - 1 & \text{otherwise}\,. \end{cases}$$

Section 2 contains definitions and auxiliary results. The rest of the paper is mostly dedicated to proving Theorem 1.4. The proof of Theorem 1.4 is divided into three technical sections and a concluding section, namely, Sections 3-6. Section 7 concludes the paper with examples of interest to the auxiliary results in Section 2 as well as a conjecture on the value of $ms(K_{s(n)})$ and $cms(K_{s(n)})$ for complete multi-partite graphs $K_{s(n)}$.

2 Preliminary definitions and auxiliary results

For technical reasons we will, contrary to the introduction, define hypergraphs without the use of "multisets" in the following manner. A hypergraph $\mathcal{H} = (V, E)$ is a pair consisting of two sets, the set of *vertices* V of $\mathcal{H}$ and the set of *edges* E of $\mathcal{H}$, where each edge $e \in E$ has associated to it a prescribed set of vertices. Each such associated vertex $v \in V$ is said to be *incident* with $e \in E$ and this is denoted by $v \in e$. Here, the two distinct edges $e, e' \in E$ can be incident with the same set of vertices, in which case e and e' are *parallel*. We can thus view the edges of a hypergraph as a family of distinctly labelled sets comprising not necessarily distinct collections of vertices.

For an integer n, let $[n] := \{0, 1, \ldots, n-1\}$. An *ordering* or *labelling* of a hypergraph $\mathcal{H} = (V, E)$ is a bijective function $\ell : E \to [|E|]$. The image of e under ℓ is called the *label* of e. A sequence of edges $e_0, \ldots, e_{s-1}$ is *consecutive* in ℓ if the labels of $e_0, \ldots, e_{s-1}$ are consecutive integers, respectively. For a sequence S of edges, define $\mathcal{H}(S)$ to be the hypergraph whose edges are those in the sequence S and whose vertices are the vertices incident with these edges.

For an ordering ℓ of a hypergraph $\mathcal{H}$, let $ms_r(\ell)$ denote the maximum integer s such that, for every sequence S of s consecutive edges of ℓ, $\Delta(\mathcal{H}(S)) \leq r$. Define the *$r$-matching sequencibility* of $\mathcal{H}$, denoted by $ms_r(\mathcal{H})$, to be the maximum value of $ms_r(\ell)$ over all orderings ℓ of $\mathcal{H}$. In particular, the special case $ms_1(\mathcal{H})$, which we denote as $ms(\mathcal{H})$, is the same invariant as presented in the Introduction.

A sequence of edges $e_0, \ldots, e_{s-1}$ of a hypergraph $\mathcal{H} = (V, E)$ is *cyclically consecutive* in ℓ if the labels of $e_0, \ldots, e_{s-1}$ are consecutive integers modulo $|E|$, respectively.

We define $cms_r(\ell)$ and $cms_r(\mathcal{H})$ analogously to $ms_r(\ell)$ and $ms_r(\mathcal{H})$, respectively, where we now consider sequences of cyclically consecutive edges. We first consider cases when $r < \Delta(\mathcal{H})$, as the cases when $r \geq \Delta(\mathcal{H})$ are somewhat different and will be dealt with in Subsection 2.1. The following lemma was presented in [6] and we shall give a proof for completeness.

Lemma 2.1. *For a hypergraph $\mathcal{H}$ with ordering ℓ and integers r_1, r_2 with $r_1 r_2 < \Delta(\mathcal{H})$,*

$$r_2 \; ms_{r_1}(\mathcal{H}) \leq ms_{r_1 r_2}(\mathcal{H}) \quad \text{and} \quad r_2 \; cms_{r_1}(\mathcal{H}) \leq cms_{r_1 r_2}(\mathcal{H}) \, .$$

Proof. Let ℓ be a labelling of $\mathcal{H}$ such that $cms_{r_1}(\ell) = cms_{r_1}(\mathcal{H})$. Any sequence S of $r_2 \; cms_{r_1}(\ell)$ cyclically consecutive edges of ℓ consists of r_2 subsequences of $cms_{r_1}(\ell)$ cyclically consecutive edges of ℓ and each subsequence forms a hypergraph for which every vertex has degree at most r_1. Thus, every vertex has degree at most $r_1 r_2$ in $\mathcal{H}(S)$. Hence,

$$cms_{r_1 r_2}(\mathcal{H}) \geq cms_{r_1 r_2}(\ell) \geq r_2 \; cms_{r_1}(\ell) = r_2 \; cms_{r_1}(\mathcal{H}) \, .$$

The non-cyclic case is similar and, therefore, omitted. □

For edge-disjoint hypergraphs $\mathcal{H}_0, \ldots, \mathcal{H}_{a-1}$ on the same vertex set V, with labellings $\ell_0, \ldots, \ell_{a-1}$, respectively, let $\ell_0 \vee \cdots \vee \ell_{a-1}$ denote the ordering ℓ of $G = \big(V, \bigcup_{i=0}^{a-1} E(\mathcal{H}_i)\big)$ defined by $\ell(e_{i,j}) = \ell_j(e_{i,j}) + \sum_{l=0}^{j-1} |E(\mathcal{H}_l)|$ where $e_{ij} \in E(\mathcal{H}_j)$ for all i and j. Let s be an integer and $\mathcal{H}$ and $\mathcal{H}'$ be edge-disjoint hypergraphs on the same vertex set V, each having at least $s-1$ edges. Also, let $\mathcal{H}$ and $\mathcal{H}'$ have labellings ℓ and ℓ', respectively, and let $\mathcal{H}_s$ be the subhypergraph of $\big(V, E(\mathcal{H}) \cup E(\mathcal{H}')\big)$ that consists of the last $s-1$ edges of ℓ and the first $s-1$ edges of ℓ'. Then we will let $\ell \vee_s \ell'$ denote the ordering of $\mathcal{H}_s$ for which the edges of $\mathcal{H}_s$ appear in the same order as they do in $\ell \vee \ell'$. We now define $ms_r(\ell, \ell')$ to be the largest integer s such that $ms_r(\ell \vee_s \ell') \geq s$.

A *matching* of a hypergraph $\mathcal{H}$ is a subhypergraph $\mathcal{M}$ in which every vertex has degree 1. A *matching decomposition* of a hypergraph $\mathcal{H} = (V, E)$ is a set of matchings of $\mathcal{H}$ that partition the edge set E. The following proposition, presented in [6], gives a lower bound on the r-cyclic matching sequencibility, given that a matching decomposition with certain properties exists. In the proposition, the subscripts of the orderings ℓ_i are taken modulo t: $\ell_{i+r} = \ell_{i'}$ holds exactly when $i' \equiv i + r \pmod{t}$.

Proposition 2.2. *Let $\mathcal{H}$ be a hypergraph that decomposes into matchings $\mathcal{M}_0, \ldots, \mathcal{M}_{t-1}$, each with n edges and orderings $\ell_0, \ldots, \ell_{t-1}$, respectively. Suppose, for some $x \in [n]$ and $r < \Delta(G)$, that $ms(\ell_i, \ell_{i+r}) \geq n - x$ for all $i \in [t-r]$. Then $ms_r(G) \geq rn - x$, and if $ms(\ell_i, \ell_{i+r}) \geq n - x$ for all $i \in [t]$, then $cms_r(G) \geq rn - x$.*

The following definitions are used here and throughout the paper. For a hypergraph $\mathcal{H}$ with ordering ℓ, $S_\ell(\mathcal{H})$ denotes the sequence of edges of $\mathcal{H}$ listed in the same order as ℓ, and ℓ *corresponds* to $S_\ell(\mathcal{H})$; i.e., if $e_0, \ldots, e_{k-1}$ is a sequence of the edges of $\mathcal{H}$, then ℓ corresponds to that sequence if $\ell(e_i) = i$ for all $i \in [k]$. We will omit the subscript ℓ if the ordering is clear. Also, for edge disjoint graphs $\mathcal{H}_0, \dots, \mathcal{H}_{a-1}$ with labellings $\ell_0, \dots, \ell_{a-1}$, respectively, one can check that the ordering $\ell = \ell_0 \vee \cdots \vee \ell_{a-1}$ corresponds to sequence $S_{\ell_0}(\mathcal{H}_0) \vee \cdots \vee S_{\ell_{a-1}}(\mathcal{H}_{a-1})$. Proposition 2.2 was proven for graphs in [6]. We provide the details for hypergraphs for completeness.

Proof of Proposition 2.2. We consider only the cyclic case, as the non-cyclic case is similar. Let ℓ be the ordering corresponding to $S_{\ell_0}(\mathcal{M}_0) \vee \cdots \vee S_{\ell_{t-1}}(\mathcal{M}_{t-1})$. Consider a sequence S of $rn - x$ consecutive edges of ℓ. The sequence S is of the form

$$\underbrace{e_1, \dots, e_j}_{\text{edges in } \mathcal{M}_i}, S_{\ell_{i+1}}(\mathcal{M}_{i+1}) \vee \cdots \vee S_{\ell_{i+r-a}}(\mathcal{M}_{i+r-a}), \underbrace{e_{j+1}, \dots, e_{an-x}}_{\text{edges in } \mathcal{M}_{i+r+1-a}},$$

for some $i \in [t]$, $j \in [n+1]$, and a. If S contains edges from only one matching $\mathcal{M}_l$, then S is a subsequence of $S_\ell(\mathcal{M}_l)$ and $r = 1$. Then we are done, as $\mathcal{H}(S)$ is clearly a matching. Hence, without loss of generality, we can assume that S contains edges from each of $\mathcal{M}_i$ and $\mathcal{M}_{i+r+1-a}$. Let S' be the sequence of the edges of S which are in either $\mathcal{M}_i$ or $\mathcal{M}_{i+r+1-a}$, in order with respect to S. There are $0 < an - x \leq 2n$ edges in S'. Therefore, $a = 1$ or $a = 2$.

If $a = 2$, then S is a subsequence of $S(\mathcal{M}_i) \vee \cdots \vee S(\mathcal{M}_{i+r-1})$, and, hence, $\Delta(\mathcal{H}(S)) \leq r$. If $a = 1$, then the first j edges and last $n - j - x$ edges of S and thus S' form the sequence of the last j edges of ℓ_i and the first $n - j - x$ edges of ℓ_{i+u+1}, respectively. Therefore, the $j + n - j - x = n - x$ edges of S' are consecutive in $\ell_i \vee_{n-x} \ell_{i+r}$. By assumption, $ms(\ell_i, \ell_{i+r}) \geq n_1 - x$, so $\mathcal{H}(S')$ must be a matching. The edges of S not in S' are from the $r - 1$ matchings $\mathcal{M}_{i+1}, \dots, \mathcal{M}_{i+r-1}$. Thus, $\Delta(\mathcal{H}(S)) \leq r$. □

An ordering of a set A is a bijective function $\sigma : A \to [|A|]$. Many of the matching decompositions that we will use henceforth have a natural indexing which is not directly compatible with Proposition 2.2. In such cases we will find it useful to be able to find an ordering of the set of indices, with particular properties. To do this, we will make use of the following lemma, first given in [6].

Lemma 2.3. *Let $s < t$ be integers and set $d := \gcd(s, t)$. Define $a_{i,j} := \left(j \bmod \frac{t}{d}\right) + \left(i\frac{t}{d} \bmod t\right)$ for all integers i and j. Then some ordering σ of $[t]$ satisfies $\sigma(a_{i,j+1}) = (\sigma(a_{i,j}) + s) \bmod t$ for all $i \in [d]$ and $j \in \left[\frac{t}{d}\right]$.*

Proof. We check that the function $\sigma : [t] \to [t]$ defined by $\sigma(a_{i,j}) = (i + js) \pmod t$ for $i \in [d]$ and $j \in \left[\frac{t}{d}\right]$ will suffice. Suppose that $i + js \equiv i' + j's \pmod t$ for some $i, i' \in [d]$ and $j, j' \in \left[\frac{t}{d}\right]$. Then $i - i' \equiv (j' - j)s \pmod t$. As d divides s and t, any multiple of s modulo t is also a multiple of d. Thus, $i - i'$ is a multiple of d, while $0 \leq |i - i'| \leq d - 1$. This is only possible if $i = i'$ and so $(j - j')s \equiv 0 \pmod t$. As $0 \leq |j' - j| \leq \frac{t}{d} - 1$ and $\operatorname{lcm}(s, t) = \frac{st}{d}$, we must also have that $j = j'$. Thus, σ is injective and so bijective; σ is thus an ordering of $[t]$. For any $i \in [d]$ and $j \in \left[\frac{t}{d}\right]$,

$$\sigma(a_{i,j+1}) = (i + (j+1)s) \pmod t = (\sigma(a_{i,j}) + s) \pmod t\,.$$

Hence, σ has the required properties. □

The function σ in the lemma also satisfies an analogous non-cyclic property, as follows.

Corollary 2.4. *Let $s < t$ be integers. Then there exists an ordering τ of $[t]$ with the property that, if $\tau(a) \leq t - s - 1$, then $\tau(a+1) = \tau(a) + s$.*

We use Lemma 2.3 to give an analogous version of Proposition 2.2 for the cyclic case.

Proposition 2.5. *Let $\mathcal{H}$ be a hypergraph that decomposes into matchings $\mathcal{M}_{i,j}$, each with n edges and orderings $\ell_{i,j}$ for $i \in [d]$ and $j \in [c]$, respectively. Suppose, for some $x \in [n]$ and $r < \Delta(\mathcal{H})$, that $\gcd(dc, r) = d$ and $ms(\ell_{i,j}, \ell_{i,j+1}) \geq n - x$ for all $i \in [d]$ and $j \in [c]$. Then $cms_r(\mathcal{H}) \geq rn - x$.*

Proof. Let $a_{i,j}$ and σ be as defined in Lemma 2.3 for $s = r$ and $t = cd$. Set $\mathcal{M}_{\sigma(a_{i,j})} := \mathcal{M}_{i,j}$ and $\ell_{\sigma(a_{i,j})} := \ell_{i,j}$ for all $i \in [d]$ and $j \in [c]$. For $l \in [t]$, let $l = \sigma(a_{i,j})$. By Lemma 2.3, $\sigma(a_{i,j+1}) \equiv \sigma(a_{i,j}) + r \equiv l + r \pmod{t}$. Hence, $ms(\ell_l, \ell_{l+r}) = ms(\ell_{i,j}, \ell_{i,j+1}) \geq n - x$. Therefore, the conditions of Proposition 2.2 are satisfied and the result follows. □

One could also use Corollary 2.4 to create an analogous version of Proposition 2.2 for the non-cyclic case, but we will not require this.

2.1 Non-trivial definitions of $ms_r(\mathcal{H})$ and $cms_r(\mathcal{H})$ for all $r \geq 1$

If $\mathcal{H}$ is a hypergraph with maximum degree $\Delta(\mathcal{H})$ and $r \geq \Delta(\mathcal{H})$, then one might say that, trivially, $ms_r(\mathcal{H}) = |E(\mathcal{H})|$, as clearly any sequence of edges containing all the edges of $\mathcal{H}$ form $\mathcal{H}$, which has no vertex of degree greater than r. Somewhat implicitly, the definition of cyclic r-matching sequencibility allows $r \geq \Delta(\mathcal{H})$, and $cms_r(\mathcal{H})$ is non-trivial in general. However, when $r < \Delta(\mathcal{H})$, $ms_r(\mathcal{H})$ and $cms_r(\mathcal{H})$ have the intuitive relationship $cms_r(\mathcal{H}) \leq ms_r(\mathcal{H})$ for any $\mathcal{H}$. Thus, to preserve that relationship for all r and make the determination of $ms_r(\mathcal{H})$ for hypergraphs with $r \geq \Delta(\mathcal{H})$ of interest, we will give a definition of $ms_r(\mathcal{H})$ which is non-trivial in general, for all $r \geq 1$.

Let $\mathcal{H} = (V, E)$ be a hypergraph with an ordering ℓ and, to use the notation of Bondy and Murty [2], let $\varepsilon := |E|$. First, recall the notion of cyclically consecutive edges. A sequence $S = e_0, \ldots, e_{s-1}$ of edges in E is cyclically consecutive in ℓ if the labels of $e_0, \ldots, e_{s-1}$ are cyclically consecutive integers modulo ε, respectively. In particular, a sequence of $s > \varepsilon$ edges can be cyclically consecutive, where e_i and $e_{i+\varepsilon}$ must be the same edge, for all $i \in [s - \varepsilon]$. We define $\mathcal{H}(S)$ to be the hypergraph with (distinctly labelled) edges $e_0, \ldots, e_{s-1}$

We now define $ms_r(\mathcal{H})$ for all $r \geq 1$. For an integer s, let a be the integer such that $a\varepsilon \leq s < (a+1)\varepsilon$. A sequence $e_0 \ldots, e_{s-1}$ of edges of $\mathcal{H}$ is *consecutive* in ℓ if $\ell(e_0) \leq (a+1)\varepsilon - s$ and the labels of $e_0 \ldots, e_{s-1}$ are cyclically consecutive integers modulo ε, respectively. The definition of consecutive edges, given earlier in the section, is recovered by setting $a = 0$. Define $ms_r(\ell)$ to be the largest value s

such that, for every sequence S of s consecutive edges in ℓ, $\Delta(\mathcal{H}(S)) \leq r$. Define $ms_r(\mathcal{H})$ to be the largest value of $ms_r(\ell)$ over all orderings ℓ of $\mathcal{H}$. As the edges in a sequence $S = e_0, \ldots, e_{s-1}$ of consecutive edges of ℓ are also cyclically consecutive under the restriction $\ell(e_0) \leq (a+1)\varepsilon - s$, it follows that $cms_r(\ell) \leq ms_r(\ell)$ and, thus, $cms_r(\mathcal{H}) \leq ms_r(\mathcal{H})$ for all positive integers r and hypergraphs $\mathcal{H}$.

We now demonstrate that $ms_r(\mathcal{H})$, as defined above, is non-trivial in general. For a hypergraph $\mathcal{H} = (V, E)$ and positive integer λ, let $\lambda\mathcal{H}$ be the hypergraph $\mathcal{H}' = (V, E')$ where E' is formed from E by replacing each $e \in E$ with λ distinct edges parallel to e. For an ordering ℓ of $\mathcal{H}$ and integer a, let $a\ell := \ell \vee \cdots \vee \ell$, where ℓ occurs a times. That is, $a\ell$ corresponds to the sequence $e_0, \ldots, e_{a\varepsilon-1}$ of edges of $\mathcal{H}$ such that $S_\ell(\mathcal{H}) = e_0, \ldots, e_{\varepsilon-1}$, and e_i and $e_{i+\varepsilon}$ are the same edge for all $i \in [(a-1)\varepsilon]$. In particular, for an integer s such that $a\varepsilon \leq s < (a+1)\varepsilon$, the set of all sequences S of s consecutive edges of ℓ is the set of all sequences S' of s consecutive edges of $(a+1)\ell$. Also, the hypergraph formed by the sequence corresponding to $b\ell$ is $b\mathcal{H}$ for all positive integers b. So, for any r, if a is the integer such that $a\Delta(\mathcal{H}) \leq r < (a+1)\Delta(\mathcal{H})$, then $ms_r(\mathcal{H}) = s$ for some s such that $a\varepsilon \leq s < (a+1)\varepsilon$ and, in general, the value s is non-trivial for any $r \geq 1$ and hypergraph $\mathcal{H}$.

The two following lemmas will each be used in several parts of the proof of Theorem 1.4.

Lemma 2.6. *Let $\mathcal{H}$ be a hypergraph with ε edges and maximum degree Δ, and $r = a\Delta + b$ for non-negative integers a and b with $b \in [\Delta]$. Then*

$$a\varepsilon + ms_b(\mathcal{H}) \leq ms_r(\mathcal{H}) \quad \text{and} \quad a\varepsilon + cms_b(\mathcal{H}) \leq cms_r(\mathcal{H})\,.$$

Proof. Let $s = a\varepsilon + ms_b(\mathcal{H})$ and ℓ be an ordering of $\mathcal{H}$ satisfying $ms_b(\ell) = ms_b(\mathcal{H})$. Consider a sequence $S = e_0, \ldots, e_{s-1}$ of s consecutive edges of ℓ. As $e_i = e_{i+\varepsilon}$ for all $i \in [s-\varepsilon]$, $a+1$ copies of the edge e_j occur in the sequence S if $j \in [s-a\varepsilon]$, and a copies of the edge e_j occur if $s - a\varepsilon \leq j \leq \varepsilon - 1$. In particular, $\mathcal{H}(S)$ is the hypergraph obtained by adding to $a\mathcal{H}$ an edge parallel to e for each edge e in the sequence $S' := e_0, \ldots, e_{s-a\varepsilon-1}$. The sequence S' is consecutive in ℓ, as $\ell(e_0) \leq (a+1)\varepsilon - s$. Since $ms_b(\ell) = s - a\varepsilon$, $\Delta(\mathcal{H}(S')) \leq b$. Thus, the degree of a vertex v in $\mathcal{H}(S)$ is at most $a\deg_{\mathcal{H}}(v) + b \leq a\Delta + b = r$. Hence, $ms_r(\mathcal{H}) \geq ms_r(\ell) \geq s = a\varepsilon + ms_b(\mathcal{H})$. The cyclic case is similar. □

Lemma 2.7. *For a hypergraph $\mathcal{H}$ and $\lambda \geq 1$, $cms_r(\lambda\mathcal{H}) \geq cms_r(\mathcal{H})$.*

Proof. Let ℓ be an ordering of $\mathcal{H}$ satisfying $cms_r(\ell) = cms_r(\mathcal{H})$. For an edge $e \in E(\mathcal{H})$, let $e'_0, \ldots, e'_{\lambda-1}$ be the corresponding edges parallel to e in $E(\lambda\mathcal{H})$. By identifying each of $e'_0, \ldots, e'_{\lambda-1}$ with a unique copy of e in the sequence $S_{\lambda\ell}(\mathcal{H})$, we can define $\ell' = \lambda\ell$ to be an ordering of $\lambda\mathcal{H}$. For any sequence S of s cyclically consecutive edges of ℓ and the corresponding sequence S' of s cyclically consecutive edges of ℓ', clearly $\mathcal{H}(S) = \mathcal{H}(S')$. Therefore, $cms_r(\ell') = cms_r(\ell)$ and, thus, $cms_r(\lambda\mathcal{H}) \geq cms_r(\mathcal{H})$. □

An analogous result to Lemma 2.7 in the non-cyclic case does not hold; see Section 7.

3 Proof of Theorem 1.4: Part I

Theorem 1.4 will be proved by a set of lemmas that fall into three separate categories, each to be addressed in this and the next two sections. The first two of these lemmas are given in the present section.

We start by introducing the following notation, which will be used in the remainder of the paper. Let $\lambda \geq 1$, $1 \leq n_1 \leq \cdots \leq n_k$ and u be the largest integer such that $n_1 = n_u$. Let $N = \prod_{i=2}^{k} n_i$, $N' = \prod_{i=u+1}^{k} n_i$, $r = r_1\lambda N + r_2$ and $\lambda N = ar_2 + b$ for integers a, b, r_1 and r_2 such that $r_2 \in [\lambda N]$ and $b \in [r_2]$.

Recall from the Introduction that the complete k-partite k-hypergraph, denoted by $\mathcal{K}_{n_1,\dots,n_k}$, is the hypergraph whose vertex set V is the union of disjoint sets $N_1, \dots, N_k$ of sizes $n_1, \dots, n_k$, respectively, and whose edge set E is the family of all k-edges that have exactly one endpoint in N_i for all i. We note that the inequality $ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) \leq rn_1$ is trivial for all r as every edge incident with one of the n_1 vertices of N_1 and, therefore, a sequence of at most rn_1 edges of $\lambda\mathcal{K}_{n_1,\dots,n_k}$ can form a hypergraph with maximum degree at most r. Thus, the inequalities $cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) \leq ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) \leq rn_1$ will always hold.

The following claim is an immediate necessary condition for an ordering ℓ of $\lambda\mathcal{K}_{n_1,\dots,n_k}$ to satisfy $ms_r(\ell) = rn_1$ or $cms_r(\ell) = rn_1$.

Claim 3.1. *Let ℓ be an ordering of $\lambda\mathcal{K}_{n_1,\dots,n_k}$. If $ms_r(\ell) = rn_1$, then the edges $\ell^{-1}(j)$ and $\ell^{-1}(r_2n_1 + j)$ are incident with the same vertex in N_i for all $i = 1, \dots, u$ and $j \in [\lambda Nn_1 - r_2n_1]$. If $cms_r(\ell) = rn_1$, then the edges $\ell^{-1}(j)$ and $\ell^{-1}((r_2n_1 + j) \bmod \lambda Nn_1)$ are incident with the same vertex in N_i for all $i = 1, \dots, u$ and $j \in [\lambda N]$.*

Proof. We only prove the non-cyclic case as the cyclic case is similar. Let ℓ be an ordering of $\lambda\mathcal{K}_{n_1,\dots,n_k}$ such that $ms_r(\ell) = rn_1$, and let $\varepsilon := |E(\lambda\mathcal{K}_{n_1,\dots,n_k})| = \lambda Nn_1$.

Consider a sequence $S = e_0, \dots, e_{rn_1}$ of consecutive edges of ℓ, where, by definition, $j := \ell(e_0) \in [(r_1 + 1)\varepsilon - rn_1] = [\varepsilon - r_2n_1]$. The sequence $S' = e_1, \dots, e_{rn_1-1}$ consists of $rn_1 - 1$ consecutive edges of ℓ and so $(\mathcal{H}(S')) \leq r$.

As every edge in $E(\lambda\mathcal{K}_{n_1,\dots,n_k})$ is incident with a vertex in each of $N_1, \dots, N_u$ and $|N_i| = n_1$ for $i \leq u$, every vertex in each of $N_1, \dots, N_u$ must have degree exactly r in $\mathcal{H}(S')$, except for some $v_1 \in N_1, \dots, v_u \in N_u$ which each have degree $r - 1$. Thus, in order for the hypergraphs formed by the sequences $S_0 = e_0, \dots, e_{rn_1-1}$ and $S_1 = e_1, \dots, e_{rn_1}$ to each have maximum degree at most r, the edges e_0 and e_{rn_1} must be incident with each of $v_1 \in N_1, \dots, v_u \in N_u$. As $e_i = e_{i+\varepsilon}$ for all $i \in [rn_1 - \varepsilon]$, $e_{rn_1} = e_{r'}$ for $r' = rn_1 \bmod \varepsilon$. Since $r = r_1\lambda N + r_2$ and $\varepsilon = \lambda Nn_1$, it follows that e_0 and $e_{r'} = e_{r_2n_1}$ are incident with $v_1, \dots, v_u$; i.e., the edges $\ell^{-1}(j)$ and $\ell^{-1}(r_2n_1 + j)$ are incident with the same vertex in N_i for all $i = 1, \dots, u$, as required. □

Lemma 3.2. *If $ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = rn_1$, then $n_1^{u-1} \mid r_2$ or*

$$\left(\left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor + 1\right) \left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor \leq \lambda \prod_{i=u+1}^{k} n_i \leq \left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor \left(\left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor + 1\right).$$

Proof. Let ℓ be an ordering of $\lambda\mathcal{K}_{n_1,\dots,n_k}$ such that $ms_r(\ell) = rn_1$. Let $S_\ell(\lambda\mathcal{K}_{n_1,\dots,n_k}) = e_0, \dots, e_{\lambda N n_1 - 1}$ and let $S = e'_0, \dots, e'_{\lambda N n_1 - 1}$ be the sequence of edges from $E(\mathcal{K}_{n_1,\dots,n_u})$ such that if e_i is incident with each of $v_1 \in N_1, \dots, v_u \in N_u$, then e'_i is the edge in $E(\mathcal{K}_{n_1,\dots,n_u})$ incident with each of $v_1, \dots, v_u$. For an edge $e \in E(\mathcal{K}_{n_1,\dots,n_u})$, let $d(e)$ be the number of times that e appears among the first $r_2 n_1$ edges of S. Similarly, let $d'(e)$ number of times that e appears among the first bn_1 edges of S, where $d'(e)$ is 0 for all e if $b = 0$.

We count in two ways the number times that an edge $e \in E(\mathcal{K}_{n_1,\dots,n_u})$ appears in S. For all $j \in [\lambda N n_1 - r_2 n_1]$, Claim 3.1 implies that the edges e_j and $e_{r_2 n_1 + j}$ are incident with the same vertex in N_i for $i = 1, \dots, u$. Therefore, $e'_j = e'_{r_2 n_1 + j}$ for all $j \in [\lambda N n_1 - r_2 n_1]$. In particular, $e'_j = e'_{a r_2 n_1 + j}$ for $j \in [bn_1]$, where $[bn_1] = [0] = \emptyset$ if $b = 0$. As $\lambda N = a r_2 + b$, the first bn_1 edges and the last bn_1 edges of S (in order) are therefore the same. Thus, the edge $e \in (\mathcal{K}_{n_1,\dots,n_u})$ appears $ad(e) + d'(e)$ times in the sequence S. On the other hand, as ℓ is an ordering of $\lambda\mathcal{K}_{n_1,\dots,n_k}$, any vertices $v_1 \in N_1, \dots, v_u \in N_u$ are incident with exactly $\lambda N'$ edges in the sequence $S_\ell(\mathcal{H})$. Thus, each edge $e \in E(\mathcal{K}_{n_1,\dots,n_u})$ appears $\lambda N'$ times in S. Hence,

$$\lambda N' = ad(e) + d'(e) \tag{2}$$

for all $e \in E(\mathcal{K}_{n_1,\dots,n_u})$.

We now establish the upper inequality of the lemma. As the first bn_1 edges of S are contained in the first $r_2 n_1$ edges of S, clearly $d'(e) \le d(e)$ for all e. So, by (2), we have that $(a+1)d(e) \ge \lambda N'$ for all $e \in E(\mathcal{K}_{n_1,\dots,n_u})$. In particular, $(a+1)d_{\min} \ge \lambda N'$, where $d_{\min}$ is the minimum of $d(e)$ over all edges $e \in E(\mathcal{K}_{n_1,\dots,n_u})$. Clearly,

$$\sum_{e \in E(\mathcal{K}_{n_1,\dots,n_u})} d(e) = r_2 n_1 \,, \tag{3}$$

and so, by the Pigeonhole Principle, $d_{\min} \le \left\lfloor \frac{r_2 n_1}{n_1^u} \right\rfloor = \left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor$. Thus,

$$(a+1)\left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor \ge (a+1)d_{\min} \ge \lambda N' \,,$$

which is equivalent to

$$\left(\left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor + 1 \right) \left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor \ge \lambda \prod_{i=u+1}^{k} n_i \,.$$

This establishes the upper inequality of the lemma.

We now establish the lower inequality of the lemma. Since $d'(e) \ge 0$, (2) implies that $\lambda N' \ge ad(e)$ for all $e \in E(\mathcal{K}_{n_1,\dots,n_u})$. In particular, $\lambda N' \ge ad_{\max}$, where $d_{\max}$ is the maximal value of $d(e)$ for edges $e \in E(\mathcal{K}_{n_1,\dots,n_u})$. By (3) and the Pigeonhole Principle, $d_{\max} \ge \left\lceil \frac{r_2}{n_1^{u-1}} \right\rceil$. Thus,

$$\lambda \prod_{i=u+1}^{k} n_i = \lambda N' \ge a \left\lceil \frac{r_2}{n_1^{u-1}} \right\rceil = \left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor \left\lceil \frac{r_2}{n_1^{u-1}} \right\rceil \,,$$

which establishes the lower inequality of the lemma if $n_1^{u-1} \nmid r_2$. Otherwise, $n_1^{u-1} \mid r_2$, and we are done. $\square$

Lemma 3.3. *If $cms_r(\lambda\mathcal{K}_{n_1,\ldots,n_u}) = rn_1$, then $n_1^{u-1} \mid r_2$.*

Proof. Let ℓ be an ordering of $\lambda\mathcal{K}_{n_1,\ldots,n_k}$ such that $cms_r(\ell) = rn_1$. Let x and y be integers satisfying $xr_2 = y\lambda N$. Write $S_{y\ell}(\lambda\mathcal{K}_{n_1,\ldots,n_k}) = e_0, \ldots, e_{y\lambda Nn_1-1}$ and let $S = e'_0, \ldots, e'_{y\lambda Nn_1-1}$ be the sequence of edges from $E(\mathcal{K}_{n_1,\ldots,n_u})$ such that, if e_i is incident with each of $v_1 \in V_1, \ldots, v_u \in V_u$, then e'_i is the edge in $E(\mathcal{K}_{n_1,\ldots,n_u})$ incident with each of $v_1, \ldots, v_u$. For an edge $e \in E(\mathcal{K}_{n_1,\ldots,n_u})$ let $d(e)$ be the number of times that e appears among the first r_2n_1 edges of S.

We count in two ways the number of times that an edge $e \in E(\mathcal{K}_{n_1,\ldots,n_u})$ appears in S. For all j, Claim 3.1 implies that edges e_j and $e_{j'}$ are incident with the same vertex in N_i for $i = 1, \ldots, u$, where $j' := (r_2n_1 + j) \bmod \lambda Nn_1$. So, $e'_j = e'_{j'}$ for all $j \in [y\lambda Nn_1 - r_2n_1]$. Therefore, each edge $e \in E(\mathcal{K}_{n_1,\ldots,n_u})$ appears $xd(e)$ times in the sequence S, as $xr_2 = y\lambda N$. On the other hand, $e \in E(\mathcal{K}_{n_1,\ldots,n_u})$ appears $\lambda N'$ times in the sequence $S_\lambda(\mathcal{H})$, as ℓ is an ordering of $\lambda\mathcal{K}_{n_1,\ldots,n_k}$. Thus, e appears $y\lambda N'$ times in the sequence S. Therefore, $xd(e) = y\lambda N'$ for all $e \in E(\mathcal{K}_{n_1,\ldots,n_u})$, and $d(e)$ is therefore constant. By (3), $d(e)n_1^u = r_2n_1$; hence, $n_1^{u-1} \mid r_2$. $\square$

4 Proof of Theorem 1.4: Part II

The next lemma required for the proof of Theorem 1.4 is Lemma 4.2 below. Before presenting this lemma, however, let us first introduce notation used in this section and the next.

Recall that the representation of an integer x in base m is $x = (x_l, \ldots, x_0)_m$, where $x = \sum_{i=0}^{l} x_i m^i$ and $x_i \in \mathbb{Z}_m$ for all i. We consider the following generalisation of this representation. Let $m_1, \ldots, m_k$ be arbitrary positive integers and set $M := \prod_{i=2}^{k} m_i$. The representation of each integer $x \in \mathbb{Z}_M$ in base $\overline{m} := (m_1, \ldots, m_k)$ is the k-vector $\langle x \rangle_{\overline{m}} := (0, x_2, \ldots, x_k) \in \{0\} \times \prod_{i=2}^{k} \mathbb{Z}_{m_i}$ that satisfies

$$x = \sum_{i=2}^{k} \left(x_i \prod_{j=i+1}^{k} m_j \right). \tag{4}$$

By the following lemma, this representation is indeed well defined. Note that the 0 in the first coordinate is technically useful as it will align with notation used later in the paper.

Lemma 4.1. *The representation $\langle x \rangle_{\overline{m}} := (0, x_2, \ldots, x_k)$ of each $x \in \mathbb{Z}_M$ exists and is unique. Furthermore, $\langle x + 1 \rangle_{\overline{m}} = (0, x_2, \ldots, x_{t-1}, x_t + 1 \ldots, x_k + 1)_{\overline{m}}$ for some $2 \le t \le k$.*

Proof. Let $x \in \mathbb{Z}_M$ be an integer with representation $\langle x \rangle_{\overline{m}} = (0, x_2, \ldots, x_k)$. Clearly, $x_k \equiv x \pmod{m_k}$. Suppose, by induction, that $x_{l+1}, \ldots, x_k$ are uniquely determined by x. Then, as $x \equiv \sum_{i=l}^{k} x_i \prod_{j=i+1}^{k} m_j \pmod{\prod_{i=l}^{k} m_i}$ for any $2 \le l \le k$, we can determine x_l uniquely given x and $x_{l+1}, \ldots, x_k$. Thus, if an integer in $\mathbb{Z}_M$ has a representation $\langle x \rangle_{\overline{m}}$, then it is unique. As there are M k-tuples, each of which represents an integer satisfying (4), every integer in $\mathbb{Z}_M$ has a unique representation as a k-tuple.

If $x_k \ne m_k - 1$, then clearly $\langle x+1 \rangle = (0, x_2, \ldots, x_{k-1}, x_k + 1)$, as required. Otherwise, let t' be the smallest positive integer such that $x_j = m_j - 1$ for all $t' < j \le k$. Then $x = \sum_{i=2}^{t'}(x_i \prod_{j=i+1}^{k} m_j) + \sum_{i=t'+1}^{k}\big((m_i - 1)\prod_{j=i+1}^{k} m_j\big)$, and so

$$\begin{aligned} x+1 &= \sum_{i=2}^{t'}\left(x_i \prod_{j=i+1}^{k} m_j\right) + \sum_{i=t'+1}^{k} \prod_{j=i}^{k} m_j - \sum_{i=t'+1}^{k}\left(\prod_{j=i+1}^{k} m_j\right) + 1 \\ &= \sum_{i=2}^{t'}\left(x_i \prod_{j=i+1}^{k} m_j\right) + \prod_{j=t'+1}^{k} m_j \,. \end{aligned}$$

Hence, $x+1 = M$ if $t' = 1$, and, if $t' \ge 2$, then

$$x+1 = \sum_{i=2}^{t'-1}\left(x_i \prod_{j=i+1}^{k} m_j\right) + (x_t' + 1) \prod_{j=t'+1}^{k} m_j \,.$$

Thus, $\langle x+1 \rangle = \langle M \rangle_{\overline{m}} = \langle 0 \rangle_{\overline{m}} = (0, x_2 + 1, \ldots, x_k + 1)$ when $t' = 1$ and, when $t' \ge 2$, $\langle x+1 \rangle_{\overline{m}} = (0, x_2, \ldots, x_{t'-1}, x_{t'} + 1, \ldots, x_k + 1)$. In particular, $\langle x+1 \rangle_{\overline{m}} = (0, x_2, \ldots, x_{t-1}, x_t + 1, \ldots, x_k + 1)$ for some t, namely $t = t'$ if $t' \ge 2$, and $t = 2$ if $t' = 1$. $\square$

Lemma 4.2. *For all* $1 \le n_1 \le n_2 \le \cdots \le n_k$ *and* $r, \lambda \ge 1$,

$$rn_1 - 1 \le cms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) \le ms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) \le rn_1 \,.$$

To prove Lemma 4.2, we need only consider cases, according to the following claim.

Claim 4.3. *If Lemma 4.2 is true for all* $1 \le r < N$ *and* $\lambda = 1$, *then Lemma 4.2 is true for all* $r, \lambda \ge 1$.

Proof. Suppose that Lemma 4.2 is true for all $r < N$ and $\lambda = 1$. Write r as $r = r_1 N + r_2$ Then,

$$cms_r(\mathcal{H}) \ge r_1 n_1 N + cms_{r_2}(\mathcal{H}) \ge r_1 n_1 N + r_2 n_1 + 1 = rn_1 - 1 \,,$$

by Lemma 2.6. Thus, by Lemma 2.7, $cms_r(\lambda\mathcal{H}) \ge rn_1 - 1$ for all $\lambda \ge 1$, and we can conclude that $rn_1 - 1 \le cms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) \le ms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) \le rn_1$, as the two upper inequalities are trivially true. $\square$

To prove Lemma 4.2, it therefore suffices to consider $\mathcal{K}_{n_1,\ldots,n_k}$. More notation is however needed, so let d be a positive factor of N, and let $m_1, \ldots, m_k$ be integers satisfying $d = \prod_{i=2}^{k} m_i$ where $m_1 = n_1$ and $m_i \mid n_i$ for all $2 \le i \le\ k$. Define $N_i :=$

$\mathbb{Z}_{m_i} \times \mathbb{Z}_{n_i/m_i}$ for $1 \leq i \leq k$, $\overline{m} := (m_1, \ldots, m_k)$ and $\overline{n/m} := (n_1/m_1, \ldots, n_k/m_k)$. Without loss of generality, we can identify the edges of $\mathcal{K}_{n_1,\ldots,n_k}$ with the elements of $\mathcal{N} := \prod_{i=1}^k N_i$; in particular, each edge of $\mathcal{K}_{n_1,\ldots,n_k}$ is identified with a vector $\big((x_1, y_1), \ldots, (x_k, y_k)\big)_{\overline{m},\overline{n/m}}$. The sum of two elements $((x_1, y_1), \ldots, (x_k, y_k))$, $((x'_1, y'_1), \ldots, (x'_k, y'_k)) \in \mathcal{N}$ is defined as $\big((x_1+x'_1, y_1+y'_1), \ldots, (x_k+x'_k, y_k+y'_k)\big)_{\overline{m},\overline{n/m}}$. The difference of two such elements is defined analogously.

For integers $x \in \mathbb{Z}_d$ and $y \in \mathbb{Z}_{N/d}$, define $\langle(x, y)\rangle_{\overline{m},\overline{n/m}} := ((0, 0), (x_2, y_2) \ldots, (x_k, y_k))_{\overline{m},\overline{n/m}}$, where $\langle x\rangle_{\overline{m}} = (0, x_2, \ldots, x_k)_{\overline{m}}$ and $\langle y\rangle_{\overline{n/m}} = (0, y_2, \ldots, y_k)_{\overline{n/m}}$. Also, for each integer $x \in [n_1]$, define $\langle x^*\rangle_{\overline{m},\overline{n/m}} := \big((x_{1,1}, x_{1,2}), \ldots, (x_{k,1}, x_{k,2})\big)_{\overline{m},\overline{n/m}}$, where $x_{i,1} \in [m_i]$ and $x_{i,2} \in [\frac{n_i}{m_i}]$ satisfy $x = x_{i,1}\frac{n_i}{m_i} + x_{i,2}$ for all $1 \leq i \leq k$. It is easily checked that each $x_{i,j}$ is uniquely determined by x, and so $\langle x^*\rangle_{\overline{m},\overline{n/m}}$ is well defined. Note that the first entry of $\langle x^*\rangle_{\overline{m},\overline{n/m}}$ is not necessarily equal to $(0, 0)$. The subscript $\overline{m}, \overline{n/m}$ will be omitted if the context is implicitly clear.

For $i \in [d]$ and $j \in [\frac{N}{d}]$, define $\mathcal{M}_{i,j} := \left\{\langle x^*\rangle_{\overline{m},\overline{n/m}} + \langle(i, j)\rangle_{\overline{m},\overline{n/m}} \ : \ x \in [n_1]\right\}$.

Claim 4.4. *The set* $\left\{\mathcal{M}_{i,j} \ : \ i \in [d], j \in [\frac{N}{d}]\right\}$ *is a matching decomposition of* $\mathcal{K}_{n_1,\ldots,n_k}$.

Proof. We first check that each $\mathcal{M}_{i,j}$ is a matching. Let $\langle x^*\rangle = \big((x_{1,1}, x_{1,2}), \ldots, (x_{k,1}, x_{k,2})\big)$, $\langle y^*\rangle = \big((y_{1,1}, y_{1,2}), \ldots, (y_{k,1}, y_{k,2})\big)$ and $\langle(i, j)\rangle = \big((i_1, j_1), \ldots, (i_k, j_k)\big)$ for distinct $x, y \in [n_1]$. Suppose that the edges $\langle x^*\rangle + \langle(i, j)\rangle$ and $\langle y^*\rangle + \langle(i, j)\rangle$ in $\mathcal{M}_{i,j}$ have the same l-th entry for some $1 \leq l \leq k$; i.e.,

$$x_{l,1} + i_l \equiv y_{l,1} + i_l \pmod{m_l} \qquad \text{and} \qquad x_{l,2} + j_l \equiv y_{l,2} + j_l \left(\operatorname{mod} \frac{n_l}{m_l}\right).$$

Then $x_{l,1} = y_{l,1}$ and $x_{l,2} = y_{l,2}$. Hence, $x = x_{l,1}\frac{n_l}{m_l} + x_{l,2} = y_{l,1}\frac{n_l}{m_l} + y_{l,2} = y$, a contradiction. Thus, $\mathcal{M}_{i,j}$ is a matching for all $i \in [d], j \in [\frac{N}{d}]$.

We now verify that the matchings $\mathcal{M}_{i,j}$ for $i \in [d], j \in [\frac{N}{d}]$ partition $E(\mathcal{K}_{n_1,\ldots,n_k})$. As there are clearly N matchings $\mathcal{M}_{i,j}$, each containing n_1 edges, we need only show that no two distinct $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i',j'}$ contain a common edge. Suppose, otherwise, that there are distinct $(i, j), (i', j')$ such that $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i',j'}$ contain a common edge. By considering first entries, it is easy to check that if $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i',j'}$ contain a common edge, then that edge is of the form $\langle x^*\rangle + \langle(i, j)\rangle = \langle x^*\rangle + \langle(i', j')\rangle$ for some $x \in [n_1]$. Let $\langle x^*\rangle = ((x_{1,1}, x_{1,2}), \ldots, (x_{k,1}, x_{k,2}))$, $\langle(i, j)\rangle = ((i_1, j_1), \ldots, (i_k, j_k))$ and $\langle(i', j')\rangle = ((i'_1, j'_1), \ldots, (i'_k, j'_k))$. Then, by equating the l-th entries of $\langle x^*\rangle + \langle(i, j)\rangle$ and $\langle x^*\rangle + \langle(i', j')\rangle$, we see that, for $1 \leq l \leq k$,

$$x_{l,1} + i_l \equiv x_{l,1} + i'_l \pmod{m_i} \qquad \text{and} \qquad x_{l,2} + j_l \equiv x_{l,2} + j'_l \left(\operatorname{mod} \frac{n_i}{m_i}\right).$$

Then $(i_l, j_l) = (i'_l, j'_l)$ for all $1 \leq l \leq k$, and so

$$\langle(i, j)\rangle = ((i_1, j_1), \ldots, (i_k, j_k)) = ((i'_1, j'_1), \ldots, (i'_k, j'_k)) = \langle(i', j')\rangle\,,$$

contradicting our assumption that $(i, j) \neq (i', j')$. Hence, the matchings $\mathcal{M}_{i,j}$ for $i \in [d], j \in [\frac{N}{d}]$ are disjoint and, by the number of their edges, partition $E(\mathcal{K}_{n_1,\ldots,n_k})$. $\square$

Let $\ell_{i,j}$ be the ordering of $\mathcal{M}_{i,j}$ defined by $\ell_{i,j}\left(\langle x^*\rangle + \langle (i,j)\rangle\right) = x$ for all $x \in [n_1]$, and set $\ell_{i,\frac{N}{d}} := \ell_{i,0}$ and (thus) $\mathcal{M}_{i,\frac{N}{d}} := \mathcal{M}_{i,0}$ for all $i \in [d]$.

Lemma 4.5. *For all $i \in [d]$ and $j \in [\frac{N}{d}]$, $ms(\ell_{i,j}, \ell_{i,j+1}) \geq n_1 - 1$.*

Proof. Let $\ell = \ell_{i,j} \vee_{n_1-1} \ell_{i,j}$. Consider a sequence S of $n_1 - 1$ consecutive edges in ℓ. We check that $\mathcal{H}(S)$ is a matching of $\mathcal{K}_{n_1,\ldots,n_k}$. Let $1 \leq s \leq n_1 - 2$ be the number of edges in S which are from $\mathcal{M}_{i,j}$. There are then $n_1 - 1 - s$ edges in S from $\mathcal{M}_{i,j+1}$, and the edges in S which are from $\mathcal{M}_{i,j}$ are $\langle x^*\rangle + \langle (i,j)\rangle$ for $n_1 - s \leq a \leq n_1 - 1$, and the edges in S from $\mathcal{M}_{i,j+1}$ are $\langle y^*\rangle + \langle (i,j+1)\rangle$ for $0 \leq y \leq n_1 - s - 2$. As $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i,j+1}$ are each matchings, $\mathcal{H}(S)$ is not a matching only if there is an edge from $\mathcal{M}_{i,j}$ in S and another from $\mathcal{M}_{i,j+1}$ in S that have a common entry. So, suppose that $\langle x^*\rangle + \langle (i,j)\rangle$ and $\langle y^*\rangle + \langle (i,j+1)\rangle$ have the same l-th entry for some $1 \leq l \leq k$, $n_1 - s \leq x \leq n_1 - 1$ and $0 \leq y \leq n_1 - s - 2$. Let $\langle x^*\rangle = \big((x_{1,1}, x_{1,2}), \ldots, (x_{k,1}, x_{k,2})\big)$, $\langle y^*\rangle = \big((y_{1,1}, y_{1,2}), \ldots, (y_{k,1}, y_{k,2})\big)$ and $\langle (i,j)\rangle = \big((i_1, j_1), \ldots, (i_k, j_k)\big)$. By Lemma 4.1, $\langle (i,j+1)\rangle = \big((i_1, j_1), \ldots, (i_{t-1}, j_{t-1}), (i_t, j_t + 1), \ldots, (i_k, j_k + 1)\big)$ for some $2 \leq t \leq k$. Thus, by equating the ℓth entries of $\langle x^*\rangle + \langle (i,j)\rangle$ and $\langle y^*\rangle + \langle (i,j+1)\rangle$, we see that

$$(x_{\ell,1}{+}i_l, x_{\ell,2}{+}j_\ell) = \begin{cases} \big((y_{\ell,1} + i_\ell) \mod m_\ell, (y_{\ell,2} + j_\ell) \mod \frac{n_l}{m_l}\big) & \text{if } l < t \\ \big((y_{\ell,1} + i_\ell) \mod m_\ell, (y_{\ell,2} + j_\ell + 1) \mod \frac{n_l}{m_l}\big) & \text{otherwise.} \end{cases} \tag{5}$$

By equating the entries of the pairs in (5), we see that $x_{l,1} = y_{l,1}$ and either $x_{l,2} = y_{l,2}$ or $x_{l,2} \equiv y_{l,2} + 1 \pmod{\frac{n_l}{m_l}}$. If the former is true, then $x = x_{l,1}\frac{n_l}{m_l} + x_{l,2} = y_{l,1}\frac{n_l}{m_l} + y_{l,2} = y$, a contradiction. Hence, $x_{l,2} \equiv y_{l,2} + 1 \pmod{\frac{n_l}{m_l}}$. If $x_{l,2} = y_{l,2} + 1$, then, using a similar argument, we arrive at the contradiction $x = y + 1$. We are then left with the case in which $x_{l,2} = 0$ and $y_{l,2} = \frac{n_l}{m_l} - 1$, also a contradiction, as, otherwise, $x = x_{l,1}\frac{n_l}{m_l} < y_{l,1}\frac{n_l}{m_l} + \frac{n_l}{m_l} - 1 = y$. Hence, $\mathcal{H}(S)$ is a matching, as required. □

We can now prove Lemma 4.2.

Proof of Lemma 4.2. Let $r < N$ and $d = \gcd(N, r)$. By Claim 4.4 and Lemma 4.5, the assumptions of Proposition 2.5 are met for the hypergraph $\mathcal{K}_{n_1,\ldots,n_k}$ with matchings $\mathcal{M}_{i,j}$ ordered by $\ell_{i,j}$ for $i \in [d]$ and $j \in [\frac{N}{d}]$, respectively. Thus, $cms_r(\mathcal{K}_{n_1,\ldots,n_k}) \geq rn_1 - 1$ when $r < N$. By Claim 4.3, Lemma 4.2 is true for all $r \geq 1$ and $\lambda \geq 1$. □

5 Proof of Theorem 1.4: Part III

We now present the remaining lemmas required for the proof of Theorem 1.4, namely, Lemmas 5.1 and 5.2.

Lemma 5.1. *If $n_1^{u-1} \mid r_2$, then $cms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) = rn_1$.*

Lemma 5.2. *If* $n_1^{u-1} \mid r_2$ *or*

$$\left(\left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor + 1\right) \left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor \leq \lambda \prod_{i=u+1}^{k} n_i \leq \left\lfloor \frac{r_2}{n_1^{u-1}} \right\rfloor \left(\left\lfloor \frac{\lambda}{r_2} \prod_{i=2}^{k} n_i \right\rfloor + 1\right), \tag{6}$$

then $ms_r(\mathcal{K}_{n_1,\ldots,n_k}) = rn_1$.

The rest of this section serves to prove these lemmas.

First note that we can immediately reduce Lemma 5.1 to a single case for λ, as follows.

Claim 5.3. *If Lemma 5.1 is true for* $r = n_1^{u-1}$ *and* $\lambda = 1$*, then Lemma 5.1 is true for all* $r \geq n_1^{u-1}$ *and* $\lambda \geq 1$.

Proof. Suppose that Lemma 5.1 is true for $r = n_1^{u-1}$ and $\lambda = 1$. Then, by Lemma 2.1, Lemma 5.1 is true for $\lambda = 1$ and all $r < N$ such that $n_1^{u-1} \mid r_2 = r$. Thus, for any $r = r_1 N + r_2$ such that $n_1^{u-1} \mid r_2$, it follows from Lemma 2.6 that

$$cms_r(\mathcal{K}_{n_1,\ldots,n_k}) \geq r_1 n_1 N + cms_{r_2}(\mathcal{H}) = r_1 n_1 N + r_2 n_1 = rn_1 .$$

By Lemma 2.7, the cases in which $\lambda > 1$ follow from the case in which $\lambda = 1$, and we are done. □

Claim 5.4. *If Lemma 5.2 is true for* $1 \leq r < \lambda N$*, then Lemma 5.2 is true for all* $r \geq 1$.

Proof. Suppose that Lemma 5.2 is true for $r < \lambda N$ and that either $n_1^{u-1} \mid r_2$ or equation (6) holds. Then, for each $r \geq 1$,

$$ms_r(\lambda\mathcal{K}_{n_1,\ldots,n_k}) \geq r_1 n_1 \lambda \prod_{i=2}^{k} n_i + ms_{r_2}(\lambda\mathcal{K}_{n_1,\ldots,n_k}) = r_1 \lambda n_1 \prod_{i=2}^{k} n_i + r_2 n_1 = rn_1 ,$$

by Lemma 2.6. □

Set $N_i := [n_i]$ for all $1 \leq i \leq k$. By the natural isomorphism between $[n_i]$ and $[n_i] \times [1]$ for all i, it follows that the sets N_i are, up to isomorphism, the same sets as those defined in Section 4 for $d = N$; i.e., when $m_i = n_i$ for all $1 \leq i \leq k$. We will therefore use the definitions and notation of the previous section, where, for simplicity, we identify the edges of $\mathcal{K}_{n_1,\ldots,n_k}$ with the elements of $\prod_{i=1}^{k} \mathbb{Z}_{n_i}$. Then $\langle x^* \rangle_{\overline{n}} := \langle x^* \rangle_{\overline{m},\overline{n/m}}$, as defined in Section 4, will be identified with the element $(x, \ldots, x) \in \prod_{i=1}^{k} \mathbb{Z}_{n_i}$ for each $x \in \mathbb{Z}_{n_1}$.

Let ℓ' be a labelling of $\mathcal{K}_{n_1,\ldots,n_u}$ such that the edges $(\ell')^{-1}(xn_1), \ldots, (\ell')^{-1}(xn_1 + n_1 - 1)$ form a matching for all $x \in [n_1^{u-1}]$. That is, let ℓ' be an ordering which corresponds to $S(\mathcal{M}_0) \vee \cdots \vee S(\mathcal{M}_{n_1^{u-1}-1})$ for some matching decomposition $\mathcal{M}_0, \ldots, \mathcal{M}_{n_1^{u-1}-1}$ of $\mathcal{K}_{n_1,\ldots,n_u}$, where each $\mathcal{M}_i$ is ordered arbitrarily. Let $\overline{n}' :=$

$(n_1, n_{u+1}, \ldots, n_k)$. For $i \in [N']$, let $\mathcal{M}'_i := \{\langle x^* \rangle_{\overline{n}'} - \langle i \rangle_{\overline{n}'} \ : \ x \in [n_1]\}$ and $\overline{\mathcal{M}'_i} := \{(x_{u+1}, \ldots, x_k) \ : \ (x_1, x_{u+1}, \ldots, x_k) \in \mathcal{M}'_i\}$. It is easy to check that $\mathcal{M}'_i$ and, therefore, $\overline{\mathcal{M}'_i}$ is a matching, by using a similar argument to the proof of Claim 4.4. Let ℓ'_i be the ordering of $\mathcal{M}'_i$ defined by $\ell'_i(\langle x^* \rangle_{\overline{n}'} - \langle i \rangle_{\overline{n}'}) = x$ for all $x \in [n_1]$. Also let $\overline{\ell'_i}$ be the analogous ordering for $\overline{\mathcal{M}'_i}$. Identify each element $(x_1, \ldots, x_k) \in \prod_{i=1}^k \mathbb{Z}_{n_i}$ with element $((x_1, \ldots, x_u), (x_{u+1}, \ldots, x_k)) \in (\prod_{i=1}^u \mathbb{Z}_{n_i}) \times (\prod_{i=u+1}^k \mathbb{Z}_{n_i})$. For $i \in [n_1^{u-1}]$ and $j \in [\lambda N']$, let $\mathcal{M}'_{i,j}$ be a set containing an edge parallel to the edge $((\ell')^{-1}(in_1 + x), (\overline{\ell'_j})^{-1}(x))$ for each $x \in [n_1]$, where, for simplicity, we let $\ell'_j = \ell'_{j'}$ for $j' \in [N']$ with $j' \equiv j \pmod N$.

Claim 5.5. *The set* $\{\mathcal{M}'_{i,j} \ : \ i \in [n_1^{u-1}], j \in [\lambda N']\}$ *is a matching decomposition of* $\lambda \mathcal{K}_{n_1,\ldots,n_k}$.

Proof. Each $\mathcal{M}'_{i,j}$ is a matching since $(\ell')^{-1}(in_1), \ldots, (\ell')^{-1}(in_1 + n_1 - 1)$ form the matching $\mathcal{M}_i$ and $(\ell'_j)^{-1}(0), \ldots, (\ell'_j)^{-1}(n_1 - 1)$ form the matching $\overline{\mathcal{M}_i}$. For $i \in [n_1^{u-1}]$ and $j \in [N']$, there are λ matchings whose edges are parallel to the same as those in $\mathcal{M}'_{i,j}$, namely, $\mathcal{M}'_{i,j}, \ldots, \mathcal{M}'_{i,j+(\lambda-1)N'}$. Therefore, it suffices to show that $\{\mathcal{M}_{i,j} \ : \ i \in [n_1^{u-1}], j \in [N']\}$ is a matching decomposition of $\mathcal{K}_{n_1,\ldots,n_k}$.

We see that the matching $\mathcal{M}'_i$ is isomorphic to the matching $\mathcal{M}_{0,N'-i}$ defined in Section 4 for $d = N'$ and $\mathcal{K}_{n_1,n_{u+1},\ldots,n_k}$, by noting that $\langle x^* \rangle_{\overline{n}'} + \langle -i \rangle_{\overline{n}'} = \langle x^* \rangle_{\overline{n}'} - \langle i \rangle_{\overline{n}'}$ for any $x \in [n_1]$ and by setting $\mathcal{M}_{0,N'} := \mathcal{M}_{0,0}$. Thus, by Claim 4.4, $\{\mathcal{M}'_j \ : \ j \in [N']\}$ is a matching decomposition of $\mathcal{K}_{n_1,n_{u+1},\ldots,n_k}$. The edges of $\mathcal{M}'_{i,j}$ are isomorphic to edges in $\mathcal{M}'_j$ by identifying $((\ell')^{-1}(in_1 + x), (\overline{\ell'_j})^{-1}(x))$ with $(x, (\overline{\ell'_j})^{-1}(x))$ for all $x \in [n_1]$. Hence, $\{\mathcal{M}'_{i,j} \ : \ j \in [N']\}$ is a matching decomposition of $\mathcal{M}_i \times \mathcal{K}_{n_{u+1},\ldots,n_k}$ for any $i \in [n_1^{u-1}]$. As every edge of $\mathcal{K}_{n_1,\ldots,n_u}$ appears in exactly one $\mathcal{M}_i$, the set $\{\mathcal{M}'_{i,j} \ : \ i \in [n_1^{u-1}], j \in [N']\}$ is a matching decomposition of $\mathcal{K}_{n_1,\ldots,n_k}$, as required. ☐

Let $\ell'_{i,j}$ be the ordering of $\mathcal{M}'_{i,j}$ defined by $\ell'_{i,j}\left(((\ell')^{-1}(in_1 + x), (\overline{\ell'_j})^{-1}(x))\right)$ for $x \in [n_1]$, and set $\ell'_{i,\lambda N} := \ell'_{i,0}$ and (thus) $\mathcal{M}'_{i,\lambda N} := \mathcal{M}'_{i,0}$.

Lemma 5.6. *For all* $i \in [n_1^{u-1}]$ *and* $j \in [\lambda N']$, $ms(\ell'_{i,j}, \ell'_{i,j+1}) \geq n_1$ *holds.*

Proof. Let $\ell = \ell'_{i,j} \vee_{n_1} \ell'_{i,j+1}$. Consider a sequence S of n_1 consecutive edges in ℓ. The edges of S that appear in the matching $\mathcal{M}_{i,j}$ (in order with respect to ℓ) are

$$\left((\ell')^{-1}(in_1 + x), (\overline{\ell'_j})^{-1}(x)\right), \ldots, \left((\ell')^{-1}(in_1 + n_1 - 1), (\overline{\ell'_j})^{-1}(n_1 - 1)\right)$$

and the edges of S that appear in the matching $\mathcal{M}_{i,j+1}$ (in order with respect to ℓ) are

$$\left((\ell')^{-1}(in_1), (\overline{\ell'_{j+1}})^{-1}(0)\right), \ldots, \left((\ell')^{-1}(in_1 + x - 1), (\overline{\ell'_{j+1}})^{-1}(x - 1)\right)$$

for some $1 \leq x \leq n_1 - 1$. The edges $(\ell')^{-1}(in_1 + 0), \ldots, (\ell')^{-1}(in_1 + n_1 - 1)$ form the matching $\mathcal{M}_i$ and, in particular, every vertex in $[n_l]$ for $1 \leq l \leq u$ has degree 1 in $\mathcal{H}(S)$.

So without loss of generality, we consider the degree of vertices in $[n_l]$ for $u+1 \leq l \leq k$ in the hypergraph $\mathcal{H}(S')$, where $S' = (\overline{\ell'_j})^{-1}(x), \ldots, (\overline{\ell'_j})^{-1}(n_1 - 1)$,

$(\overline{\ell'_{j+1}})^{-1}(0), \ldots, (\overline{\ell'_{j+1}})^{-1}(x-1)$. As we are not concerned with the degree of vertices in $[n_1]$, we can consider the hypergraph formed by the edges

$$\begin{gathered}\langle x^*\rangle_{\overline{n}'} - \langle j\rangle_{\overline{n}'}, \ldots, \langle (n_1-1)^*\rangle_{\overline{n}'} - \langle j\rangle_{\overline{n}'}, \\ \langle 0^*\rangle_{\overline{n}'} - \langle j+1\rangle_{\overline{n}'}, \ldots, \langle (x-1)^*\rangle_{\overline{n}'} - \langle j+1\rangle_{\overline{n}'}\,,\end{gathered} \tag{7}$$

by ignoring the first entry of each edge. Let $\langle j\rangle_{\overline{n}'} = (j_1, j_{u+1}, \ldots, j_k)_{\overline{n}'}$. By Lemma 4.1 $\langle j+1\rangle_{\overline{n}'} = (j_1, j_{u+1}, \ldots, j_{u+t-1}, j_{u+t}+1, \ldots, j_k+1)_{\overline{n}'}$ for some $1 \le t \le k-u$. For $u+1 \le l \le u+t-1$, the $(l-u+1)$-th entry of the edges in (7) are, modulo n_l, $x-j_l, \ldots, n_1-1-j_l$ and $-j_l, 1-j_l, \ldots, x-1-j_l$, which are clearly distinct as $n_l > n_1$. For $u+t \le l \le k$, the $(l-u+1)$-th entry of the edges in (7) modulo n_l are $x-j_l, \ldots, n_1-1-j_l$ and $-j_l-1, -j_l, \ldots, x-2-j_l$, which are distinct since $n_l > n_1$. Thus, every vertex in $[n_l]$ for $u+1 \le l \le k$ is incident with at most one edge in (7) and thus at most one edge in S. Hence, $\mathcal{H}(S)$ is a matching. □

Proof of Lemma 5.1 . By Claim 5.3, we only need to consider the case in which $r = n_1^{u-1}$ and $\lambda = 1$. By Claim 5.5, $\{\mathcal{M}'_{i,j} : i \in [n_1^{u-1}], j \in [N']\}$ is a matching decomposition of $\mathcal{K}_{n_1,\ldots,n_k}$. By Lemma 5.6, $ms(\ell'_{i,j}, \ell'_{i,j+1}) \ge n_1$ for all $i \in [n_1^{u-1}]$ and $j \in [N']$. Hence, by Proposition 2.5, we have that $cms_r(\mathcal{K}_{n_1,\ldots,n_k}) \ge rn_1$, and so $cms_r(\mathcal{K}_{n_1,\ldots,n_k}) = rn_1$ as required. □

The remainder of this section is devoted to proving Lemma 5.2. We assume that $n_1^{u-1} \nmid r_2$, as the case in which $n_1^{u-1} \mid r_2$ has been shown in Lemma 5.1. Let $r < \lambda N$ be a positive integer and write $r = pn_1^{u-1} + q$ for non-negative integers p and q such that $0 < q < n_1^{u-1}$, and recall that $\lambda N = ar + b$. Then (6) can be expressed as

$$(p+1)a \le \lambda N' \le p(a+1)\,. \tag{8}$$

As we are proving Lemma 5.2, we will assume that (8) holds and thus that $p \neq 0$.

Let $\alpha = p$, $\beta = (p+1)$, $\gamma = (\lambda N' - ap)$, $\delta = \big(\lambda N' - a(p+1)\big)$ and $\nu = n_1^{u-1} - q$. The identities $r = pn_1^{u-1} + q$ and $n_1^{u-1}\lambda N' = ra + b$ easily yield the following expressions:

$$\gamma\nu + \delta q = b \tag{9}$$

$$(\alpha - \gamma)\nu + (\beta - \delta)q = r - b \tag{10}$$

$$a\alpha + \gamma = \lambda N' = a\beta + \delta\,. \tag{11}$$

By (8), each of the numbers $\gamma, \delta, \alpha - \gamma$ and $\beta - \delta$ is non-negative.

Let $\sigma : [\lambda N'] \to [\lambda N']$ be a function with the properties given in Corollary 2.4 with $s = \alpha$ and $t = \lambda N'$. Similarly, let $\tau : [\lambda N'] \to [\lambda N']$ be a function with the properties given in Corollary 2.4 with $s = \beta$ and $t = \lambda N'$. For a fixed pair $(i, j) \in [n_1^{u-1}] \times [\lambda N']$, let $s_{i,j}$ and $t_{i,j}$ be the integers that satisfy

$$\begin{cases} \sigma(j) = s_{i,j}\alpha + t_{i,j} \text{ with } t_{i,j} \in [\alpha] & \text{if } i \in [\nu]\,; \\ \tau(j) = s_{i,j}\beta + t_{i,j} \text{ with } t_{i,j} \in [\beta] & \text{otherwise.} \end{cases}$$

Let $\rho : [n_1^{u-1}] \times [\lambda N'] \to [\lambda N]$ be defined by

$$\rho(i,j) = \begin{cases} s_{i,j}r + \nu t_{i,j} + i & \text{if } t_{i,j} \in [\gamma] \text{ and } i \in [\nu]\,; \\ s_{i,j}r + \nu\gamma + qt_{i,j} + i - \nu & \text{if } t_{i,j} \in [\delta] \text{ and } i \in [\nu+q]-[\nu]\,; \\ s_{i,j}r + b + \nu(t_{i,j}-\gamma) + i & \text{if } t_{i,j} \in [\alpha]-[\gamma] \text{ and } i \in [\nu]\,; \\ s_{i,j}r + b + \nu(\alpha-\gamma) + qt_{i,j} + i - \nu & \text{if } t_{i,j} \in [\beta]-[\delta] \text{ and } i \in [\nu+q]-[\nu]. \end{cases}$$

As σ and τ are bijections of $[\lambda N']$, (8) implies that $s_{i,j} \in [a+1]$ for all i and j. Furthermore by (11), if $s_{i,j} = a$, then $t_{i,j} \in [\gamma]$ if $i \in [\nu]$ and $t \in [\delta]$ otherwise. Therefore, if $i \in [\nu]$, then either $\rho(i,j) = s_{i,j}r + \nu t_{i,j} + i \le ar + \nu(\gamma-1) + \nu - 1$ or $\rho(i,j) = s_{i,j}r + b + \nu(t_{i,j}-\gamma) + i \le (a-1)r + b + \nu(\alpha-1-\gamma) + \nu - 1$. In either case, $\rho(i,j) < \lambda N$, by (9) and (10), respectively. By a similar argument, $\rho(i,j) < \lambda N$ when $i \in [\nu+q]-[\nu]$, and ρ is thus well defined.

Lemma 5.7. *The function ρ is an ordering of $[n_1^{u-1}] \times [\lambda N']$ with the property that if $\rho(i,j) \in [\lambda N - r]$, then $\rho(i,j+1) = \rho(i,j) + r$.*

Proof. We first check that ρ is an ordering of $[n_1^{u-1}] \times [\lambda N']$. Suppose that $\rho(i,j) = \rho(i',j')$. By inspection, we have that

$$\rho(i,j) - s_{i,j}r \in \begin{cases} [\nu\gamma] & \text{for } t_{i,j} \in [\gamma] \text{ and } i \in [\nu]\,; \\ [b]-[\nu\gamma] & \text{for } t_{i,j} \in [\delta] \text{ and } i \in [\nu+q]-[\nu]\,; \\ [b+\nu(\alpha-\gamma)]-[b] & \text{for } t_{i,j} \in [\alpha]-[\gamma] \text{ and } i \in [\nu]\,; \\ [r]-[b+\nu(\alpha-\gamma)] & \text{for } t_{i,j} \in [\beta]-[\delta] \text{ and } i \in [\nu+q]-[\nu]\,, \end{cases}$$

and $\rho(i',j') - s_{i',j'}r$ has the analogous property. Therefore, $s_{i,j} = s_{i',j'}$ and either $i,i' \in [\nu]$ or $i,i' \in [\nu+q]-[\nu]$. Thus, by the definition of ρ,

$$0 = \rho(i,j) - \rho(i',j') = \begin{cases} \nu(t_{i,j} - t_{i',j'}) + i - i' & \text{if } i,i' \in [\nu]\,; \\ q(t_{i,j} - t_{i',j'}) + i - i' & \text{if } i,i' \in [\nu+q]-[\nu]\,. \end{cases} \tag{12}$$

However, $|i-i'| \in [\nu]$ if $i,i' \in [\nu]$, and $|i-i'| \in [q]$ if $i,i' \in [\nu+q]-[\nu]$. Hence, (12) implies that $t_{i,j} = t_{i',j'}$ and $i = i'$. Therefore, $j = \sigma^{-1}(s_{i,j}\alpha + t_{i,j}) = \sigma^{-1}(s_{i',j'}\alpha + t_{i',j'}) = j'$ if $i \in [\nu]$, and, similarly, $j = j'$ if $i \in [\nu+q]-[\nu]$. Thus, $(i,j) = (i',j')$, and so ρ is injective. Since $|[n_1^{u-1}] \times [\lambda N']| = |[\lambda N]|$, ρ is a bijection and, hence, an ordering of $[n_1^{u-1}] \times [\lambda N']$.

We now check that ρ satisfies the property given in the lemma. Suppose that $\rho(i,j) \in [\lambda N - r]$. If $i \in [\nu]$, then $s_{i,j}r + \nu t_{i,j} + i < (a-1)r + b$ if $t \in [\gamma]$, and $s_{i,j}r + b + \nu(t_{i,j}-\gamma) + i < (a-1)r + b$ otherwise. Therefore, $s_{i,j} \le a-1$ and if $s_{i,j} = a-1$, then $t_{i,j} \in [\gamma]$. Thus, $s_{i,j}\alpha + t_{i,j} < \lambda N' - \alpha$ and so, by Corollary 2.4, $\sigma(j+1) = (s_{i,j}+1)\alpha + t_{i,j}$. By a similar argument, $\tau(j+1) = (s_{i,j}+1)\beta + t_{i,j}$ for $i \in [\nu+q]-[\nu]$. Hence, in any case, $s_{i,j+1} = s_{i,j}+1$ and $t_{i,j+1} = t_{i,j}$. By the definition of ρ, $\rho(i,j+1) - s_{i,j+1}r = \rho(i,j) - s_{i,j}r$, and so $\rho(i,j+1) - \rho(i,j) = s_{i,j+1}r - s_{i,j}r = r$. Rearranging yields the required expression. ∎

Proof of Lemma 5.2. By Claim 5.4, we only need to consider the cases in which $1 \leq r < \lambda N$. Let $\mathcal{M}_l = \mathcal{M}'_{\rho^{-1}(l)}$ and $\ell_l = \ell'_{\rho^{-1}(l)}$ for all $l \in [\lambda N]$. We check that the conditions of Proposition 2.2 are satisfied for the matchings $\mathcal{M}_0, \dots, \mathcal{M}_{\lambda N-1}$ of $\mathcal{K}_{n_1,\dots,n_k}$. By Claim 5.5, $\{\mathcal{M}_0, \dots, \mathcal{M}_{\lambda N-1}\}$ is a matching decomposition of $\mathcal{K}_{n_1,\dots,n_k}$. For $l \in [\lambda N - r]$, let $\rho^{-1}(l) = (i, j)$ and so $\rho(i, j) = l$. By Lemma 5.7, $\rho(i, j + 1) = \rho(i, j) + r = l + r$ and, as ρ is a bijection, $\rho^{-1}(l + r) = (i, j + 1)$. Hence, $ms(\ell_l, \ell_{l+r}) = ms(\ell_{i,j}, \ell_{i,j+1}) \geq n_1$, by Lemma 5.6, and the proof then follows from Proposition 2.2. $\square$

6 Proof of Theorem 1.4: Conclusion

Proof of Theorem 1.4. By Lemma 4.2, $ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k})$ and $cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k})$ are each either $rn_1 - 1$ or rn_1. By Lemmas 3.2 and 5.2, $ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = rn_1$ if and only if $n_1^{u-1} \mid r$ or (1) holds. Thus,

$$ms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = \begin{cases} rn_1 & \text{if } n_1^{u-1} \mid r_2 \text{ or (1) holds}\,; \\ rn_1 - 1 & \text{otherwise}\,. \end{cases}$$

Similarly by Lemmas 3.3 and 5.1 $cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = rn_1$ if and only if $n_1^{u-1} \mid r$ and, thus,

$$cms_r(\lambda\mathcal{K}_{n_1,\dots,n_k}) = \begin{cases} rn_1 & \text{if } n_1^{u-1} \mid r_2\,; \\ rn_1 - 1 & \text{otherwise}\,. \end{cases}$$

$\square$

7 Concluding Remarks

One can show, for the special case in which $p = 1$, $q = 0$, $\lambda = 1$, and where σ is the identity function on $[\lambda N']$, that the function ρ defined in Section 5 reduces to the much simpler function $\rho(i, j) = jr + i$ for all $i \in [n_1^{u-1}]$ and $j \in [N']$ and, furthermore, that it satisfies a cyclic analogue of Lemma 5.7, namely $\rho(i, j + 1) = (\rho(i, j) + r)$ modulo N for all $i \in [n_1^{u-1}]$ and $j \in [N']$. The given proof of Lemma 5.1 implicitly uses this ρ: Proposition 2.5 uses Lemma 2.3. The cyclic construction in the previous section is thus a very special case of the non-cyclic construction.

Though the hypergraphs in this paper attain the lower bounds in Lemma 2.6, there are hypergraphs which do not. Consider the graph G below. First, we check that $cms(G) = 1$. Suppose otherwise that $cms(\ell) = 2$ for some ordering ℓ of G. As G has 6 edges and the vertex v has degree 3, the edges incident with v are, without loss of generality, labelled as depicted in Figure 1. However, for any choice of a label for the edge e, there will be two cyclically consecutive edges incident with a common vertex. Thus, $cms(G) = 1$. On the other hand, it is easy to check that, for any ordering ℓ of G with the edges incident with v labelled as depicted, $cms_4(\ell) \geq 8$. As $\Delta(G) = 3$ and $|E(G)| = 6$, the lower bound of Lemma 2.6 for G when $r = 4$ is

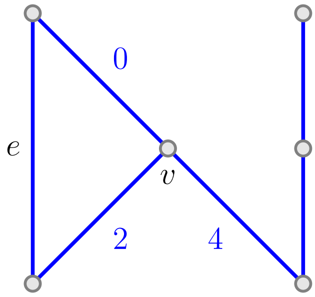

Figure 1: The graph G

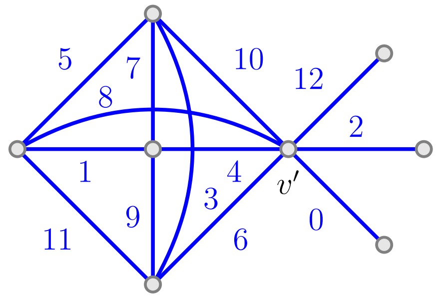

Figure 2: The graph H

$1 \times 6 + cms_1(G) = 7 < 8 \leq cms_4(G)$. By similar reasoning, the graph G' obtained from G by removing the edge e' satisfies $ms(G') = 1$ and $ms_4(G') \geq 7$, which is strictly above the lower bound given by Lemma 2.6. The bounds in Lemma 2.6 are thus not always achieved.

We can also show that Lemma 2.7 is no longer true if cyclic-sequencibility is replaced by non-cyclic sequencibility. Consider the graph H in Figure 2. It is easy to verify that the ordering ℓ of H depicted in Figure 2 satisfies $ms(\ell) = 2$ and, in particular, that $ms(G) \geq 2$. The graph $2H$ has 24 edges, 14 of which are incident with v. Therefore, for any ordering ℓ' of $2H$ corresponding to the sequence of edges $e_0, \ldots, e_{23}$, at least one of the 12 pairs of edges e_{2i}, e_{2i+1} for $i \in [12]$ has both of its edges incident with v, by the Pigeonhole Principle. Thus, no ordering ℓ' of $2H$ can satisfy $ms(\ell') \geq 2$, and so $ms(2H) = 1 < 2 = ms(H)$. So, there is no non-cyclic sequencibility analogue of Lemma 2.7.

We end the paper with the following conjecture on the matching sequencibility of complete multi-partite graphs. Let $K_{s(n)}$ be the complete s-partite graph with parts of size n.

Conjecture 7.1. For any integers $n \geq 2$ and $s \geq 2$,

$$ms(K_{s(n)}) = cms(K_{s(n)}) = \left\lfloor \frac{sn}{2} \right\rfloor - 1\,.$$

Acknowledgements

I thank my supervisor Thomas Britz for very helpful discussions during the writing of this paper.

References

[1] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Combin. Appl.* **52** (2008), 7–20.

[2] J. A. Bondy, and U. S. R. Murty, Graph theory with applications, *American Elsevier Publishing Co., Inc., New York* (1976).

[3] R. A. Brualdi, K. P. Kiernan, S. A. Meyer and M. W. Schroeder, Cyclic matching sequencibility of graphs, *Australas. J. Combin.* **53** (2012), 245–256.

[4] G. O. H. Katona, Constructions via Hamiltonian theorems, *Discrete Math.* **303** (2005), 87–103.

[5] D. L. Kreher, A. Pastine and L. Tollefson, A note on the cyclic matching sequencibility of graphs, *Australas. J. Combin.* **61** (2015), 142–146.

[6] A. Mammoliti, The r-matching sequencibility of complete graphs, *Electron. J. Combin.* **25**(1) (2018), #P1.6, 23 pp.

(Received 3 Sep 2018; revised 26 Apr 2019)