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Protected cells in bargraphs

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Abstract

For bargraphs the notion of *protected cells* is introduced in close analogy to the notion of protected points for trees and interior vertices in set partitions, which have been considered previously in the literature. The generating function for the number of bargraphs according to the number of cells, the number of columns, and the number of protected cells is found. Furthermore, similar generating functions for empty cells and empty protected cells are discussed.

1 Introduction

Bargraphs have been studied recently from several perspectives and have found various refined enumerations. Recall that a *bargraph* is a first quadrant lattice path starting at the origin and ending upon its first return to the x-axis which consists of three types of steps—an up step $u = (0, 1)$, a down step $d = (0, -1)$ and a horizontal step $h = (1, 0)$ — such that the first step is a u and where no d directly follows a

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u and vice versa, see Figure 1. Bargraphs go by different names, such as *wall polyominoes* [6] and *skylines* [7], and have proven to be an effective way of visualizing compositions [1–4] and other discrete structures [11, 12].

Given a bargraph B, four points (x, y), $(x+1, y)$, $(x+1, y+1)$, $(x, y+1)$ that lie within the area $t(B)$ it encloses in the first quadrant determine what is called a *cell* of B. If B contains m horizontal steps, then it can be identified as a sequence of columns $t(B) = t_1 t_2 \cdots t_m$ such that the jth column from the left contains exactly t_j cells. In this case, the bargraph contains $\sum_{j=1}^{m} t_j$ cells. Given a cell s of B, we denote its left (respectively, right) neighbour by $L(s)$ (respectively, $R(s)$).

A *rectangle* (a, b) in the first quadrant is defined by the coordinates $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$ in the lattice $\mathbb{Z}^2$. A bargraph B defines in unique way a rectangle (a, b), denoted by $R(B)$, if we let a be the number of columns of B and let b be the maximal height of its columns. Writing $B \equiv t(B) = t_1 t_2 \cdots t_m$ as above, then $a = m$ and $b = \max_j t_j$.

In [5] protected points in ordered trees and in [10] interior vertices in set partitions were considered. In the same fashion, protected points in k-ary trees were considered in [9]. We now introduce an analogous notion to being protected for bargraphs. Let us call a cell s in $R(B)$ a *protected cell* (respectively, *empty cell*, *empty protected cell*) of the bargraph B if and only if the cells s, $L(s)$ and $R(s)$ lie in $t(B)$ (respectively, if the cell s lies in $R(B) \setminus t(B)$, if the cells s, $L(s)$ and $R(s)$ lie in $R(B) \setminus t(B)$). For instance, the bargraph $B = 136246252154$ has 16 protected cells, $12 \cdot 6 - 41 = 31$ empty cells, and 4 empty protected cells, see Figure 1. Note that the first and last column of B never contribute protected cells.

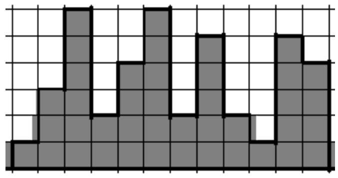

Figure 1: The bargraph $B = 136246252154$ with 16 protected cells and 4 empty protected cells.

2 Protected cells in bargraphs

For a given bargraph B, let $f_i(B)$ be the number of protected cells in the ith row. Define

$$F(x, y) \equiv F(x, y; q_1, q_2, \ldots) = \sum_{B} x^{\#\text{cells in B}} y^{\#\text{columns in B}} \prod_{i \geq 1} q_i^{f_i(B)},$$

where the sum is over all bargraphs including the empty bargraph. Since each bargraph is either empty, or has exactly one column, or has at least two columns, we can write

$$F(x,y) = 1 + \frac{xy}{1-x} + \tilde{F}(x,y), \tag{2.1}$$

where $\tilde{F}(x,y) \equiv \tilde{F}(x,y;q_1,q_2,\ldots)$ is the generating function for the number of bargraphs with at least two columns according to the number of cells, the number of columns, and the statistics $f_i(B)$.

2.1 Number of protected cells

Each bargraph with at least two columns is of exactly one of the following four types: (1) each column has height at least two, (2) the height of first column is one, (3) the height of the last column is one and all other columns have height at least two, (4) there exists a j, $2 \le j \le m-1$ (where m denotes the number of columns of the bargraph), such that the height of the ith column is at least two for $i = 1,2,\ldots,j-1$ and the height of the jth column is equal to 1. Thus, according to the types (1) - (4), we have

$$\begin{aligned}
\tilde{F}(x,y) =& \frac{1}{q_1^2}\tilde{F}(x,q_1xy) \\
&+ \frac{x^2y^2}{1-x} + q_1xy\tilde{F}(x,y) \\
&+ \frac{x^3y^2}{1-x} + \frac{xy}{q_1}\tilde{F}(x,q_1xy) \\
&+ \frac{q_1x^4y^3}{(1-x)^2} + \frac{q_1^2x^3y^2}{1-x}\tilde{F}(x,y) + \frac{x^2y^2}{1-x}\tilde{F}(x,q_1xy) + q_1xy\tilde{F}(x,y)\tilde{F}(x,q_1xy),
\end{aligned}$$

where here $\tilde{F}(x,q_1xy)$ stands for $\tilde{F}(x,q_1xy;q_2,q_3,\ldots)$. Hence,

$$\tilde{F}(x,y) = \frac{\frac{1}{q_1^2}\tilde{F}(x,q_1xy) + \frac{x^2y^2}{1-x} + \frac{x^3y^2}{1-x} + \frac{xy}{q_1}\tilde{F}(x,q_1xy) + \frac{q_1x^4y^3}{(1-x)^2} + \frac{x^2y^2}{1-x}\tilde{F}(x,q_1xy)}{1 - q_1xy - \frac{q_1^2x^3y^2}{1-x} - q_1xy\tilde{F}(x,q_1xy)}.$$

By (2.1), we have $\tilde{F}(x,y) = F(x,y) - 1 - \frac{xy}{1-x}$, which implies the following result.

Theorem 2.1 *We have*

$$\begin{aligned}
&F(x,y;q_1,q_2,\ldots) \\
&= 1 + \frac{xy}{1-x} + \frac{((1-x)(1+q_1xy) + q_1^2x^2y^2)F(x,q_1xy;q_2,q_3,\ldots) - 1 + x - q_1xy}{q_1^2(1-x)(1-q_1xyF(x,q_1xy;q_2,q_3,\ldots))}.
\end{aligned}$$

In particular, if $f(x,y) = F(x,y;1,1,\ldots)$, then Theorem 2.1 gives

$$\begin{aligned}
f(x,y) &= 1 + \frac{xy}{1-x} + \frac{((1-x)(1+xy) + x^2y^2)f(x,xy) - 1 + x - xy}{(1-x)(1-xyf(x,xy))} \\
&= \frac{f(x,xy)}{1-xyf(x,xy)},
\end{aligned}$$

which, by taking reciprocals and iterating, implies that the generating function for the number of bargraphs according to the number of cells and the number of columns is given by $f(x,y) = \frac{1-x}{1-x-xy}$, as expected (see [8]).

The generating function $F(x,y;q_1,q_2,\ldots)$ can be written as continued fraction:

$$\begin{aligned}
&F(x,y;q_1,q_2,\ldots)\\
&= 1 + \frac{xy}{1-x} + \frac{((1-x)(1+q_1xy)+q_1^2x^2y^2)F(x,q_1xy;q_2,q_3,\ldots)-1+x-q_1xy}{q_1^2(1-x)(1-q_1xyF(x,q_1xy;q_2,q_3,\ldots))}\\
&= \alpha(x,y;q_1,q_2,\ldots) + \frac{1}{q_1^3xy - q_1^4x^2y^2F(x,q_1xy;q_2,q_3,\ldots)},
\end{aligned}$$

where $\alpha(x,y;q_1,q_2,\ldots) = 1 - \frac{(1-x)(1+q_1xy)+q_1^2(1-q_1)x^2y^2}{q_1^3(1-x)xy}$. Define, for all $i \geq 0$,

$$G_i \equiv F(x,q_1q_2\cdots q_ix^iy;q_{i+1},q_{i+2},\ldots) \text{ and } \alpha_i \equiv \alpha(x,q_1q_2\cdots q_ix^iy;q_{i+1},q_{i+2},\ldots).$$

Thus, for all $i \geq 0$,

$$G_i = \alpha_i + \frac{1}{q_1q_2\cdots q_iq_{i+1}^3x^{i+1}y - q_1^2q_2^2\cdots q_i^2q_{i+1}^4x^{2i+2}y^2G_{i+1}}.$$

Introducing furthermore, for $i \geq 0$,

$$\beta_i \equiv q_1q_2\cdots q_iq_{i+1}^3x^{i+1}y - q_1^2q_2^2\cdots q_i^2q_{i+1}^4x^{2i+2}y^2\alpha_{i+1},$$

we can state the following result.

Theorem 2.2 *The generating function $F(x,y;q_1,q_2,\ldots)$ can be expressed in terms of continued fractions as*

$$\alpha_0 + \cfrac{1}{\beta_0 - \cfrac{q_1^4x^2y^2}{\beta_1 - \cfrac{q_1^2q_2^4x^4y^2}{\beta_2 - \cfrac{q_1^2q_2^2q_3^4x^6y^2}{\beta_3 - \cfrac{q_1^2q_2^2q_3^2q_4^4x^8y^2}{\beta_4 - \ddots}}}}}.$$

Example 2.3 Let us find a formula for $F(x,y;0,0,0,\ldots)$. Since each bargraph with at least three columns has at least one protected cell in the first row, we obtain (here, assuming $0^0 = 1$)

$$F(x,y;0,0,0,\ldots) = 1 + \frac{xy}{1-x} + \frac{x^2y^2}{(1-x)^2},$$

which is the generating function for the number of bargraphs of n with $0,1,2$ columns. This agrees with Theorem 2.2 where $q_i = 0$ for all $i \geq 1$.

Example 2.4 By Theorem 2.1, the generating function for the number of bargraphs without protected cells in the second row is given by

$$\begin{aligned}
&F(x,y;1,0,0,0,\ldots)\\
&=1+\frac{xy}{1-x}+\frac{((1-x)(1+xy)+x^2y^2)F(x,xy;0,0,0,\ldots)-1+x-xy}{(1-x)(1-xyF(x,xy;0,0,0,\ldots))}\\
&=1+\frac{(1-x+x^2y+x^4y^2)xy}{(1-x)^2-x(1-x)^2y-x^3(1-x)y^2-x^5y^3}.
\end{aligned}$$

Now, let us find a formula for the generating function $F(x,y;q_1,q_2,\ldots)$ with $q_j=1$ for all $j=1,2,\ldots,i-1$ and $q_j=0$ for all $j\geq i$. In this case, we write the generating function as $\frac{a_i(x,y)}{b_i(x,y)}$. Then Theorem 2.1 shows that for all $i\geq 2$,

$$\frac{a_i(x,y)}{b_i(x,y)}=1+\frac{xy}{1-x}+\frac{((1-x)(1+xy)+x^2y^2)a_{i-1}(x,xy)-(1-x+xy)b_{i-1}(x,xy)}{(1-x)(b_{i-1}(x,xy)-xya_{i-1}(x,xy))}.$$

This holds if and only if

$$\begin{aligned}
a_i(x,y)&=(1-x)a_{i-1}(x,xy),\\
b_i(x,y)&=(1-x)b_{i-1}(x,xy)-x(1-x)ya_{i-1}(x,xy).
\end{aligned}$$

By Example 2.3, we have $a_1(x,y)=(1-x)^2+x(1-x)y+x^2y^2$ and $b_1(x,y)=(1-x)^2$. Let $a_i(x,y)=(1-x)^ia'_i(x,y)$ and $b_i(x,y)=(1-x)^ib'_i(x,y)$. Then

$$\begin{aligned}
a'_i(x,y)&=a'_{i-1}(x,xy),\\
b'_i(x,y)&=b'_{i-1}(x,xy)-xya'_{i-1}(x,xy),
\end{aligned}$$

with $a'_1(x,y)=1-x+xy+\frac{x^2}{1-x}y^2$ and $b'_1(x,y)=1-x$. By induction over i, one has

$$a'_i(x,y)=a'_1(x,x^{i-1}y)=1-x+x^iy+\frac{x^{2i}}{1-x}y^2,$$

as well as

$$b'_i(x,y)=1-x-\sum_{j=1}^{i-1}x^{i-j}ya'_j(x,x^{i-j}y).$$

Hence, we have the following result.

Corollary 2.5 *The generating function for the number of bargraphs without protected cells in the ith row according to the number of cells and the number of columns is given by*

$$\frac{a_i(x,y)}{b_i(x,y)}=\frac{1-x+x^iy+\frac{x^{2i}}{1-x}y^2}{1-x-y(1-x+x^iy+\frac{x^{2i}}{1-x}y^2)\frac{x-x^i}{1-x}}.$$

2.2 Total number of protected cells

Define $f(x,y;q) = F(x,y;q,q,\ldots)$. By Theorem 2.1 with $q = q_1 = q_2 = \cdots$, we have

$$f(x,y;q) = 1 + \frac{xy}{1-x} + \frac{((1-x)(1+qxy)+q^2x^2y^2)f(x,qxy;q) - 1 + x - qxy}{q^2(1-x)(1-qxyf(x,qxy;q))}.$$

Note that $\frac{d}{dq}f(x,qxy;q)\mid_{q=1} = x\frac{d}{dy}f(x,xy;1) + \frac{d}{dq}f(x,xy;q)\mid_{q=1}$ and $f(x,y;1) = \frac{1-x}{1-x-xy}$. Thus,

$$\frac{d}{dq}f(x,y;q)\mid_{q=1} = \frac{x^3y^3}{(1-x-xy)^2(1-x)} + \frac{(1-x-x^2y)^2}{(1-x-xy)^2}\frac{d}{dq}f(x,xy;q)\mid_{q=1}.$$

By iterating this an infinite number of times, we obtain

$$\frac{d}{dq}f(x,y;q)\mid_{q=1} = \sum_{n\geq 0}\frac{x^{3i+3}y^3}{(1-x-xy)^2(1-x)} = \frac{x^3y^3}{(1-x-xy)^2(1-x)(1-x^3)}.$$

Hence, we can state the following result.

Corollary 2.6 *The generating function for the total number of protected cells in all bargraphs according to the number of cells and the number of columns is given by*

$$\frac{x^3y^3}{(1-x-xy)^2(1-x)(1-x^3)}.$$

As consequence, we have that the generating function for the total number of protected cells in all bargraphs according to the number of cells is given by

$$\frac{x^3}{(1-2x)^2(1-x)(1-x^3)}.$$

Thus, asymptotically, the total number of protected cells in all bargraphs with n cells is given by $\frac{n}{7}2^{n+1}$.

3 Empty cells

A *d-bargraph* is a bargraph where the height of all columns is at most d and which has at least one column of height d. Let $G_d(x,y) \equiv G_d(x,y;q_1,q_2,\ldots,q_d)$ be the generating function for the number of d-bargraphs according to the number of cells, the number of columns, and the number of empty cells in the ith row, that is,

$$G_d(x,y) = \sum_{\pi} x^{\#\text{cells in }\pi}y^{\#\text{columns in }\pi}\prod_{i=1}^{d}q_i^{\#\text{empty cells in }i\text{th row in }\pi},$$

where the sum is over all d-bargraphs π.

Note that each d-bargraph can be written as $\pi = \pi^{(1)}d\cdots\pi^{(s)}d\pi^{(s+1)}$ where $s \geq 1$ and $\pi^{(j)}$ is a bargraph such that the height of each column is at most $d-1$. For example, if s=1 we have

$$x^d y(\sum_{j=0}^{d-1} G_j(x, q_{j+1}q_{j+2}\cdots q_d y; q_1, q_2, \ldots, q_j))^2,$$

and if $s = 2$, then we have

$$x^d y(\sum_{j=0}^{d-1} G_j(x, q_{j+1}q_{j+2}\cdots q_d y; q_1, q_2, \ldots, q_j))^3,$$

and so on. Thus, we can state the following recurrence.

Proposition 3.1 *For all $d \geq 2$,*

$$G_d(x, y; q_1, q_2, \ldots, q_d) = \frac{x^d y\left(\sum_{j=0}^{d-1} G_j(x, q_{j+1}q_{j+2}\cdots q_d y; q_1, q_2, \ldots, q_j)\right)^2}{1 - x^d y\left(\sum_{j=0}^{d-1} G_j(x, q_{j+1}q_{j+2}\cdots q_d y; q_1, q_2, \ldots, q_j)\right)}$$

with $G_0(x, y) = 1$ and $G_1(x, y) = \frac{xy}{1-xy}$.

Example 3.2 By Proposition 3.1, we have

$$\begin{aligned} G_2(x, y; q_1, q_2) &= \frac{x^2 y}{(1 - q_2 xy - x^2 y)(1 - q_2 xy)}, \\ G_3(x, y; q_1, q_2, q_3) &= \frac{x^3 y}{(1 - q_2 q_3 xy - q_3 x^2 y - x^3 y)(1 - q_2 q_3 xy - q_3 x^2 y)}. \end{aligned}$$

By induction over d, we may deduce the following formula.

Theorem 3.3 *The generating function $G_d(x, y; q_1, q_2, \ldots, q_d)$ is given, for all $d \geq 1$, by*

$$\frac{x^d y}{\left(1 - y\sum_{i=1}^{d} q_{i+1}q_{i+2}\cdots q_d x^i\right)\left(1 - y\sum_{i=1}^{d-1} q_{i+1}q_{i+2}\cdots q_d x^i\right)}.$$

Moreover, the generating function for the number of bargraphs according to the number of cells, the number of columns, and the number of empty cells in each row is given by

$$1 + \sum_{d\geq 1} \frac{x^d y}{\left(1 - y\sum_{i=1}^{d} q_{i+1}q_{i+2}\cdots q_d x^i\right)\left(1 - y\sum_{i=1}^{d-1} q_{i+1}q_{i+2}\cdots q_d x^i\right)}.$$

From Theorem 3.3, one concludes

$$G_d(x,y;1,1,\ldots,1)=\frac{x^dy}{(1-y(x+x^2+\cdots+x^{d-1}))(1-y(x+x^2+\cdots+x^d))},$$

implying

$$\sum_{j=0}^{d}G_j(x,y;1,1,\ldots,1)=\frac{1}{1-y(x+x^2+\cdots+x^d)} \tag{3.1}$$

for the generating function for the number of bargraphs whose columns have height at most d according to the number of cells and the number of columns, as expected.

Theorem 3.3 with $q_d=q$ and $q_1,q_2,\ldots,q_{d-1}=1$ yields the following result.

Corollary 3.4 *The generating function* $1+\sum_{d\geq1}G_d(x,y;1,\ldots,1,q)$ *for the number of bargraphs according to the number of cells, the number of columns, and the number of empty cells in the top row is given by*

$$1+\sum_{d\geq1}\frac{x^dy}{(1-qy(x+x^2+\cdots+x^{d-1}))(1-qy(x+x^2+\cdots+x^{d-1})-x^dy)}.$$

By the preceding corollary, $H\equiv\frac{d}{dq}\sum_{d\geq1}G_d(x,1;1,\ldots,1,q)\mid_{q=1}$ is given by

$$H=\sum_{d\geq1}\left(\frac{x+x^2+\cdots+x^{d-1}}{(1-x-x^2-\cdots-x^d)^2}-\frac{x+x^2+\cdots+x^{d-1}}{(1-x-x^2-\cdots-x^{d-1})^2}\right).$$

Thus, the generating function for the total number of empty cells in the top row of the bargraphs with n cells is given by

$$H=\sum_{d\geq1}\frac{(x+x^2+\cdots+x^{d-1})(2-2x-\cdots-2x^{d-1}-x^d)x^d}{(1-x-x^2-\cdots-x^d)^2(1-x-x^2-\cdots-x^{d-1})^2}.$$

4 Empty protected cells

Let $E_d(x,y)\equiv E_d(x,y;q_1,q_2,\ldots,q_d)$ be the generating function for the number of d-bargraphs according to the number of cells, the number of columns, and the number of empty protected cells in the ith row, that is,

$$E_d(x,y)=\sum_{\pi}x^{\#\text{cells in }\pi}y^{\#\text{columns in }\pi}\prod_{i=1}^{d}q_i^{\#\text{empty protected cells in }i\text{th row in }\pi},$$

where the sum is over all d-bargraphs π. Using the same decomposition argument as for Proposition 3.1, we obtain the following result.

Proposition 4.1 *Define*

$$F_j \equiv x^j y + \frac{1}{q_{j+1}^2 q_{j+2}^2 \cdots q_d^2}(E_j(x, q_{j+1}q_{j+2}\cdots q_d y; q_1, q_2, \ldots, q_j) - x^j q_{j+1}q_{j+2}\cdots q_d y).$$

Then the generating function $E_d(x, y; q_1, q_2, \ldots, q_d)$ satisfies, for all $d \geq 2$,

$$E_d(x, y; q_1, q_2, \ldots, q_d) = \frac{x^d y \left(1 + \sum_{j=1}^{d-1} F_j\right)^2}{1 - x^d y \left(1 + \sum_{j=1}^{d-1} F_j\right)},$$

with $E_0(x, y) = 1$ and $E_1(x, y) = \frac{xy}{1-xy}$.

Example 4.2 By Proposition 4.1, we have

$$E_2(x, 1; 1, q) = \frac{\left(1 - (q-1)(x + x^2)\right)^2 x^2}{(1 - qx - x^2 + (q-1)x^2(x + x^2))(1 - qx)},$$

$$E_3(x, 1; 1, 1, q) = \frac{\left(1 - (q-1)(x + 2x^2 + 2x^3 + x^4)\right)^2 x^3}{(1 - qx - qx^2 - x^3 + (q-1)x^3(x + 2x^2 + 2x^3 + x^4))(1 - qx - qx^2)}.$$

By induction over d, we derive the following result.

Theorem 4.3 *Let $d \geq 1$. Then the generating function $E_d(x, 1; 1, \ldots, 1, q)$ for the number of d-bargraphs according to the number of cells and the number of empty protected cells in the top row is given by*

$$\frac{(1 - (q-1)f_d(x))^2 x^d}{\left(1 - q\frac{x-x^d}{1-x} - x^d + (q-1)x^d f_d(x)\right)\left(1 - q\frac{x-x^d}{1-x}\right)},$$

where $f_d(x) = x + 2x^2 + \cdots + (d-1)x^{d-1} + (d-1)x^d + \cdots + 2x^{2d-3} + x^{2d-2}$.

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