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Anti-van der Waerden numbers of graph products

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Abstract

In this paper, anti-van der Waerden numbers on Cartesian products of graphs are investigated and a conjecture made by Schulte, et al. is answered. In particular, the anti-van der Waerden number of the Cartesian product of two graphs has an upper bound of four. This result is then used to determine the anti-van der Waerden number for any Cartesian product of two paths.

1 Introduction

The anti-van der Waerden number on $[n] = \{1, \ldots, n\}$ is the fewest number of colors that must be assigned to the elements of $[n]$ to guarantee an arithmetic progression of length 3 (or more) where each element of the progression is a unique color. The anti-van der Waerden number was first defined in [7]. Many results on arithmetic progressions of $[n]$ and the cyclic groups $\mathbb{Z}_n$ were considered in [4] and a function $f(n)$ was established in [3] such that $\mathrm{aw}([n], 3) = f(n)$ for all $n \in \mathbb{N}$. Results

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on colorings of the integers with no rainbow 3-term arithmetic progressions were also studied in [1] and [2]. Colorings and 3-term arithmetic progressions have been extended to groups (see [8]) and graphs (see [6]). The authors in [6] were inspired to investigate the anti-van der Waerden number of graphs by extending results on the anti-van der Waerden number of $[n]$ and $\mathbb{Z}_n$ to paths and cycles, respectively. In particular, they noticed that the set of arithmetic progressions on $[n]$ is isomorphic to the set of non-degenerate arithmetic progressions on P_n. Similarly, the set of arithmetic progressions on $\mathbb{Z}_n$ is isomorphic to the set of non-degenerate arithmetic progressions on C_n. Therefore, considering the anti-van der Waerden number of $[n]$ or $\mathbb{Z}_n$ is equivalent to studying the anti-van der Waerden number of paths or cycles, respectively. The authors of [6] made a conjecture about graph products and this conjecture is proven in this paper. First, some terminology and notation is introduced.

A *graph*, G, is a collection of vertices, $V(G)$, and edges, $E(G)$, and will be denoted as $G = (V, E)$. The edge set E is a set of pairs of vertices that indicate the two vertices are connected. Thus, if there is an edge connecting vertices u and v, then $\{u, v\}$ is an edge or uv is an edge for short. Graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of G is one formed by deleting vertices of G and keeping all possible edges. For the purposes of this paper all graphs are simple (loop free, undirected, no edge weights, no multiple edges) and connected. The *distance* between vertex u and v in graph G is denoted $\mathrm{d}_G(u, v)$, $\mathrm{d}(u, v)$ will be used when the context is clear. If $G = (V, E)$ and $H = (V', E')$ then the *Cartesian product*, written $G\Box H$, has vertex set $\{(x, y) \,|\, x \in V \text{ and } y \in V'\}$ and (x, y) and (x', y') are adjacent in $G\Box H$ if either $x = x'$ and $yy' \in E'$ or $y = y'$ and $xx' \in E$. In this paper, P_n denotes the path graph on n vertices.

The vertex set of $P_m\Box P_n$ is given by $\{v_{i,j} \,|\, 1 \le i \le m \text{ and } 1 \le j \le n\}$. This graph is visually represented as a grid and can be thought of as having m rows and n columns. Further, $v_{i,j}$ can be found at the intersection of the ith row and jth column of $P_m\Box P_n$. This convention allows for the computation of distances in grid graphs based on the subscripts of the vertices. In particular, if $v_{i,j}$ and $v_{\ell,k}$ are in $P_m\Box P_n$ then $\mathrm{d}(v_{i,j}, v_{\ell,k}) = |i - \ell| + |j - k|$.

A *k-term arithmetic progression of a graph G*, k-AP, is a subset of k vertices of G of the form $\{v_1, v_2, \dots, v_k\}$, where $\mathrm{d}(v_i, v_{i+1}) = d$ for all $1 \le i < k$. Throughout the remainder of this paper the order the subset is written will denote the order of the arithmetic progression. A k-term arithmetic progression is *degenerate* if $v_i = v_j$ for any $i \ne j$.

An *exact r-coloring of a graph G* is a surjective function $c : V(G) \to \{1, 2, \dots, r\}$. A set of vertices $S \subseteq V(G)$ is *rainbow* under coloring c, if for any $v_i, v_j \in S$, $c(v_i) \ne c(v_j)$ when $v_i \ne v_j$. Note that degenerate k-APs will not be rainbow. Given a set of vertices $S \subseteq V(G)$, $c(S) = \{c(v) | v \in S\}$, is the set of colors used on the vertices of S.

The *anti-van der Waerden number of a graph G*, denoted by $\mathrm{aw}(G, k)$, is the least positive integer r such that every exact r-coloring of G contains a non-degenerate rainbow k-AP. If G has n vertices and no coloring of G contains non-degenerate k-APs, then $\mathrm{aw}(G, k) = n + 1$. For a graph G, if $\mathrm{aw}(G, k) = r$, then an *extremal coloring* is an exact $(r - 1)$-coloring of G that avoids rainbow 3-APs.

Notice that at least k colors are needed to have a rainbow k-AP. This paper also includes the convention that since a graph cannot be colored with more colors than it has vertices the anti-van der Waerden number of a graph is bounded above by one more than its order. In the case that $k \geq |G| + 1$, then $\mathrm{aw}(G, k) = |G| + 1$. This is formally stated in Observation 1.1.

Observation 1.1. *If G is a graph on n vertices, then $k \leq \mathrm{aw}(G, k) \leq n + 1$. If $k \geq n + 1$, then $\mathrm{aw}(G, k) = n + 1$.*

In Section 2, results that will be used throughout the paper are established. In Section 3, results are established on $P_m \square P_n$ where $m = 2$ or $m = 3$. In Section 4, these results are used to prove Conjecture 1.2 from a paper authored by Schulte, et al.

Conjecture 1.2 ([6])**.** *If G and H are connected graphs, then*

$$\mathrm{aw}(G \square H, 3) \leq 4.$$

The result from Conjecture 1.2 is used in Section 5 to find the anti-van der Waerden number of $P_m \square P_n$ for all m and n.

2 Fundamental Tools

In this section, preliminary results are established that are applicable throughout the remainder of the paper. A subgraph H of G is *isometric* if for all $u, v \in V(H)$, $\mathrm{d}_H(u, v) = \mathrm{d}_G(u, v)$.

Lemma 2.1. *If H is an isometric subgraph of G, then a k-AP in H is a k-AP in G. If there exists a k-AP in G that only contains vertices of H, then it is also a k-AP in H.*

Proof. Let $\{x_1, x_2, \ldots, x_k\}$ be a k-AP in H. Since this is a k-AP, then $\mathrm{d}_H(x_i, x_{i+1}) = d$ for $1 \leq i \leq k - 1$. By the definition of isometric subgraph, $\mathrm{d}_H(x_i, x_{i+1}) = \mathrm{d}_G(x_i, x_{i+1})$. Hence, $\{x_1, x_2, \ldots, x_k\}$ is a k-AP in G. Now suppose $\{x_1, x_2, \ldots, x_k\}$ is a k-AP in G and $x_i \in V(H)$ for $1 \leq i \leq k$. Since $\{x_1, x_2, \ldots, x_k\}$ is a k-AP in G, it follows that $\mathrm{d}_G(x_i, x_{i+1}) = d'$ for $1 \leq i \leq k - 1$. Since H is an isometric subgraph, $\mathrm{d}_G(x_i, x_{i+1}) = \mathrm{d}_H(x_i, x_{i+1})$ for all $1 \leq i \leq k - 1$, and therefore, $\{x_1, x_2, \ldots, x_k\}$ is a k-AP in H. □

Proposition 2.2. *If H is an isometric subgraph of G and c is an exact r-coloring of G that avoids rainbow k-APs, then H contains at most $\mathrm{aw}(H, k) - 1$ colors.*

Proof. Suppose by way of contradiction, $|c(H)| \geq \text{aw}(H, k)$. This implies H has a rainbow k-AP, namely $\{x_1, x_2, \ldots, x_k\}$, since every $\text{aw}(H, k)$-coloring of H must have a rainbow k-AP by definition. By Lemma 2.1, $\{x_1, x_2, \ldots, x_k\}$ is also a k-AP in G, a contradiction. Hence, any isometric subgraph H of G has at most $\text{aw}(H, k) - 1$ colors. □

Note that Proposition 2.2 ensures that whenever there exists a rainbow 3-AP in an isometric subgraph of G, there is a corresponding rainbow 3-AP in G. This fact is used frequently without citation in the remainder of this paper.

Lemma 2.3. *Let $G = P_m \square P_n$ and c be an exact r-coloring of G with $r \geq 3$ that avoids rainbow 3-APs. If $c(v_{i,j}) = red$ and $c(v_{i-1,j+1}) = blue$, then $c(v_{k,\ell}) \in \{red, blue\}$ when $k \geq i$ and $\ell \geq j + 1$ or $k \leq i - 1$ and $\ell \leq j$. Further, if $c(v_{i,j}) = red$ and $c(v_{i-1,j-1}) = blue$, then $c(v_{k,\ell}) \in \{red, blue\}$ when $k \geq i$ and $\ell \leq j - 1$ or $k \leq i - 1$ and $\ell \geq j$.*

Proof. Consider the case when $c(v_{i,j}) = red$ and $c(v_{i-1,j+1}) = blue$ (see Figure 1). Define $v_{k,\ell}$ so that $k \geq i$ and $\ell \geq j + 1$. Notice that $\text{d}(v_{k,\ell}, v_{i,j}) = \text{d}(v_{k,\ell}, v_{i-1,j+1}) = k - i + \ell - j$. This means $\{v_{i,j}, v_{k,\ell}, v_{i-1,j+1}\}$ is a 3-AP and since c avoids rainbow 3-APs $c(v_{k,\ell}) \in \{red, blue\}$. A similar argument can be made in the other three situations. □

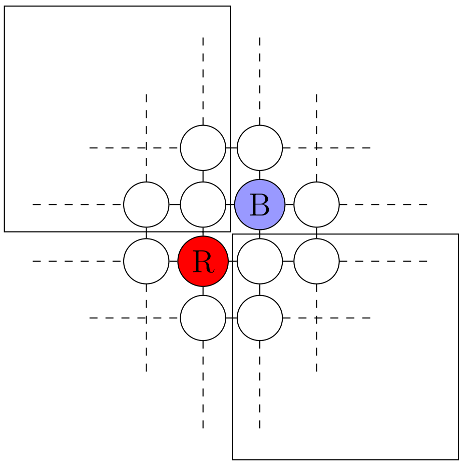

Figure 1: Vertex R (or $v_{i,j}$) being red and vertex B (or $v_{i-1,j+1}$) being blue force the Northwest and Southeast blocks to be *red* or *blue*.

Lemma 2.4. *Let $G = P_m \square P_n$ and c be an exact r-coloring of G such that c avoids rainbow 3-APs and $r \geq 3$. If $c(v_{i,k}) = \{red\}$ for fixed i and $1 \leq k \leq n$, $S_1 = \{v_{s,t} \,|\, 1 \leq s < i,\ 1 \leq t \leq n\}$ and $S_2 = \{v_{s,t} \,|\, i < s \leq m,\ 1 \leq t \leq n\}$, then $|c(S_i) \cup \{red\}| \leq 2$.*

Proof. Assume, without loss of generality, that $c(v_{\ell,j}) = blue$ for some j and $i < \ell \leq m$ and rows $i + 1$ to $\ell - 1$ are monochromatic *red*. By Lemma 2.3, if $c(v_{s,t}) = green$ for $\ell \leq s \leq m$, $1 \leq t \leq n$ and $t \neq j$, then either $\{v_{\ell,j}, v_{s,t}, v_{\ell-1,j-1}\}$ or $\{v_{\ell,j}, v_{s,t}, v_{\ell-1,j+1}\}$ is rainbow. This implies that for $t \neq j$, $c(v_{s,t}) \in \{red, blue\}$.

However, using Lemma 2.3 with $v_{s,j}$, one of $\{v_{s,j}, v_{\ell,j}, v_{s-1,j-1}\}$, $\{v_{s,j}, v_{\ell,j}, v_{s-1,j+1}\}$, $\{v_{s,j}, v_{\ell-1,j}, v_{s-1,j-1}\}$ or
$\{v_{s,j}, v_{\ell-1,j}, v_{s-1,j+1}\}$ exists and is rainbow. Thus, no such $v_{s,t}$ is *green*. A similar argument applies when $1 \leq \ell < i$ and rows $\ell + 1$ to $i - 1$ are monochromatic *red*. □

Lemma 2.4 says that if there is a monochromatic row in some $P_m \square P_n$, then at most one new color can be introduced below the monochromatic row and at most one new color can be introduced above the monochromatic row. Note that the argument can be easily applied to monochromatic columns. Corollary 2.5 states this result.

Corollary 2.5. *Let $G = P_m \square P_n$ and c be an exact r-coloring of G such that c avoids rainbow 3-APs and $r \geq 3$. If $c(v_{i,k}) = \{red\}$ for fixed k and $1 \leq i \leq m$, $S_1 = \{v_{s,t} \,|\, 1 \leq s \leq m,\ 1 \leq t < k\}$ and $S_2 = \{v_{s,t} \,|\, 1 \leq s \leq m,\ k < t \leq n\}$, then $|c(S_i) \cup \{red\}| \leq 2$.*

Lemma 2.6 will be useful in combination with Proposition 2.2 in determining the anti-van der Waerden number. In particular, Lemma 2.6 establishes the possible 3-colorings of $P_2 \square P_{2k+1}$ that avoid rainbow 3-APs. These colorings are achieved when two non-adjacent corners are given unique colors and the remainder of the graph is colored with the third color.

Lemma 2.6. *If $G = P_2 \square P_{2k+1}$ and $k \geq 1$, then there are precisely two exact 3-colorings of G that avoid rainbow 3-APs.*

Proof. Let c be an exact 3-coloring of G that avoids rainbow 3-APs. Without loss of generality, let $c(v_{1,1}) = red$. If $c(v_{2,1}) = red$, then by Corollary 2.5, G is colored with at most two colors. Thus, $c(v_{2,1}) = blue$. If $c(v_{1,2}) = green$, then $\{v_{1,2}, v_{1,1}, v_{2,1}\}$ is a rainbow 3-AP. Now, consider the following cases.

Case 1: $c(v_{1,2}) = red$
By Lemma 2.3, $c(v_{2,j}) \in \{red, blue\}$ for $2 \leq j \leq 2k+1$. If $c(v_{2,2}) = blue$, then Lemma 2.3 forces both the top and bottom rows to be colored *red* or *blue* contradicting that c was an exact 3-coloring. Thus, $c(v_{2,2})$ must be *red*. By Corollary 2.5, columns 3 through $2k + 1$ must be *red* and *green*, but the bottom row is also *red* and *blue*; thus, $c(v_{2,j}) = red$ for $3 \leq j \leq 2k + 1$. This means $c(v_{1,i}) = green$ for some $3 \leq i \leq 2k + 1$. If $i \neq 2k + 1$, then $\{v_{1,i}, v_{2,1}, v_{2,i+1}\}$ is a rainbow 3-AP. Thus, for $i < 2k + 1$, $c(v_{1,i}) = red$ and $c(v_{1,2k+1}) = green$. This is an exact 3-coloring that avoids rainbow 3-APs.

Case 2: $c(v_{1,2}) = blue$
If $c(v_{2,2}) = green$ then there exists an obvious rainbow 3-AP. If $c(v_{2,2}) = blue$ apply an argument similar to Case 1 and achieve a symmetric coloring. Consider if $c(v_{2,2}) = red$. Let $c(v_{i,j}) = green$ such that j is minimal. By Corollary 2.5, column j cannot be monochromatic since *red* and *blue* appear in column 1. If $c(v_{1,j}) = red$ and $c(v_{2,j}) = green$, then $c(v_{1,j-1}) \neq green$ by minimality of j, $c(v_{1,j-1}) \neq blue$ by the rainbow 3-AP $\{v_{1,j-1}, v_{2,2}, v_{2,j}\}$ and $c(v_{1,j-1}) \neq red$ by the rainbow 3-AP $\{v_{1,j-1}, v_{2,1}, v_{2,j}, \}$. If $c(v_{1,j}) = blue$ and $c(v_{2,j}) = green$, a similar argument can be made. Finally, the symmetry of column 1 and 2 demonstrate that $c(v_{1,j}) \neq green$.

Therefore, there are two exact 3-colorings on G that avoid rainbow 3-APs. □

3 Analysis of $P_2 \square P_n$ and $P_3 \square P_n$

In this section, results on $P_2 \square P_n$ and $P_3 \square P_n$ are established. These results are used in conjunction with Proposition 2.2 to obtain other results including Theorem 4.5. To begin, first consider the smallest non-trivial $P_m \square P_n$.

Observation 3.1. *The anti-van der Waerden number of* $P_2 \square P_2$ *is* 3, *that is,* $\mathrm{aw}(P_2 \square P_2, 3) = 3$.

Almost all of the results in this section require an arbitrary coloring of a graph. Lemma 2.6 allows the elimination of one (or more) colors from half of the graph under the right circumstances.

Proposition 3.2. *For every* $k \geq 1$, $\mathrm{aw}(P_2 \square P_{2k}, 3) = 3$.

Proof. Consider the graph $P_2 \square P_{2k}$ and proceed by induction on k. First, if $k = 1$, then Observation 3.1 gives $\mathrm{aw}(P_2 \square P_{2k}, 3) = 3$.

For the inductive hypothesis, assume that $\mathrm{aw}(P_2 \square P_{2k}, 3) = 3$. Now consider $P_2 \square P_{2k+2}$ with an exact 3-coloring c which avoids rainbow 3-APs. The graph $P_2 \square P_{2k+2}$ can be thought of as the union of the two isometric subgraphs formed by columns 1 through 3 and columns 3 through $2k + 2$. More technically, let $G_1 = P_2 \square P_3$ with $V(G_1) = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}\}$ and let $G_2 = P_2 \square P_{2k}$ with $V(G_2) = \{v_{1,3}, v_{2,3}, v_{1,4}, v_{2,4}, \ldots, v_{1,2k+2}, v_{2,2k+2}\}$. Then $V(G_1) \cap V(G_2) = \{v_{1,3}, v_{2,3}\}$ and $G_1 \cup G_2 = P_2 \square P_{2k+2}$ (see Figure 2). For the following cases, let c be an exact 3-coloring and let $S = \{v_{1,3}, v_{2,3}\}$.

Case 1: $|c(S)| = 2$.
Without loss of generality, let $c(v_{1,3}) = \textit{blue}$ and $c(v_{2,3}) = \textit{red}$. By the inductive hypothesis, a third color cannot be introduced into G_2 such that there is no rainbow 3-AP. However, by Lemma 2.6, there exists a unique exact 3-coloring that avoids rainbow 3-AP's in G_1. Without loss of generality, consider the following coloring of G_1 where $c(v_{1,1}) = \textit{green}$, and all other vertices in G_1 are colored *blue* (see Figure 2).

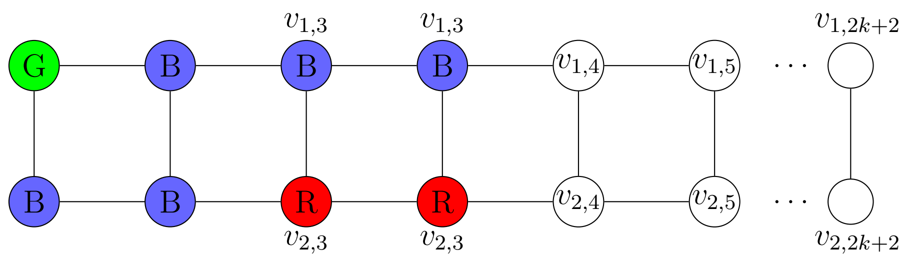

Figure 2: Note the identification of vertices $v_{1,3}$ and $v_{2,3}$ implies the figure shows $P_2 \square P_3 \cup P_2 \square P_{2k} = P_2 \square P_{2k+2}$.

Now, focusing on the vertex pairs $v_{1,1}$, $v_{2,2}$ and $v_{1,2}$, $v_{2,3}$, Lemma 2.3 forces $c(v_{1,j}) = \textit{blue}$ for $2 \leq j \leq 2k + 2$. This however yields the rainbow 3-AP, $\{v_{1,4}, v_{1,1}, v_{2,3}\}$ and this case is complete.

Case 2: $|c(S)| = 1$.
Without loss of generality let $c(S) = \{red\}$. By Lemma 2.6 at most one new color can be added to G_1 and by the induction hypothesis at most one new color can be added to G_2 while still avoiding rainbow 3-APs. Without loss of generality, assume the color introduced in G_1 is *blue* and the color introduced in G_2 is *green*. If $c(v_{1,2}) =$ *blue* then, by Lemma 2.3, $c(v_{1,j}) =$ *red* for $3 \leq j \leq 2k+2$. Now if $c(v_{2,\ell}) =$ *green* for some $4 \leq \ell \leq 2k+2$, then Lemma 2.3 says that $c(v_{2,1}) =$ *red*, but then $\{v_{2,1}, v_{2,\ell}, v_{1,2}\}$ is a rainbow 3-AP. A similar argument can be made if $c(v_{2,2}) =$ *blue*, so $c(v_{1,2}) = c(v_{2,2}) =$ *red*.

Now let $c(v_{1,1}) =$ *blue*, then by Lemma 2.3 $c(v_{1,j}) =$ *red* for $4 \leq j \leq 2k+2$. This forces $c(v_{2,\ell}) =$ *green* for some $4 \leq \ell \leq 2k+2$. If $4 \leq \ell \leq 2k+1$ then $\{v_{2,\ell}, v_{1,1}, v_{1,\ell+1}\}$ is a rainbow 3-AP. If $\ell = 2k+2$, then $\{v_{1,1}, v_{1,k+2}, v_{2,2k+2}\}$ is a rainbow 3-AP. Similarly, $c(v_{2,1}) \neq$ *blue* which means $|c(G_1)| = 1$. This in turn implies that $|c(G_2)| = 3$ which, as noted earlier, has a rainbow 3-AP via the inductive hypothesis.

It has been demonstrated that every exact 3-coloring of $P_2 \square P_{2k+2}$ will result in a rainbow 3-AP. Thus, $\text{aw}(P_2 \square P_{2k}, 3) = 3$ for all $k \geq 1$. □

Lemma 3.3. *If $G = P_m \square P_n$ and $m+n = 2k+1$ for some $k \geq 1$, then $4 \leq \text{aw}(G, 3)$.*

Proof. Consider the exact 3-coloring c where $c(v_{1,1}) =$ *red*, $c(v_{m,n}) =$ *blue* and the remaining vertices are *green*. Note $\text{d}(v_{1,1}, v_{m,n}) = m+n-2$ which, by assumption, is odd so there does not exist a vertex equidistant from both $v_{1,1}$ and $v_{m,n}$, i.e. there is no 3-AP of the form $\{v_{1,1}, v_{i,j}, v_{m,n}\}$. This means if a rainbow 3-AP exists it must be of the form $\{v_{1,1}, v_{m,n}, v_{i,j}\}$ (or similarly $\{v_{m,n}, v_{1,1}, v_{i,j}\}$ which implies there is some vertex $v_{i,j}$ that is distance $m+n-2$ from $v_{1,1}$ or $v_{m,n}$. However, this cannot happen since $v_{1,1}$ and $v_{m,n}$ are, up to isomorphism, the only two vertices distance $m+n-2$ apart. Thus, an exact 3-coloring that avoids rainbow 3-APs has been constructed, therefore $4 \leq \text{aw}(G, 3)$. □

Proposition 3.4. *For every $k \geq 1$, $\text{aw}(P_2 \square P_{2k+1}, 3) = 4$.*

Proof. Let $G = P_2 \square P_{2k+1}$. First notice that $4 \leq \text{aw}(G, 3)$ by Lemma 2.6. Now, consider the two isometric subgraphs $G_1 = P_2 \square P_2$ and $G_2 = P_2 \square P_{2k}$ with $S = V(G_1) \cap V(G_2) = \{v_{1,2}, v_{2,2}\}$. Let c be an exact 4-coloring of G. Note that G_1 and G_2 must share at least one color. If $|c(G_1)| = 2$ and $|c(G_2)| = 2$ then at most three colors have been used. This implies $|c(G_i)| = 3$ for $i = 1$ or $i = 2$, but $\text{aw}(G_1, 3) = \text{aw}(G_2, 3) = 3$ by Observation 3.1 and Proposition 3.2, respectively. Thus, there exists a rainbow 3-AP in either G_1 or G_2. Therefore, $\text{aw}(G, 3) = 4$. □

Proposition 3.5. *For every $k \geq 1$, $aw(P_3 \square P_{2k}, 3) = 4$.*

Proof. Consider the graph $G = P_3 \square P_{2k}$. Since $3+2k = 2(k+1)+1$, then $4 \leq \text{aw}(G, 3)$ by Lemma 3.3. Let c be an exact 4-coloring of G. Now consider the two isometric subgraphs G_1 and G_2 each of which are $P_2 \square P_{2k}$ graphs where $V(G_1) \cap V(G_2) = \{v_{2,1}, v_{2,2}, \ldots, v_{2,2k}\}$. By Proposition 3.2, G_1 and G_2 must have at most two colors to avoid a rainbow 3-APs. This means the coloring c must give a rainbow 3-AP, thus $\text{aw}(G, 3) = 4$. □

Lemma 3.6. *The anti-van der Waerden number of* $P_3 \square P_3$ *is* 3*, that is,* $\text{aw}(P_3 \square P_3, 3) = 3$.

Proof. Let $G = P_3 \square P_3$ and note that Observation 1.1 gives $3 \le \text{aw}(G, 3)$. Let c be an exact 3-coloring. Consider the two isometric subgraphs G_1 and G_2 each of which are $P_2 \square P_3$ graphs. Let $S = V(G_1) \cap V(G_2) = \{v_{2,1}, v_{2,2}, v_{2,3}\}$. If each of these vertices is assigned a different color, then G clearly has a rainbow 3-AP.

Case 1: $|c(S)| = 1$.
Suppose, without loss of generality, that $c(S) = \{red\}$. By Lemma 2.4, neither G_1 nor G_2 can have three colors. Let $c(v_{1,1}) = blue$ and suppose $c(v_{3,j}) = green$ for some $j \in \{1, 2, 3\}$. Then, either $\{v_{1,1}, v_{2,1}, v_{3,1}\}$, $\{v_{3,2}, v_{1,1}, v_{2,3}\}$ or $\{v_{1,1}, v_{2,2}, v_{3,3}\}$ is a rainbow 3-AP. Therefore, $c(v_{1,1}) = red$ and by symmetry

$$c(v_{1,1}) = c(v_{1,3}) = c(v_{3,1}) = c(v_{3,3}) = red.$$

This leaves only $v_{1,2}$ and $v_{3,2}$ uncolored and assigning them the colors *blue* and *green* yields the rainbow 3-AP $\{v_{1,2}, v_{2,2}, v_{3,2}\}$.

Case 2: $|c(S)| = 2$.
Without loss of generality, let $c(S) = \{blue, green\}$. A coloring described in Lemma 2.6 indicates that if a third color is added to G_1 or G_2, without loss of generality, $c(\{v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}\}) = \{blue\}$, $c(v_{1,1}) = red$, and $c(v_{2,3}) = green$. If $c(v_{3,1}) = blue$, $c(v_{3,1}) = green$ or $c(v_{3,1}) = red$, then $\{v_{1,1}, v_{2,3}, v_{3,1}\}$, $\{v_{1,1}, v_{2,1}, v_{3,1}\}$ or $\{v_{1,2}, v_{3,1}, v_{2,3}\}$ is a rainbow 3-AP, respectively.

Therefore, $\text{aw}(G, 3) = 3$. □

Proposition 3.7. *For every* $k \ge 1$, $\text{aw}(P_3 \square P_{2k+1}, 3) = 3$.

Proof. First, consider when $k = 1$. From Lemma 3.6, $\text{aw}(P_3 \square P_{2k+1}, 3) = 3$. Assume that $\text{aw}(P_3 \square P_{2k+1}, 3) = 3$ and now consider the graph $P_3 \square P_{2k+3}$. Recall that $3 \le \text{aw}(P_3 \square P_{2k+3}, 3)$ by Observation 1.1. Let c be an exact 3-coloring of $P_3 \square P_{2k+3}$ and consider the two isometric subgraphs $G_1 = P_3 \square P_3$ and $G_2 = P_3 \square P_{2k+1}$ where $S = V(G_1) \cap V(G_2) = \{v_{1,3}, v_{2,3}, v_{3,3}\}$. Note that $\text{aw}(G_1, 3) = \text{aw}(G_2, 3) = 3$ by the base case and induction hypothesis, respectively. Notice that $|c(S)| \ne 2$, otherwise adding a third color to either G_1 or G_2 would yield a rainbow 3-AP. Clearly $|c(S)| \ne 3$, so suppose $|c(S)| = 1$. Without loss of generality, let $c(S) = \{red\}$, $c(V(G_1)) = \{red, blue\}$, and $c(V(G_2)) = \{red, green\}$.

If $c(v_{1,2}) = blue$, then $c(v_{1,j}) = red$ for $4 \le j \le 2k+3$ by Lemma 2.3. If $c(v_{2,\ell}) = green$ for some $4 \le \ell \le 2k+2$, then $\{v_{2,\ell}, v_{1,2}, v_{1,\ell+1}\}$ is a rainbow 3-AP so $c(v_{2,\ell}) = red$. If $c(v_{2,2k+3}) = green$, then $\{v_{1,2}, v_{1,k+3}, v_{2,2k+3}\}$ is a rainbow 3-AP. Thus, the color *green* must only appear in the third row. A similar argument, using 3-AP $\{v_{3,\ell}, v_{1,2}, v_{2,\ell+1}\}$, shows that $c(v_{3,\ell}) = red$. If $c(v_{3,2k+3}) = green$, then $c(v_{2,1})$ must be *blue* since $\{v_{2,1}, v_{3,2k+3}, v_{1,2}\}$ is a 3-AP. However, this creates the rainbow 3-AP $\{v_{3,2k+3}, v_{2,1}, v_{1,2k+3}\}$. This implies $c(v_{1,2}) = red$ and by symmetry $c(v_{3,2}) = red$.

Now, if $c(v_{2,2}) = blue$, then by Lemma 2.3, a third color cannot be introduced in G_2. Therefore, all of column two is colored *red*.

If $c(v_{2,1}) = blue$, then by Lemma 2.3 a third color cannot be introduced in G_2. Thus $c(v_{2,1}) = red$. If $c(v_{1,1}) = blue$, then by Lemma 2.3 $c(v_{1,j}) = red$ for $4 \leq j \leq 2k+3$. If $c(v_{2,\ell}) = green$ for some $4 \leq \ell \leq 2k+2$, then $\{v_{1,\ell+1}, v_{1,1}, v_{2,\ell}\}$ is a rainbow 3-AP. If $c(v_{2,2k+3}) = green$, then by Lemma 2.3 $c(v_{3,1}) = red$ which yields the rainbow 3-AP $\{v_{1,1}, v_{2,2k+3}, v_{3,1}\}$. Thus, $c(v_{2,2k+3}) = red$. If $c(v_{3,\ell}) = green$ for $4 \leq \ell \leq 2k+2$, then $\{v_{3,\ell}, v_{1,1}, v_{2,\ell+1}\}$ is a rainbow 3-AP. Therefore, $c(v_{3,\ell}) = red$. Finally, if $c(v_{3,2k+3}) = green$, then $\{v_{1,1}, v_{2,k+2}, v_{3,2k+3}\}$ is a rainbow 3-AP. Therefore, any 3-coloring of G yields a rainbow 3-AP. □

4 General Products

In this section, the main result is that the anti-van der Waerden number of Cartesian products of graphs are bounded above by 4. The section begins with Lemma 4.1 which limits the number of colors that can be introduced in a Cartesian product of graphs.

Lemma 4.1. *[6, Lemma 4.3] Let G be a connected graph on m vertices and H be a connected graph on n vertices. Let c be an exact r-coloring of $G\Box H$ with no rainbow 3-APs. If $G_1, G_2, \ldots, G_n$ are the labeled copies of G in $G\Box H$, then $|c(V(G_j)) \setminus c(V(G_i))| \leq 1$ for all $1 \leq i, j \leq n$.*

It will be useful to find isometric graphs that have at least three colors. This is made possible by the following lemma.

Lemma 4.2. *If G is a connected graph on at least three vertices with an exact r-coloring c where $r \geq 3$, then there exists a subgraph H in G with at least three colors where H is either an isometric path or $H = C_3$.*

Proof. Choose $u, v \in V(G)$ such that $uv \in E(G)$, $c(u) = red$ and $c(v) = blue$. Further, let $w \in V(G)$ such that $\mathrm{d}(v, w)$ is minimal and $c(w) = green$, P_v be a shortest path from v to w, and P_u be a shortest path from u to w. If $P_u \subseteq P_v$ or $P_v \subseteq P_u$, then one of P_u or P_v is H. Now assume $P_u \nsubseteq P_v$ and $P_v \nsubseteq P_u$. If the lengths of P_u and P_v differ by two or more, then there is a contradiction about the minimality of P_u or P_v. If the lengths of P_u and P_v differ by one, then extending P_u to contain v or P_v to contain u gives H. If the lengths of P_u and P_v are the same, define $V(P_x) = \{x, x_1, x_2, \ldots, x_\ell, w\}$ for $x \in \{u, v\}$. If $c(u_i) = green$, then either $\mathrm{d}(v, w) = \mathrm{d}(v, u_i)$ or $\mathrm{d}(v, u_i) < \mathrm{d}(v, w)$. In the former case, the shortest path from v to u to u_i is H, and the latter case contradicts the minimality of P_v. If $c(u_i) \neq red$, then $P_u = H$ and if $c(v_i) \neq blue$, then $P_v = H$. Thus, the last situation to consider is if $c(u_i) = red$ and $c(v_i) = blue$ for all i. However, the subgraph of G induced by $\{u_\ell, w, v_\ell\}$ is H. □

Lemma 4.3. *Assume G and H are connected and consider the graph $G\Box H$. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, and suppose c is an exact r-coloring such that $r \geq 3$, c avoids rainbow 3-APs and $|c(V(G_i))| \leq 2$ for $1 \leq i \leq n$. If $v_iv_j \in E(H)$, then $|c(V(G_i) \cup V(G_j))| \leq 2$.*

Proof. If G_i is monochromatic and G_j is monochromatic then the result is immediate. If G_i is monochromatic and G_j is bichromatic then either $|c(V(G_i) \cup V(G_j))| \leq 2$ or $|c(V(G_j)) \setminus c(V(G_i))| = 2$. The former is the desired result and the latter contradicts Lemma 4.1. Now consider the case where at least one of G_i or G_j has three or more colors. Without loss of generality, assume G_i has three or more colors. Then there exists an C_3 subgraph or an isometric path with at least three colors, by Lemma 4.2, in G_i. If G_i has a C_3 subgraph with three colors, then there is an immediate rainbow 3-AP in $G\Box H$. If G_i has an isometric path with at least three colors, let $\rho^{(i)}$ be the shortest such path G_i and $\rho^{(j)}$ be the corresponding path in G_j. This creates a $P_2\Box P_y$ where y is the length of $\rho^{(i)}$. By Lemma 2.6 and Proposition 3.2 there exists a rainbow 3-AP in $G\Box H$.

Assume G_i and G_j each have two colors with $|c(V(G_i) \cup V(G_j))| \geq 3$. Since $|c(V(G_i)) \setminus c(V(G_j))| \leq 1$, by Lemma 4.1, then they must share a color. Without loss of generality, let $c(V(G_i)) = \{red, blue\}$ and $c(V(G_j)) = \{blue, green\}$. Pick a *red* vertex, say $v_{i,\alpha}$, in G_i with a *blue* neighbor, namely v. Also, choose $v_{j,\beta}$ in G_j such that $c(v_{j,\beta}) = green$. Let $v_{i,\beta}$ be the vertex in G_i that corresponds to $v_{j,\beta}$ and let $P^{(i)}$ be a shortest path from $v_{i,\alpha}$ to $v_{i,\beta}$ in G_i and $P^{(j)}$ be the corresponding path in G_j. Notice that $P^{(i)}$ and $P^{(j)}$ form an isometric $P_2\Box P_x$ in $G\Box H$ where x is the length of $P^{(i)}$. If $P_2\Box P_x$ has no *blue* vertices, then $\{v, v_{i,\alpha}, v_{j,\alpha}\}$ is a rainbow 3-AP. If $P_2\Box P_x$ has a *blue* vertex and x is even, then there is a rainbow 3-AP since $\text{aw}(P_2\Box P_{2k}, 3) = 3$ by Proposition 3.2. If x is odd, then by Lemma 2.6, so $c(v_{j,\alpha}) = c(v_{i,\beta}) = blue$. Now, extend to $P_2\Box P_x$ to include a corresponding path from G_k where $v_jv_k \in E(H)$, which gives a $P_3\Box P_x$ subgraph. If $P_3\Box P_x$ is an isometric subgraph of $G\Box H$, then there is a rainbow 3-AP since $\text{aw}(P_3\Box P_{2k+1}, 3) = 3$, by Proposition 3.7. If $P_3\Box P_x$ is not an isometric subgraph of $G\Box H$, then it must correspond to an isometric subgraph $C_3\Box P_x$ of $G\Box H$. Let $v_{k,\beta}$ be the vertex in G_k that corresponds to $v_{j,\beta}$. However, $c(v_{k,\beta})$ cannot be *red*, *blue* or *green* due to 3-APs $\{v_{i,\beta}, v_{j,\beta}, v_{k,\beta}\}$, $\{v_{k,\beta}, v_{i,\alpha}, v_{j,\beta}\}$ or $\{v_{i,\alpha}, v_{k,\beta}, v_{j,\alpha}\}$. ☐

Lemma 4.4. *If H is connected and $|H| \geq 2$, then* $\text{aw}(P_2\Box H, 3) \leq 4$.

Proof. Let c be an exact 4-coloring of $P_2\Box H$ with H_1 and H_2 labeled copies of H. If $|c(V(H_1))| \geq 3$, then, by Lemma 4.2, there exists an isometric C_3 or a shortest isometric path P with at least three colors in H_1. If $C_3 \subseteq H_1$ has three colors, then there is an immediate rainbow 3-AP in $P_2\Box H$. In the other case, $P_2\Box P$ is an isometric subgraph of $P_2\Box H$. By Lemma 2.6 and Proposition 3.2 there exists a rainbow 3-AP in $P_2\Box H$ In the case where $|c(V(H_1))| = 1$, then $|c(V(H_2))| \geq 3$, which is the previous situation. Finally, consider the case where $|c(V(H_1))| = 2$. Since c is an exact 4-coloring of $P_2\Box H$, $|c(V(H_1))\setminus c(V(H_2))| = 2$ so by Lemma 4.1 there is a rainbow 3-AP. ☐

The results established thus far come together to show an extremely useful bound on the Cartesian products of graphs in Theorem 4.5. This bound demonstrates that the anti-van der Waerden number of any Cartesian product is either 3 or 4.

Theorem 4.5. *If G and H are connected graphs and $|G|, |H| \geq 2$, then* $\text{aw}(G\square H, 3) \leq 4$.

Proof. If $|H| = |G| = 2$ then $G\square H = P_2\square P_2$ and by Observation 3.1, $\text{aw}(G\square H, 3) = 3 \leq 4$. Let c be an exact 4-coloring of $G\square H$ with $V(H) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, assume $|H| \geq 3$. If $|G| = 2$, then by Lemma 4.4 there is a rainbow 3-AP. Now, suppose $|H|, |G| \geq 3$ and define $G_1, G_2, \ldots, G_n$ as the labeled copies of G in $G\square H$. Let $\mathcal{P}$ be an isometric path that contains the most colors in some G_i, further, let it be the shortest such path.

Case 1: $\mathcal{P}$ has 3 or 4 colors.
Let $\mathcal{P}$ have x vertices and $v_iv_j \in E(H)$. Also, let $\mathcal{P}'$ be the path in G_j that corresponds to $\mathcal{P}$, note this creates an isometric subgraph $P_2\square P_x$ in $G\square H$. If x is even, then there is a rainbow 3-AP since $\text{aw}(P_2\square P_{2k}, 3) = 3$ for all $k \geq 1$ by Proposition 3.2 . If x is odd, then a rainbow 3-AP is guaranteed by Lemma 2.6 since path $\mathcal{P}$ has 3 or 4 colors.

Case 2: $\mathcal{P}$ is monochromatic.
This implies that each G_i is monochromatic by the definition of $\mathcal{P}$. Since $G\square H$ has 4 colors, there exists either an isometric C_3 or an isometric shortest path $\mathcal{P}'$ in a copy of H that has at least 3-colors by Lemma 4.2. If there is an isometric C_3, then there is an immediate rainbow 3-AP. In the other case, this is just Case 1 with the roles of G and H reversed.

Case 3: $\mathcal{P}$ has two colors.
This means that some copy of G has exactly two colors, call this copy G_d and assume the two colors are *red* and *blue*. By Lemma 4.1, when the remaining two new colors appear they must both appear either with colors *red* or *blue*. Let *yellow* and *green* be the two additional colors that are introduced and, without loss of generality, suppose they both appear with *red*. In particular, let $c(V(G_e)) = \{red, green\}$ and $c(V(G_f)) = \{red, yellow\}$. Now, create an auxiliary coloring c' of H defined by

$$c'(v_\ell) = \begin{cases} red & \text{if } c(V(G_\ell)) = \{red\} \\ \mathcal{C} & \text{if } c(V(G_\ell)) = \{\mathcal{C}, red\} \end{cases} .$$

Subcase 1: There is no path in H, under coloring c', that contains the colors *blue*, *green* and *yellow*.

Find the smallest subgraph of H that contains *blue*, *green* and *yellow*, say $c'(v_i) =$ *blue*, $c'(v_j) =$ *green* and $c'(v_k) =$ *yellow* and call this smallest subgraph K. This guarantees that v_i, v_j and v_k are leaves in the subgraph K. Without loss of generality, assume $\text{d}(v_i, v_j) \leq \text{d}(v_j, v_k)$. Let $v_{i,\alpha} \in G_i$ such that $c(v_{i,\alpha}) =$ *blue*, $v_{j,\beta} \in G_j$ such that $c(v_{j,\beta}) =$ *green* and $v_{i,\beta}$ be the vertex in G_i that corresponds to $v_{j,\beta}$. Let $v_{k,\alpha}$ be the vertex in G_k that corresponds to $v_{i,\alpha}$ and find a shortest path P from $v_{j,\beta}$ to $v_{k,\alpha}$ whose only vertex in G_j is $v_{j,\beta}$. Now, consider the 3-AP, $\{v_{i,\alpha}, v_{j,\beta}, u\}$, such that u is a vertex on P since $\text{d}(v_i, v_j) \leq \text{d}(v_j, v_k)$. If $c(u) =$ *blue* or $c(u) =$ *green* this contradicts the minimality of K or the assumption of the subcase. Therefore, $c(u) \in \{red, yellow\}$ and this 3-AP is rainbow.

Subcase 2: There is a path in H, under coloring c', that contains *blue*, *green* and *yellow*.

Let $\mathbb{P}$ be the shortest path in H that contains *blue*, *green* and *yellow* and, without loss of generality, assume the path has leaves v_i and v_k with $c'(v_i) = blue$ and $c'(v_k) = yellow$. Further, assume v_j is the closest *green* vertex to v_i on $\mathbb{P}$ and $\mathrm{d}(v_i, v_j) \leq \mathrm{d}(v_j, v_k)$. Note, there are no other *blue* or *yellow* vertices on $\mathbb{P}$, otherwise $\mathbb{P}$ would not be the shortest path that contains *blue*, *green* and *yellow*.

Let $v_{i,\alpha}$ and $v_{j,\beta}$ be in G_i and G_j, respectively, so that they are the closest two vertices with $c(v_{i,\alpha}) = blue$ and $c(v_{j,\beta}) = green$ (see Figure 3 for the following construction). Let P be a shortest path from $v_{i,\alpha}$ to $v_{i,\beta}$ in G_i and P' be a shortest path from $v_{i,\beta}$ to $v_{j,\beta}$. Notice that, by minimality of distance from v_i to v_j, $P \Box P'$ is an isometric subgraph of $G \Box H$. Note that the length of P' is 1 then there is a rainbow 3-AP by Lemma 4.3. Assume the length of P' is at least 2. If $\mathrm{d}(v_{i,\alpha}, v_{j,\beta})$ is even, then there is a *red* vertex in $P \Box P'$, say u, such that $\mathrm{d}(v_{i,\alpha}, u) = \mathrm{d}(u, v_{j,\beta})$ which creates a rainbow 3-AP.

Now, consider the case where $\mathrm{d}(v_{i,\alpha}, v_{j,\beta}) = 2x + 1$. Let $v_{k,\gamma}$ be a vertex in G_k such that $\mathrm{d}(v_{j,\beta}, v_{k,\gamma})$ is minimal and $c(v_{k,\gamma}) = yellow$. Let ρ be a shortest path from $v_{j,\beta}$ to $v_{j,\gamma}$ in G_j and ρ' be a shortest path from $v_{j,\gamma}$ to $v_{k,\gamma}$, then $\rho \Box \rho'$ is an isometric subgraph of $G \Box H$. Note that $c(V(G_{k-1})) = \{red\}$ and $c(V(H_{\gamma-1})) = \{red\}$ by Lemma 4.3. Define $D_a = \{v \in V(\rho \Box \rho') \,|\, \mathrm{d}(v, v_{j,\beta}) = a\}$ and note that this means $D_0 = \{v_{j,\beta}\}$. Define y so that $D_y = \{v_{k,\gamma}\}$. Further, define the distance from D_s to D_t to be $|s-t|$. If $y < 2x+1$, let u be the vertex on P' or P such that $\mathrm{d}(u, v_{j,\beta}) = y$. Then, $c(u) \in \{red, blue\}$ and $\{v_{k,\gamma}, v_{j,\beta}, u\}$ is a rainbow 3-AP. This means $D_{2x+1} \neq \emptyset$, further, $c(D_{2x+1}) = \{green\}$ because if $v \in D_{2x+1}$, then $\{v_{i,\alpha}, v_{j,\beta}, v\}$ is a 3-AP. This implies that the distance from D_y to either D_0 or D_{2x+1} is even. If $y - 0$ is even, then either

$$\{v_{k,\gamma}, v_{k-1,\gamma-(y/2-1)}, v_{j,\beta}\} \text{ or } \{v_{k,\gamma}, v_{k-(y/2-1),\gamma-1}, v_{j,\beta}\}$$

is a rainbow 3-AP since $c(v_{k-1,\gamma-(y/2-1)}) = c(v_{k-(y/2-1),\gamma-1}) = red$. Similarly, if $y - 2x - 1$ is even and $z = \frac{y-2x-1}{2}$, then either

$$\{v_{k,\gamma}, v_{k-1,\gamma-(z-1)}, v_{k-1-z,\gamma-(z-1)}\} \text{ or } \{v_{k,\gamma}, v_{k-(z-1),\gamma-1}, v_{k-(z-1),\gamma-1-z}\}$$

is a rainbow 3-AP.

Therefore, each case yields a rainbow 3-AP so $\mathrm{aw}(G \Box H, 3) \leq 4$. □

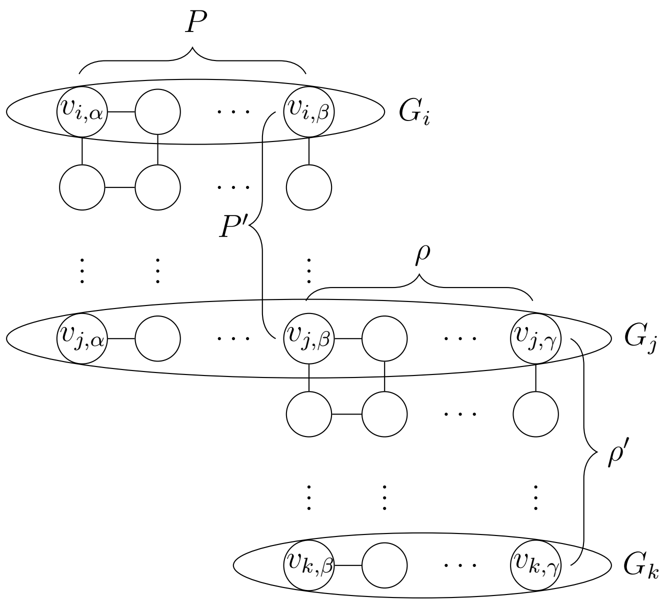

Figure 3: Construction of isometric subgraphs of $G\square H$.

5 Application to $P_m\square P_n$

In Sections 2 and 3 results for $m = 2$ and $m = 3$ were established. The result of Theorem 4.5 is used, with earlier results, to determine $\text{aw}(P_m\square P_n, 3)$ for all m and n. It is interesting to note that the pattern for the small values of m does not continue when considering large values of m. Essentially, there are 'more' 3-APs which forces the anti-van der Waerden number to always be 4. First notice that Lemma 3.3 and Theorem 4.5 give the result of Corollary 5.1 immediately.

Corollary 5.1. *If* $G = P_m\square P_n$ *and* $m + n = 2k + 1$ *for some* $k \geq 1$*, then* $\text{aw}(G, 3) = 4$.

Lemma 5.2 gives the final lower bound to determine the anti-van der Waerden number for all $P_m\square P_n$.

Lemma 5.2. *If* $m \geq 4$*,* $n \geq 4$ *and* $m + n = 2k$ *for some* $k \geq 1$*, then* $4 \leq \text{aw}(P_m\square P_n, 3)$.

Proof. Let $G = P_m\square P_n$. Define

$$c(v_{i,j}) = \begin{cases} \textit{red} & \text{if } i = 1 \text{ and } j = 2 \text{ or } i = 2 \text{ and } j = 1 \\ \textit{blue} & \text{if } i = m \text{ and } j = n \\ \textit{green} & \text{otherwise} \end{cases}.$$

Note that if a rainbow 3-AP exists it must contain vertex $v_{m,n}$ and either $v_{1,2}$ or $v_{2,1}$. Let $S = \{v_{m,n}, v_{1,2}, v_{2,1}\}$. Note that $\text{d}(v_{1,2}, v_{m,n}) = \text{d}(v_{2,1}, v_{m,n}) = m + n - 3$ which, by assumption, is odd. Therefore, there does not exist a vertex equidistant from $v_{2,1}$ and $v_{m,n}$ or equidistant from $v_{1,2}$ and $v_{m,n}$. This means a rainbow 3-AP cannot exist in the order of $\{v_{2,1}, v_{i,j}, v_{m,n}\}$ or $\{v_{1,2}, v_{i,j}, v_{m,n}\}$.

This means any rainbow 3-AP must exist in the order of $\{v_{m,n}, v_{2,1}, v_{i,j}\}$ or $\{v_{m,n}, v_{1,2}, v_{i,j}\}$ (or the reverse order) where $v_{i,j} \notin S$. Note that $v_{i,j}$ must be distance $m+n-3$ from one of the vertices in S, but the only vertices distance $m+n-3$ from any vertex in S are already in S thus $v_{i,j}$ does not exist. Therefore, c avoids rainbow 3-APs so $4 \leq \mathrm{aw}(G, 3)$. □

Using Theorem 4.5, Corollary 5.1 and Lemma 5.2 gives Corollary 5.3.

Corollary 5.3. *If $m \geq 4$ and $n \geq 4$, then* $\mathrm{aw}(P_m \square P_n, 3) = 4$.

Finally, combining Propositions 3.2, 3.4, 3.5, 3.7 and Corollary 5.3 gives a function to determine $\mathrm{aw}(P_m \square P_n, 3)$ for all m and n.

Theorem 5.4. *For $m, n \geq 2$,*

$$\mathrm{aw}(P_m \square P_n, 3) = \begin{cases} 3 & m = 2 \text{ and } n \text{ is even, or } m = 3 \text{ and } n \text{ is odd;} \\ 4 & \text{otherwise.} \end{cases}$$

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References

[1] M. Axenovich and D. Fon-Der-Flaass, On rainbow arithmetic progressions, *Electr. J. Combin.* **11** (1) (2004), #R1, 7 pp.

[2] M. Axenovich and R. R. Martin, Sub-Ramsey numbers for arithmetic progressions, *Graphs and Combin.* **22** (3) (2006), 297–309.

[3] Z. Berikkyzy, A. Schulte and M. Young, Anti-van der Waerden numbers of 3-term arithmetic progressions, *Electr. J. Combin.* **24** (2) (2017), #R2.39, 9 pp.

[4] S. Butler, C. Erickson, L. Hogben, K. Hogenson, L. Kramer, R.L. Kramer, J. Lin, R. R. Martin, D. Stolee, N. Warnberg and M. Young, Rainbow Arithmetic Progressions, *J. Combinatorics* **7** (4) (2016), 595–626.

[5] V. Jungić, J. Licht (Fox), M. Mahdian, J. Nešetril and R. Radoičić, Rainbow arithmetic progressions and anti-Ramsey results, *Combin. Probab. Comput.* **12** (5-6) (2003), 599–620.

[6] A. Schulte, N. Warnberg and M. Young, Anti-van der Waerden numbers on graphs, https://arxiv.org/abs/1802.01509 (preprint).

[7] K. Uherka, *An introduction to Ramsey theory and anti-Ramsey theory on the integers*, Master's Creative Component (2013), Iowa State University.

[8] M. Young, Rainbow Arithmetic Progressions in Finite Abelian Groups, https://arxiv.org/pdf/1603.08153.pdf (preprint).

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