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Order ideals in a product of chains

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Abstract

We give a combinatorial proof of the rank-unimodality of the poset of order ideals of a product of chains of lengths 2, n, and m, and find a symmetric chain decomposition in the case where $n = 2$. A general method is given for showing that the corresponding poset for more and/or larger chains is also rank-unimodal, with its applicability limited only by available computational power.

1 Introduction

An *order ideal* in a partially ordered set is a subset I such that if $x \in I$ and $y \leq x$, then $y \in I$. Let $L(n_1, n_2, \ldots, n_k)$ denote the poset of order ideals of the poset $\mathbf{n_1} \times \mathbf{n_2} \times \cdots \times \mathbf{n_k}$, where $\mathbf{n}$ is an n-element chain. When $k = 2$, this is isomorphic to the poset of integer partitions with at most n_1 parts and largest part at most n_2, ordered by inclusion of Young diagrams. When $k = 3$, it is the poset of plane partitions that fit inside an $n_1 \times n_2 \times n_3$ box, again ordered by inclusion. A plane partition can be visualized as a stack of unit cubes in a corner of a room with the origin at that corner, where the cubes must be pressed into the corner in the sense that no cube can decrease its position along any one axis. Plane partitions that fit inside an $n_1 \times n_2 \times n_3$ box can be represented as integers between 0 and n_3 (inclusive) arranged in an $n_1 \times n_2$ grid with weakly increasing rows and columns, where the numbers represent the number of boxes stacked at each square on the floor.

Following the definitions in [10], a poset is *graded of rank* n if every maximal chain has the same length n. There is then a *rank function* that is 0 on minimal elements and increases by 1 at each step along a maximal chain to rank n on maximal elements, thereby assigning a nonnegative integer (rank) to each element of the poset. A poset is *rank-symmetric* if the number of elements of rank k equals the number of elements of rank $n-k$ for each k. It is *rank-unimodal* if the sequence of the number of elements

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of each rank from 0 to n is unimodal, i.e., it weakly increases up to some point and then weakly decreases after that point.

The poset $L(n_1, n_2, \ldots, n_k)$ is rank-symmetric, and is known to be rank-unimodal for $k \leq 3$ [4, 5, 8], with a combinatorial proof previously known only for $k \leq 2$ [3, 13]. The cases of $L(3, n)$ and $L(4, n)$ are also known to have a symmetric chain decomposition [2, 6, 11, 12]; expressing the poset as a disjoint union of saturated chains symmetric about the middle level implies rank-unimodality and other properties not named here. Our main result on rank-unimodality is a combinatorial proof of the following theorem.

Theorem 1. *The poset $L(2, n, m)$ is rank-unimodal.*

If a poset is graded of rank n and has p_i elements of rank i, then the *rank-generating function* of P is the polynomial $F(P, q) = \sum_{i=0}^{n} p_i q^i$ in the variable q. It is known [8] that $F(L(m, n), q)$ is the q-binomial coefficient (or Gaussian polynomial)

$$\left[\begin{array}{c} m+n \\ n \end{array} \right]_q = \frac{(1-q^{m+n})(1-q^{m+n-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}.$$

This polynomial is symmetric and unimodal in the sense that its coefficients form a symmetric and unimodal sequence, centered around degree $\frac{mn}{2}$ [3, 5]. We will also use a special case of Proposition 8.2 in [7], that the rank-generating function for $L(n_1, n_2, \ldots, n_k)$ is a sum of q-binomial coefficients multiplied by certain polynomials in q. To that end, let $P = \mathbf{n_1} \times \mathbf{n_2} \times \cdots \times \mathbf{n_{k-1}}$, $m = n_k$, and $p = n_1 n_2 \cdots n_{k-1}$ (the size of P). A *natural labeling* of P is a labeling of the elements of P from 1 to p such that larger elements of the poset have larger labels. We can fix a natural labeling ω for a product of two chains by labeling the elements from bottom to top and left to right, as shown in Figure 1 for $\mathbf{2} \times \mathbf{n}$. A *linear extension* is another term for a natural labeling, but we will use it to refer specifically to the list of labels as a way of permuting the fixed labeling (this subtlety will be illustrated by example later). The *major index*, $\mathrm{maj}(\pi)$ of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the sum of all i such that $\pi_i > \pi_{i+1}$, called *descents*. We can now define the W-polynomial $W_s(P, q) = \sum_{\pi} q^{\mathrm{maj}(\pi)}$, where the sum is over all linear extensions π of P with s descents.

The relevant formula from Stanley [7] in this case is

$$F(P, q) = \sum_{s=0}^{p-1} \left[\begin{array}{c} p+m-s \\ p \end{array} \right]_q W_s(P, q). \tag{1}$$

It is known that $\left[\begin{array}{c} p+m-s \\ p \end{array} \right]_q$ is symmetric and unimodal, centered around degree $\frac{p(m-s)}{2}$. Thus if we can show that $W_s(P, q)$ is also symmetric and unimodal, centered

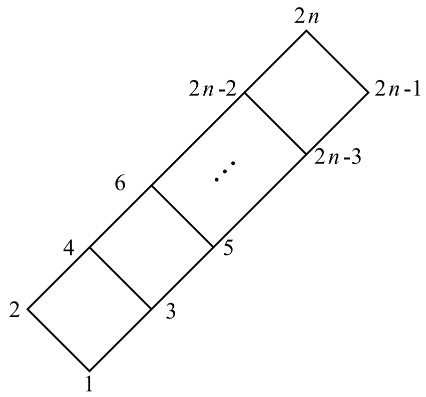

Figure 1: A natural labeling of $\mathbf{2} \times \mathbf{n}$.

around degree $\frac{ps}{2}$, then the product is symmetric and unimodal, centered around degree $\frac{pm}{2}$ [9], and hence the sum of these products, namely $F(L(n_1, n_2, \ldots, n_k), q)$, is symmetric and unimodal, centered around degree $\frac{pm}{2}$. We will now show that this works for $P = \mathbf{2} \times \mathbf{n}$, where $p = 2n$, and thereby prove Theorem 1.

2 Proof of Theorem 1

To show that $W_s(P, q)$ is symmetric and centered around degree $\frac{ps}{2}$, let π be any linear extension of $P = \mathbf{2} \times \mathbf{n}$ with s descents. Replace each number x in π by $2n + 1 - x$ and reverse the order to get a new extension π' (of the dual of P) with the same number of descents, but with $\mathrm{maj}(\pi') = 2ns - \mathrm{maj}(\pi)$. Since P is self-dual, π' is also a linear extension of P, and the central degree is $\frac{2ns}{2} = \frac{ps}{2}$. Now we prove unimodality by induction on n.

If $n = 1$, then $L(2, 1, m)$ is isomorphic to $L(2, m)$, which we already know is a q-binomial coefficient. For $n > 1$, a linear extension of $2 \times n$ can have anywhere from 0 to $n - 1$ descents, recalling that we have fixed the natural labeling ω.

Our strategy is to put a poset structure on the set of linear extensions of P with s descents. The resulting poset $W^s(P)$ has rank-generating function $W_s(P, q)$ (after factoring out the lowest power of q) for each s, then rank-unimodality is established by showing that $W^s(P)$ is isomorphic to $L(2, s, n-s-1)$. The steps will be illustrated with the five linear extensions of $\mathbf{2} \times \mathbf{3}$. Rather than drawing the diagram each time, we arrange the labels in a grid (a standard Young tableau of rectangular shape), with ω denoting our fixed labeling.

$$\omega = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\,, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\,, \text{ and } \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\,.$$

These give the permutations 123456, 123546, 132456, 132546, and 135246, respectively. Note the last one is the only one that illustrates the subtlety that we use the labels of each linear extension to determine the order in which to write the labels of

ω, which is not the same as reading the labels in the order from ω (in that case the last permutation would be 142536 and the construction would fail to be a bijection at the next step). In general, because of our choice of ω, the odd numbers and the even numbers will have to appear in increasing order, the permutation must start with 1 and end with $2n$, and each odd number has to appear before any larger even number.

Given a linear extension π with s descents, replace each element of the ith increasing run with the label $i-1$ for $1 \leq i \leq s+1$. The five permutations in the example thus become 000000, 000011, 001111, 001122, and 000111, respectively.

Next convert the labels in ω to the smaller numbers that replaced them in each permutation:

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1&1&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1&2&2\\\hline\end{array},\ \text{and}\ \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 1&1&1\\\hline\end{array}.$$

Since $P = \mathbf{2} \times \mathbf{n}$, this labeling is a $2 \times n$ array of numbers 0 to s, weakly increasing in rows and columns. By definition, this is an element of $L(2, n, s)$.

Now replace each row with a row that counts the number of entries in the original row that are at least each of s, $s-1$, …, 1. This step changes rows of length n to rows of length s, yielding an element of $L(2, s, n)$. The example becomes

$$\emptyset,\ \begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array} \in L(2,1,3),\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \in L(2,1,3),\ \begin{array}{|c|c|}\hline 0&1\\\hline 2&3\\\hline\end{array} \in L(2,2,3),\ \text{and}\ \begin{array}{|c|}\hline 0\\\hline 3\\\hline\end{array} \in L(2,1,3).$$

Now when $s > 0$ we have a minimal element, namely $\begin{array}{|c|c|c|c|}\hline 0&1&\cdots&s-1\\\hline 2&3&\cdots&s+1\\\hline\end{array}$, which can be subtracted from each of the others to yield an element of $L(2, s, n-s-1)$. The example becomes

$$\emptyset,\ \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array} \in L(2,1,1),\ \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array} \in L(2,1,1),\ \begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array} \in L(2,2,0),\ \text{and}\ \begin{array}{|c|}\hline 0\\\hline 1\\\hline\end{array} \in L(2,1,1).$$

Note that in the example we got all elements of $L(2, 1, 1)$ and $L(2, 2, 0)$, and ω has no descents (technically giving an element of $L(2, 0, 2)$).

Each step in the process described is an order-preserving bijection. The map from the set of linear extensions to the subset of $L(2, n, s)$ is a bijection, and we use the ordering on $L(2, n, s)$ to define the ordering on $W^s(P)$. The elements of $L(2, n, s)$ that we get are those where for each i, $1 \leq i \leq s$, the first i in the bottom row occurs to the left of the last $i-1$ in the top row.

The next step, which replaces the rows of n entries that are most s with rows of s entries that are at most n, is the standard bijection from $L(2, n, s)$ to $L(2, s, n)$ obtained by switching axes, or looking at the stack of boxes from a different direction. After that we subtract off the minimal element, which is possible since in $L(2, n, s)$ we had to have at least the element $\begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&\cdots&0&0&1&\cdots&s-2&s-1\\\hline 0&\cdots&1&2&3&\cdots&s&s\\\hline\end{array}$, which maps to the minimal element in $L(2,s,n)$.

Thus we have an injection into $L(2,s,n-s-1)$. To see that it is surjective, consider the inverse map. Given any element of $L(2,s,n-s-1)$, we add the minimal element to it, work our way back to $L(2,n,s)$, and observe that for each i, $1 \le i \le s$, the first i in the bottom row occurs to the left of the last $i-1$ in the top row, as desired. Thus the map described above shows that $W^s(P) \cong L(2,s,n-s-1)$.

Since $s < n$, we know $L(2,s,n-s-1)$ is rank-unimodal by induction, thus completing the proof of Theorem 1. □

3 Related questions

When $n = 2$ we have $F(L(2,2,m),q) = F(L(4,m),q) + q^2 F(L(4,m-1),q)$, which suggests a way to prove the following theorem.

Theorem 2. *$L(2,2,m)$ has a symmetric chain decomposition.*

Proof. We find a bijection from $L(2,2,m)$ to the disjoint union of $L(4,m)$ and $L(4,m-1)$, whose inverse is order-preserving, where the rank gets shifted down by 2 in the second case. An explicit SCD is known for these posets [12].

An element of $L(2,2,m)$ is a diagram of the form $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$. This element can be thought of as an ordered 4-tuple (a,b,c,d) of numbers from 0 to m, where $a \le b$, $a \le c$, $b \le d$, and $c \le d$. If $b \le c$, then this is an element of $L(4,m)$. If $b > c$, then we take $(a,c,b-1,d-1) \in L(4,m-1)$, which is the desired bijection.

Because $L(4,m-1)$ is graded of rank $4(m-1)$ while $L(2,2,m)$ goes up to rank $4m$, the elements that map to $L(4,m-1)$ traverse ranks 2 through $4m-2$ in $L(2,2,m)$. In all of these posets, one element covers another if they differ by exactly 1 in one position and are equal everywhere else. The bijection preserves cover relations since those differences are unchanged in each part, thus the saturated chains in the smaller poset SCDs correspond to saturated chains to form a SCD in $L(2,2,m)$. □

The proof Theorem 2 comes from a technique sometimes referred to as "proof by wishful thinking." It doesn't always work, but sometimes an algebraic relation can give a clue as to where to look for a combinatorial structure.

Formula 1 makes is possible to prove rank-unimodality of infinitely many cases with a finite calculation. Once again let $P = \mathbf{n_1} \times \mathbf{n_2} \times \cdots \times \mathbf{n_{k-1}}$, $m = n_k$, and $p = n_1 n_2 \cdots n_{k-1}$. As used in the proof of Theorem 1, if $W_s(P,q)$ is symmetric and unimodal, centered around degree $\frac{ps}{2}$ for each s, then $F(L(n_1,n_2,\ldots,n_k),q)$ is unimodal for fixed n_1, n_2, $\ldots$, n_{k-1} and arbitrary n_k.

Proposition 1. *The poset $L(2,2,2,m)$ is rank-unimodal for all m.*

Proof. Let $P = \mathbf{2} \times \mathbf{2} \times \mathbf{2}$, labeled as shown in Figure 2.

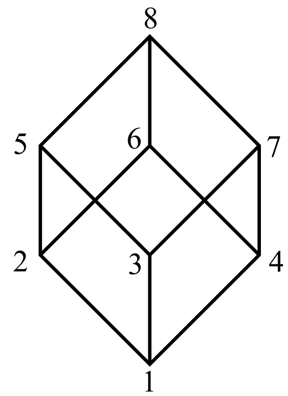

Figure 2: A natural labeling of $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$.

By computing all 48 linear extensions, we get the following polynomials.

$$
\begin{aligned}
W_0(P, q) &= 1 \\
W_1(P, q) &= 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 \\
W_2(P, q) &= q^5 + 3q^6 + 4q^7 + 8q^8 + 4q^9 + 3q^{10} + q^{11} \\
W_3(P, q) &= 2q^{10} + 2q^{11} + 3q^{12} + 2q^{13} + 2q^{14} \\
W_4(P, q) &= q^{16}
\end{aligned}
$$

Each of these is symmetric and unimodal, centered around degree $4s$, as desired. □

Some other products of more than two chains can be handled similarly with Maple in a reasonable amount of time, yielding the following new results.

Proposition 2. *The posets $L(2, 2, k, m)$ for $2 \leq k \leq 13$, $L(2, 2, 2, j, m)$ for $2 \leq j \leq 8$, $L(2, 2, 2, 2, i, m)$ for $2 \leq i \leq 4$, and $L(3, 3, h, m)$ for $3 \leq h \leq 4$ are rank-unimodal for all m.*

A natural question is whether the polynomials $W_s(P, q)$ will behave so nicely for all products of chains or for all products of two chains. There are graded lattices for which the W-polynomials are neither unimodal nor symmetric, but products of chains have much more structure. The proof of Theorem 1 does not generalize in the most obvious way, since we cannot always find a minimal element to subtract off.

Conjecture 1. *The polynomials $W_s(P, q)$ will always be unimodal where P is a product of chains, hence $L(n_1, n_2, \ldots, n_k)$ will always be rank-unimodal.*

What we can prove is the following, where $L(P)$ denotes the poset of order ideals of a poset P, ordered by inclusion.

Proposition 3. *For any poset P, $F(L(P),q) = W_1(P,q) + (1 + q + q^2 + q^3 + \cdots + q^{\deg(F(L(P),q))})$.*

Proof. Consider an order ideal in a poset P with a natural labeling. The set of vertices $\{1, 2, \ldots, k\}$ will be one such order ideal for each k. These order ideals correspond to the $1 + q + q^2 + q^3 + \cdots + q^{\deg(F(L(P),q))}$ part of the formula. For any other order ideal with k elements, we can obtain a linear extension by labeling the elements of the order ideal 1 through k in the order of their original labels, then label the rest in order. The corresponding permutation will have k numbers in increasing order, a descent at position k, and the rest of the numbers in increasing order. Any such linear extension similarly corresponds to an order ideal by taking the elements labeled 1 through k. In other words, these are exactly the permutations whose generating function is $W_1(P,q)$. □

Previously we proved that the poset of order ideals of a product of chains was rank-unimodal by showing that every W-polynomial for the product of all but the last chain was rank-unimodal. The consequence of Proposition 3 is that we can show rank-unimodality in a single case by computing only W_1 but without omitting the last chain in the product.

4 Still Open

While the posets $W^s(P)$ are useful, they have some drawbacks, such as being dependent on the natural labeling of P. There is a simple bijection between $L(n_1, n_2, \ldots, n_k, m)$ and the disjoint union of $W^s(P) \times L(p, m-s)$ (which proves the special case of (1) that we used), but in general it is not order-preserving in either direction, and the posets W^s do not always possess a symmetric chain decomposition.

Let $h_k(P)$ denote the number of linear extensions of P with k descents. For graded posets it is known that the sequence $h_0, h_1, h_2, \ldots$ is symmetric (when truncated after the last nonzero term), with a bijective proof given by J. Farley [1]. In the special case where $P = \mathbf{2} \times \mathbf{n}$, we get this bijection for free from our proof of Theorem 1, since $h_k(P) = |W^k(P)|$ and $W^s(P) \cong L(2, s, n-1-s) \cong L(2, n-1-s, s) \cong W^{n-1-s}(P)$. If we use self-duality along the way, then we get another bijection, so at least one of these will be different from Farley's. For products of more or larger chains, however, the analogous posets W^s are not isomorphic (though the polynomials W_s will be the same up to a power of q factor, so ideally a bijection would prove this as well).

The results of Proposition 2 should be feasible to extend every few years with more powerful computers, but a more general proof for products of more than three chains has thus far remained elusive.

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References

[1] J. Farley, Linear extensions of ranked posets, enumerated by descents. A problem of Stanley from the 1981 Banff conf. on ordered sets, *Adv. in Appl. Math.* 34 (2) (2005), 295–312.

[2] B. Lindström, A partition of $L(3, n)$ into saturated chains, *European J. Combin.* **1** (1980), 60–63.

[3] K. O'Hara, Unimodality of Gaussian coefficients: a constructive proof, *J. Combin. Theory Ser. A* **53** (1990), 29–52.

[4] R. Proctor, Representations of $sl(2, C)$ on posets and the Sperner property, *SIAM J. Algebraic Discrete Methods* **3** (1982), 275–280.

[5] R. Proctor, Solution of two difficult combinatorial problems with linear algebra, *Amer. Math. Monthly* **89** (1982), 721–734.

[6] W. Riess, Zwei Optimierungsprobleme auf Ordnungen, *Arbeitsberichte des Instituts für mathematische Maschinen und Datenverarbeitungen* **11** (1978).

[7] R. Stanley, Ordered structures and partitions, *Mem. Amer. Math. Soc.* **119** (1972).

[8] R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic Discrete Methods* **1** (1980), 168–184.

[9] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Annals of the New York Academy of Sciences* **576** (1989), 500–535.

[10] R. Stanley, *Enumerative Combinatorics, Vol. I*, 2nd ed., Cambridge University Press, 2012.

[11] X. Wen, Computer-generated symmetric chain decompositions for $L(4, n)$ and $L(3, n)$, *Adv. in Appl. Math.* **33** (2004), 409–412.

[12] D. B. West, A symmetric chain decomposition of $L(4, n)$, *European J. Combin.* **1** (1980), 379–383.

[13] D. Zeilberger, Kathy O'Hara's constructive proof of the unimodality of Gaussian polynomials, *Amer. Math. Monthly* **96** (1989), 590–602.

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