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On P-unique hypergraphs

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Abstract

We study hypergraphs which are uniquely determined by their chromatic, independence and matching polynomials. B. Bollobás, L. Pebody and O. Riordan [*J. Combin. Theory Ser. B* 80(2) (2000), 320–345] conjectured (BPR-conjecture) that almost all graphs are uniquely determined by their chromatic polynomials. We show that for r-uniform hypergraphs with $r \geq 3$ this is almost never the case. This disproves the analogue of the BPR-conjecture for 3-uniform hypergraphs. For $r = 2$ this also holds for the independence polynomial, as shown by J.A. Makowsky and V. Rakita (presented at 5ICC 2017 and ICGT 2018), whereas for the chromatic and matching polynomial this remains open.

1 Introduction and outline

1.1 Background

A hypergraph H consists of a set $V(H)$ together with a family of subsets $E(H)$ of $V(H)$ called hyperedges. Two vertices $u, v \in V(H)$ are adjacent if there is a hyperedge $e \in E(H)$ such that both $u \in e$ and $v \in e$. H is r-uniform if every hyperedge in $E(H)$ has exactly r elements. Two hypergraphs H_1, H_2 are isomorphic, denoted by $H_1 \simeq H_2$, if there is a bijective map $h : V(H_1) \to V(H_2)$ such that for any two vertices $u, v \in V(H_1)$ we have u and v are adjacent iff $h(u)$ and $h(v)$ are adjacent in H_2.

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We denote by $\mathfrak{H}$ ($\mathfrak{H}^r$) the class of all (r-uniform) hypergraphs, and by $\mathfrak{H}_n$ ($\mathfrak{H}_n^r$) the set of all (r-uniform) hypergraphs H with $V(H) = [n]$. Here $[n] = \{1, 2, \ldots, n\}$. We note that graphs are 2-uniform hypergraphs.

A (univariate) hypergraph polynomial $P(H; X)$ is a function $P : \mathfrak{H} \to \mathbb{Z}[X]$ which is invariant under hypergraph isomorphisms. Let $P(H; X)$ be a univariate hypergraph polynomial. A hypergraph H is P-unique if for every hypergraph H_1 with $P(H_1; X) = P(H; X)$ we have that H_1 is isomorphic to H. Similarly, a r-uniform hypergraph H is r-P-unique if for every r-uniform hypergraph H_1 with $P(H_1; X) = P(H; X)$ we have that H_1 is isomorphic to H.

A hypergraph polynomial $P(H; X)$ is *complete (for r-uniform hypergraphs)* , if all (r-uniform) hypergraphs are P-unique. Let $\mathcal{H}(n)$ be the number of non-isomorphic hypergraphs on n vertices, and let $\mathcal{H}^r(n)$ be the number of non-isomorphic r-uniform hypergraphs on n vertices. Furthermore, let $\mathcal{U}_P(n)$ ($\mathcal{U}_P^r(n)$) be the number of non-isomorphic P-unique (r-uniform) hypergraphs on n vertices.

P is *almost complete* if

$$\lim_{n\to\infty} \frac{\mathcal{U}_P(n)}{\mathcal{H}(n)} = 1.$$

It is almost complete on r-uniform hypergraphs if

$$\lim_{n\to\infty} \frac{\mathcal{U}_P^r(n)}{\mathcal{H}^r(n)} = 1.$$

P is *weakly distinguishing* if

$$\lim_{n\to\infty} \frac{\mathcal{U}_P(n)}{\mathcal{H}(n)} = 0.$$

and analogously for r-uniform hypergraphs.

No known univariate hypergraph polynomial in the literature is complete, although one should be able to construct such polynomials using some clever encoding of the isomorphism types of finite hypergraphs.

1.2 Three hypergraph polynomials

Some hypergraph polynomials studied in the literature are the chromatic polynomial $\chi(H, X)$ [15, 16, 17, 4, 5], the independence polynomial $\mathrm{Ind}(H; X)$), [18], and the matching polynomial $\mathrm{M}(H; X)$, [9]. There where also attempts to extend spectral graph theory to hypergraphs, cf. [8, 14] and the references therein. The monograph [7] summarizes what is known about graphs unique for the characteristic and the Laplacian polynomial. In [7] the authors also suggest that the characteristic polynomial is almost complete on graphs.

The chromatic polynomial. The chromatic polynomial for hypergraphs defined below generalizes the chromatic polynomial for graphs, but also show distinctly different behaviour in the case of hypergraphs, cf. [22].

Let $k \in \mathbb{N}$ and $f : V(H) \to [k]$. We say that f is a *proper coloring of H with at most k colors* if every $e \in E(H)$ contains two vertices $u, v \in e$ with $f(u) \neq f(v)$. We denote by $\chi(H; k)$ the number of proper colorings of H with at most k colors.

A set $I \subseteq V(H)$ is *independent* if there is no edge $e \subseteq I$. For $i \in \mathbb{N}$ let $b_i(H)$ be number of partitions of $V(H)$ into i independent sets. For X we denote by $X_{(i)}$ the polynomial

$$X \cdot (X-1) \cdot \dots \cdot (X-i+1).$$

Proposition 1.1 ([4, 21]) *(i) $\chi(H; k)$ is a polynomial in k, hence can be extended to a polynomial $\chi(H; X) \in \mathbb{Z}[X]$.*

(ii) $\chi(H; X) = \sum_i^n b_i(H) \cdot X_{(i)}$.

Some classes of χ-unique hypergraphs were presented compactly in [21] which we summarize here. The definitions of hypercycles, hyperpaths and sunflower hypergraphs are standard in the hypergraph literature, cf. the books by Berge [2], Voloshin [19], and Bretto [6].

Proposition 1.2 ([15, 16, 17, 21]) *The following hold:*

(i) For $r \geq 3$, r-uniform hypercycle $\mathcal{C}_m^r$ is r-χ-unique but it is not χ-unique.

(ii) For every $p, r \geq 3$, $B_{p,p}^{2,2}$ is χ-unique, where $B_{p,p}^{2,2}$ denote the hypergraphs obtained by identifying extremities x and y of $\mathcal{P}_{p-1}^r$ with distinct vertices of degree 2 in the same edge of $\mathcal{C}_p^r$.

(iii) The sunflower hypergraph $SH(n, 1, r)$ is χ-unique.

(iv) Let $n = r + (k-1)p$, where $r \geq 3, k \geq 1$ and $1 \leq p \leq r-1$. Then $SH(n, p, r)$ is r-χ-unique for every $1 \leq p \leq r-2$; for $p = r-1$, $SH(n, r-1, r)$ is r-χ-unique for $k = 1$ or $k = 2$ but it does not have this property for $k \geq 3$.

In [3] it is conjectured that the chromatic polynomial is almost complete on graphs. Our first result shows that the conjecture is not true for hypergraphs.

Theorem 1.3 *The chromatic polynomial $\chi(H; X)$ is weakly distinguishing*

(i) on hypergraphs:

$$\lim_{n\to\infty} \frac{\mathcal{U}_\chi(n)}{\mathcal{H}(n)} = 0;$$

(ii) on r-uniform hypergraphs: for every $r \geq 3$,

$$\lim_{n\to\infty} \frac{\mathcal{U}_\chi^r(n)}{\mathcal{H}^r(n)} = 0.$$

The independence polynomial. The independence polynomial for hypergraphs is defined as

$$\mathrm{Ind}(H:X) = \sum_i \mathrm{ind}_i(H) \cdot X^i,$$

where $\mathrm{ind}_i(H)$ is the number of independent sets $I \subseteq V(H)$ with $|I| = i$.

The independence polynomial for hypergraphs was studied in [18].

Theorem 1.4 *The independence polynomial* $\mathrm{Ind}(H;X)$ *is weakly distinguishing*

(i) on hypergraphs:

$$\lim_{n\to\infty} \frac{\mathcal{U}_{Ind}(n)}{\mathcal{H}(n)} = 0;$$

(ii) on r-uniform hypergraphs: for every $r \geq 2$,

$$\lim_{n\to\infty} \frac{\mathcal{U}^r_{Ind}(n)}{\mathcal{H}^r(n)} = 0.$$

The case $r = 2$ was shown in [11, 12]. The proof for $r \geq 3$ is given in Section 4.

The matching polynomial. A *k-matching m of a hypergraph H* is a set $m \subseteq E(H)$ of k disjoint hyperedges. Let $\mu_k(H)$ be the number of k-matchings of H. The *matching polynomial* $\mathrm{M}(H;X)$ of a hypergraph H is defined by

$$\mathrm{M}(H;X) = \sum_{k=1}^{\lfloor \frac{n}{k_H} \rfloor} \mu_k(H) \cdot X^k,$$

where k_H is the minimum size of the edges in $E(H)$. The matching polynomial for hypergraphs was studied in [9]. Noy [13] studied M-unique graphs.

Theorem 1.5 *The matching polynomial* $\mathrm{M}(H;X)$ *is weakly distinguishing*

(i) on hypergraphs:

$$\lim_{n\to\infty} \frac{\mathcal{U}_{M}(n)}{\mathcal{H}(n)} = 0;$$

(ii) on r-uniform hypergraphs: for every $r \geq 3$,

$$\lim_{n\to\infty} \frac{\mathcal{U}^r_{M}(n)}{\mathcal{H}^r(n)} = 0.$$

The case $r = 2$ is still open. The proof of the above theorem is also given in Section 4.

To the best of our knowledge no explicit description of Ind-unique and M-unique hypergraphs is given in the literature.

2 General strategy of the proofs

The general strategy of our proofs has been motivated by the first author's work with Rakita [11].

Let $\overline{\mathcal{H}}(n)$ and $\overline{\mathcal{H}}^r(n)$ denote the number of *labeled* hypergraphs and r-uniform hypergraphs of order n.

Lemma 2.1 *For any positive integer n:*

(i) $\overline{\mathcal{H}}(n) = 2^{2^n}$.

(ii) $\overline{\mathcal{H}}(n) \leq \mathcal{H}(n) \cdot n!$.

(iii) $\overline{\mathcal{H}}^r(n) = 2^{\binom{n}{r}}$.

(iv) $\overline{\mathcal{H}}^r(n) \leq \mathcal{H}^r(n) \cdot n!$.

Proof: (i) and (iii): The proofs of (i) and (iii) are straightforward, because any subset (of size r) of the vertex set determines one possible edge and we can have any subset of those possible edges.
(ii) and (iv): Every (r-uniform) hypergraph of order n has at most $n!$ automorphisms. $\square$

Let $P(H;X)$ be a hypergraph polynomial. We denote by $\mathcal{B}_P(n)$ ($\mathcal{B}^r_P(n)$) the number of polynomials $p(X)$ such that there is a (r-uniform) hypergraph H of order n with $p(X) = P(H;X)$.

We shall use the following observations:

Lemma 2.2 *(i)* $\mathcal{B}^r_P(n) \leq \mathcal{B}_P(n)$.

(ii) $U_P(n) \leq \mathcal{B}_P(n) \leq \mathcal{H}(n)$.

(iii) $U^r_P(n) \leq \mathcal{B}^r_P(n) \leq \mathcal{H}^r(n)$.

Proof: (i): There are more candidate polynomials for $\mathcal{B}_P(n)$ than for $\mathcal{B}^r_P(n)$.
(ii) and (iii): There cannot be more P-unique hypergraphs than polynomials in $\mathcal{B}_P(n)$ ($\mathcal{B}^r_P(n)$). There cannot be more polynomials in $\mathcal{B}_P(n)$ ($\mathcal{B}^r_P(n)$) than there are hypergraphs in $\mathcal{H}(n)$ ($\mathcal{H}^r(n)$). $\square$

To prove our Theorems we will use Lemmas 2.1 and 2.2 and estimate the numbers $\mathcal{B}^r_P(n)$ or $\mathcal{B}_P(n)$.

We also use an observation from pre-calculus:

Proposition 2.3 *Let f, g be two non-decreasing real functions.*
If $\lim_{n\to\infty} \frac{\log(f(n))}{\log(g(n))} = 0$ *then* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

3 Proof of Theorem 1.3

By Proposition 1.1(ii) the chromatic polynomial of hypergraphs can be written as

$$\chi(H;X) = \sum_{i=1}^{n} b_i(H) \cdot X_{(i)}$$

where $b_i(H)$ is the number of partitions of $V(H)$ into i non-empty independent subsets. Let $S(n,i)$ denote the Stirling number of the second kind, which counts the number of partitions of $[n]$ into i disjoint subsets.

Lemma 3.1 $\mathcal{B}_\chi(n) \leq \prod_{i=1}^{n} S(n,i)$.

Proof: Clearly, $0 \leq b_i(H) \leq S(n,i)$. □

Theorem 3.2

$$\lim_{n\to\infty} \frac{\mathcal{B}_\chi(n)}{\mathcal{H}(n)} = 0.$$

Proof: From Lemmas 2.1 and 3.1,

$$\frac{\mathcal{B}_\chi(n)}{\mathcal{H}(n)} \leq \frac{\mathcal{B}_\chi(n) \cdot n!}{\overline{\mathcal{H}}(n)} \leq \frac{\prod_{i=1}^{n} S(n,i) \cdot n!}{2^{2^n}}.$$

Since $S(n,i) < \frac{i^n}{i!} < n^n$, it follows that

$$\frac{\mathcal{B}_\chi(n)}{\mathcal{H}(n)} \leq \frac{\prod_{i=1}^{n} S(n,i) \cdot n!}{2^{2^n}} < \frac{n^{n^2} \cdot n!}{2^{2^n}} < \frac{n^{n^2} \cdot n^n}{2^{2^n}} = \frac{n^{n^2+n}}{2^{2^n}}.$$

Now we use Proposition 2.3 and take base 2 logarithms of the numerator and denominator, and obtain:

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{B}_\chi(n)}{\log_2 \mathcal{H}(n)} \leq \lim_{n\to\infty} \frac{(n^2+n)\log_2 n}{2^n} = 0, \tag{1}$$

which implies that $\lim_{n\to\infty} \frac{\mathcal{B}_\chi(n)}{\mathcal{H}(n)} = 0$. □

Now Theorem 1.3(i) follows using Lemma 2.2(ii). Theorem 1.3(ii) follows using Lemma 2.2(i) and (iii) by replacing $\mathcal{H}(n)$ by $\mathcal{H}^r(n) \leq 2^{\binom{n}{r}}$ in the proof of Theorem 3.2. We get, instead of Equation (1),

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{B}^r_\chi(n)}{\log_2 \mathcal{H}^r(n)} \leq \lim_{n\to\infty} \frac{(n^2+n)\log_2 n}{\binom{n}{r}} = 0,$$

which still holds for $r \geq 3$.

4 Proof of Theorems 1.4 and 1.5

Let $\mathcal{B}_{\text{Ind}}(n)$ be the number of polynomials $p(X)$ such that there is a hypergraph H of order n with $p(X) = \text{Ind}(H; X)$.

Theorem 4.1 *(i)* $\lim_{n\to\infty} \frac{\mathcal{B}_{\text{Ind}}(n)}{\mathcal{H}(n)} = 0$. *(ii)* $\lim_{n\to\infty} \frac{\mathcal{B}^r_{\text{Ind}}(n)}{\mathcal{H}^r(n)} = 0$.

Proof: (i): Since $0 \le \text{ind}_i(H) \le \binom{n}{i}$, it follows that

$$\mathcal{B}_{\text{Ind}}(n) \le \prod_{i=1}^{n} \binom{n}{i} < \prod_{i=1}^{n-1} n^i = n^{\frac{n^2+n}{2}}. \tag{2}$$

From Lemma 2.1, we have that

$$\frac{\mathcal{B}_{\text{Ind}}(n)}{\mathcal{H}(n)} \le \frac{\mathcal{B}_{\text{Ind}}(n) \cdot n!}{\overline{\mathcal{H}}(n)} < \frac{n^{\frac{n^2+n}{2}} \cdot n!}{2^{2^n}} < \frac{n^{\frac{n^2+n}{2}} \cdot n^n}{2^{2^n}} = \frac{n^{\frac{n^2+3n}{2}}}{2^{2^n}}. \tag{3}$$

Taking base 2 logarithms of the numerator and denominator in Equation (3), we have that

$$\frac{\log_2 \mathcal{B}_{\text{Ind}}(n)}{\log_2 \overline{H}(n)} < \frac{\frac{n^2+3n}{2} \cdot \log_2 n}{2^n}.$$

Therefore, using Proposition 2.3,

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{B}_{\text{Ind}}(n)}{\log_2 \overline{H}(n)} \le \lim_{n\to\infty} \frac{\frac{n^2+3n}{2} \cdot \log_2 n}{2^n} = 0, \tag{4}$$

which implies that

$$\lim_{n\to\infty} \frac{\mathcal{B}_{\text{Ind}}(n)}{\mathcal{H}(n)} = 0.$$

(ii): As in the case of the chromatic polynomial, we replace $\mathcal{H}(n)$ by $\mathcal{H}^r(n) \le 2^{\binom{n}{r}}$ and $\mathcal{B}_{\text{Ind}}(n)$ by $\mathcal{B}^r_{\text{Ind}}(n)$ in Equation (4) and use Lemma 2.2(i) and (iii). □

Now Theorem 1.4 (i) and (ii) follow.

Let $\mathcal{B}_{\text{M}}(n)$ be the number of polynomials $p(X)$ such that there is a hypergraph H of order n with $p(X) = \text{M}(H; X)$.

Theorem 4.2 *(i)* $\lim_{n\to\infty} \frac{\mathcal{B}_{\text{M}}(n)}{\mathcal{H}(n)} = 0$. *(ii)* $\lim_{n\to\infty} \frac{\mathcal{B}^r_{\text{M}}(n)}{\mathcal{H}^r(n)} = 0$.

Proof: (i): Since $0 \le \omega_i(\mathcal{H}) \le \binom{\lfloor \frac{n}{k} \rfloor}{i}$, from Equation (4) we have

$$\mathcal{B}_{\text{M}}(n) \le \prod_{i=1}^{n} \binom{\lfloor \frac{n}{k} \rfloor}{i} < \prod_{i=1}^{n} \binom{n}{i} < n^{\frac{n^2+n}{2}}.$$

From Lemma 2.1 and Equation (3), we have

$$\frac{\mathcal{B}_{\mathrm{M}}(n)}{\mathcal{H}(n)} \leq \frac{\mathcal{B}_{\mathrm{M}}(n) \cdot n!}{\overline{\mathcal{H}}(n)} < \frac{n^{\frac{n^2+3n}{2}}}{2^{2^n}}. \tag{5}$$

Taking base 2 logarithms of the numerator and denominator in Equation (4), we have that

$$\frac{\log_2 \mathcal{B}_{\mathrm{M}}(n)}{\log_2 \overline{\mathcal{H}}(n)} < \frac{\frac{n^2+3n}{2} \cdot \log_2 n}{2^n}.$$

Therefore, using Proposition 2.3,

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{B}_{\mathrm{M}}(n)}{\log_2 \overline{\mathcal{H}}(n)} \leq \lim_{n\to\infty} \frac{\frac{n^2+3n}{2} \cdot \log_2 n}{2^n} = 0, \tag{6}$$

which implies that $\lim_{n\to\infty} \frac{\mathcal{B}_{\mathrm{M}}(n)}{\mathcal{H}(n)} = 0$.

(ii): As in the case of the chromatic polynomial we replace $\mathcal{H}(n)$ by $\mathcal{H}^r(n) \leq 2^{\binom{n}{r}}$ and $\mathcal{B}_{\mathrm{M}}(n)$ by $\mathcal{B}^r_{\mathrm{M}}(n)$ in Equation (6) and use Lemma 2.2 (i) and (iii). □

Now Theorem 1.5 (i) and (ii) follows.

5 Conclusions and further research

We have shown that for r-uniform hypergraphs with $r \geq 3$, and hypergraphs in general, there are very few hypergraphs which are unique for χ, Ind and M. This is not so surprising as there are many more r-uniform hypergraphs of order n than graphs. Still, it is interesting to search for such graphs.

Problem 1 *Find more P-unique r-uniform hypergraphs for χ,* Ind *and* M.

For Ind and M it seems this has not been properly investigated.

In [20] the Tutte polynomial $T(G; X, Y)$ and the most general edge elimination polynomial $\xi(G; X, Y, Z)$ of [1] are generalized to hypergraphs. $T(H; X, Y)$ is a substitution instance of $\xi(H; X, Y, Z)$ both on graphs and hypergraphs. In [10] another version of a Tutte polynomial for hypergraphs is proposed.

Problem 2 *Is $T(H; X, Y)$ almost complete for r-uniform hypergraphs?*

Note that the original BPR-conjecture asserts this for graphs, and is still open.

Problem 3 *Is $\xi(H; X, Y, Z)$ almost complete for r-uniform hypergraphs?*

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