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Fractional ℓ-factors in regular graphs

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Abstract

Let G be a simple k-regular graph and let $\ell \leq k$ be a positive integer. Then, (a) for every subset E_1 of $E(G)$ having at most $\lfloor \frac{\ell}{2} \rfloor$ elements, there exists a fractional ℓ-factor of G with indicator function h such that $h(e) = 1$ for every $e \in E_1$ and (b) for every subset E_1 of $E(G)$ having at most $\lfloor \frac{k-\ell}{2} \rfloor$ elements there exists a fractional ℓ-factor of G with indicator function h such that $h(e) = 0$ for every $e \in E_1$.

1 Introduction and Terminology

All graphs considered are assumed to be simple and finite. We refer the reader to [3] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The *degree* $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. If X and Y are subsets of $V(G)$, the set of the edges of G joining X to Y is denoted by $E_G(X, Y)$. For any set X of vertices in G, the *neighbour set* of X in G is denoted by $N_G(X)$. If e is an edge of G having u and v as end-vertices, then the edge e is also denoted by uv. We say that we *insert a vertex of degree* 2 *to an edge* e *of* G when we delete e and replace it by a path of length 2 connecting its ends, the internal vertex of this path being a new vertex. A graph G is k*-regular* if $d_G(x) = k$ for all $x \in V(G)$.

A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph.

For any set S of vertices in a graph G, we denote by $G - S$ the subgraph obtained from G by deleting all the vertices belonging to S together with their incident edges. For any set X of edges in G, the subgraph of G induced by X will be denoted by $G[X]$.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$ and let $h : E(G) \mapsto [0, 1]$ be a function such that $g(x) \leq d_G^h(x) \leq f(x)$ for every $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ and $E(x)$ denotes the set of edges incident with vertex x. If we define $F_h = \{e \in E(G) : h(e) > 0\}$, then we call $G[F_h]$ a fractional (g, f)-factor of G with indicator function

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h. If $g(x) = f(x)$ for all $x \in V(G)$, then we will call such a fractional (g, f)-factor a fractional f-factor. If $f(x) = \ell$ for every $x \in V(G)$, then a fractional f-factor is called a fractional ℓ-factor. Furthermore, if function h takes only integral values (0 and 1), then a fractional f-factor and fractional ℓ-factor are called f-factor and ℓ-factor respectively. Thus, an ℓ-factor of a graph G is an ℓ-regular spanning subgraph of that graph. The following necessary and sufficient conditions for a graph to have a fractional (g, f)-factor were obtained by Anstee [2]. Liu and Zhang[5] later presented a simple proof.

Theorem 1.1 *Let G be a graph and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then, G has a fractional (g, f)-factor if and only if for any $S \subseteq V(G)$,*

$$\sum_{x\in T}(g(x) - d_{G-S}(x)) \leq \sum_{x\in S} f(x)$$

where $T = \{x \in V(G) - S : d_{G-S}(x) \leq g(x)\}$.

There exists a rich literature on the existence of an ℓ-factor in a regular graph. More specifically, many results can be found in [1, 4, 6, 7, 8] related to the existence of an ℓ-factor in a regular graph containing or excluding specified edges. The main purpose of this paper is to present a similar study on the existence of a fractional ℓ-factor in a k-regular graph containing a number of prescribed edges to which its indicator function assigns the integral value 1 for each included edge or the integral value 0 for each excluded edge. The result of this effort is the proof of the following two theorems.

Theorem 1.2 *Let G be a k-regular graph and let $\ell \leq k$ be a positive integer. Then for every subset E_1 of $E(G)$ having at most $\lfloor \frac{\ell}{2} \rfloor$ elements, there exists a fractional ℓ-factor of G with indicator function h such that $h(e) = 1$ for every $e \in E_1$.*

Theorem 1.3 *Let G be a k-regular graph and let $\ell \leq k$ be a positive integer. Then for every subset E_1 of $E(G)$ having at most $\lfloor \frac{k-\ell}{2} \rfloor$ elements there exists a fractional ℓ-factor of G with indicator function h such that $h(e) = 0$ for every $e \in E_1$.*

2 Proofs of the Main Results

For the proof of Theorem 1.2, we will use the following lemma.

Lemma 2.1 *Let G be a k-regular graph and let ℓ be a positive integer, where $\ell \leq k$. Let also S be a subset of $V(G)$ and define $T_0 = \{x \in V(G) - S : d_{G-S}(x) \leq \ell\}$. Then for every subset E_1 of $E(G)$ satisfying $|E_1| \leq \lfloor \frac{\ell}{2} \rfloor$ and every subset T of T_0,*

$$\ell|T| - \sum_{x\in T} d_{G-S}(x) \leq \ell|S| - 2|E_1 \cap E_G(S, S)| - |E_1 \cap E_G(S, V(G) - (S \cup T))|.$$

Proof: We first define $R_i = \{x \in T : d_{G-S}(x) = i\}$ and $|R_i| = r_i$ for $i = 0, 1, \dots, \ell$. If we use the fact that G is k-regular, then we obtain

$$\sum_{i=0}^{\ell}(k-i)r_i = |E_G(S,T)| = k|S| - 2|E_G(S,S)| - |E_G(S, V(G)-(S\cup T))|. \quad (2.1)$$

Thus,

$$\sum_{i=0}^{\ell}\frac{\ell}{k}(k-i)r_i = \ell|S| - \frac{\ell}{k}(2|E_G(S,S)| + |E_G(S,V(G)-(S\cup T))|)$$

and so

$$\sum_{i=0}^{\ell}(\ell-i)r_i + \sum_{i=0}^{\ell}\frac{i(k-\ell)}{k}r_i = \ell|S| - \frac{\ell}{k}(2|E_G(S,S)|+|E_G(S,V(G)-(S\cup T))|), \quad (2.2)$$

since $\frac{\ell}{k}(k-i) = (\ell - i) + \frac{i(k-\ell)}{k}$.

At this point, we consider the following cases:

Case 1: $\sum_{i=0}^{\ell} ir_i \geq 2|E_G(S,S)| + |E_G(S,V(G)-(S\cup T))|$.
Then, (2.2) implies

$$\sum_{i=0}^{\ell}(l-i)r_i \leq l|S| - (2|E_G(S,S)| + |E_G(S,V(G)-(S\cup T))|)$$

and so Lemma 2.1 holds in this case.

Case 2: $\sum_{i=0}^{\ell} ir_i < 2|E_G(S,S)| + |E_G(S,V(G)-(S\cup T))|$.
If we use the hypothesis of Case 2 and since $\sum_{i=0}^{\ell} r_i = |T|$, then (2.1) yields

$$|S| \geq |T| + 1.$$

Thus

$$\begin{aligned}\ell|T| - \sum_{i=0}^{\ell} ir_i &\leq \ell(|S|-1) - \sum_{i=0}^{\ell} ir_i \\ &\leq \ell|S| - \ell \\ &\leq \ell|S| - 2\left\lfloor\frac{\ell}{2}\right\rfloor. \end{aligned} \quad (2.3)$$

But $|E_G(S,S)\cap E_1| + |E_G(S,V(G)-(S\cup T))\cap E_1| \leq |E_1| \leq \left\lfloor\frac{\ell}{2}\right\rfloor$, so we can obtain from (2.3),

$$\sum_{i=0}^{\ell}(\ell-i)r_i \leq \ell|S| - 2|E_G(S,S)\cap E_1| - |E_G(S,V(G)-(S\cup T))\cap E_1|.$$

Hence Lemma 2.1 also holds in this case. □

Proof of Theorem 1.2

Suppose that Theorem 1.2 does not hold. Then there exists a k-regular graph G and a set $E_1 \subseteq E(G)$ satisfying the conditions of the hypothesis of Theorem 1.2, without G possessing a fractional ℓ-factor having the properties implied by Theorem 1.2. Define G_1 to be the graph obtained from G by inserting a vertex of degree 2 to every edge belonging to E_1. Clearly, G has a fractional ℓ-factor with indicator function h such that $h(e) = 1$ for every $e \in E_1$ if and only if G_1 has a fractional f-factor satisfying $f(x) = 2$ for every $x \in V(G_1) - V(G)$ and $f(x) = \ell$ for every $x \in V(G)$. So, graph G_1 does not possess a fractional f-factor having the properties mentioned above. Thus by using Theorem 1.1, there exists a subset S_1 of $V(G_1)$ such that

$$\sum_{x \in T_1} (f(x) - d_{G_1 - S_1}(x)) > \sum_{x \in S_1} f(x) \tag{2.4}$$

where $T_1 = \{x \in V(G_1) - S_1 : d_{G_1 - S_1}(x) \leq f(x)\}$.
We assume that S_1 is minimal with respect to (2.4). If this is the case, we can prove the following claim.

Claim 1 *For every $x \in V(G_1) - V(G)$, $x \notin S_1$.*

Proof: Suppose that there exists $u \in V(G_1) - V(G)$ such that $u \in S_1$. Define $S_1^* = S_1 - \{u\}$. Clearly $d_{G_1 - S_1^*}(u) \leq 2$ since $d_{G_1 - S_1^*}(u) \leq d_{G_1}(u)$ and $d_{G_1}(u) = 2$. So u is an element of the set $T_1^* = \{x \in V(G_1) - S_1^* : d_{G_1 - S_1^*}(x) \leq f(x)\}$. Furthermore $T_1^* \subseteq T_1 \cup \{u\}$ since $d_{G_1 - S_1^*}(x) \geq d_{G_1 - S_1}(x)$ for every $x \in V(G_1) - S_1$; $(f(x) - d_{G_1 - S_1}(x)) = 0$ for every $x \in T_1 - T_1^*$, $(f(u) - d_{G_1 - S_1*}(u)) \geq 0$ and

$$\begin{aligned}
\sum_{x \in T_1^* - \{u\}} (f(x) - d_{G_1 - S_1^*}(x)) &= \sum_{x \in T_1^* - \{u\}} (f(x) - d_{G_1 - S_1}(x)) - |N_{G_1}(u) \cap T_1^*| \\
&\geq \sum_{x \in T_1^* - \{u\}} (f(x) - d_{G_1 - S_1}(x)) - 2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{x \in T_1^*} (f(x) - d_{G_1 - S_1^*}(x)) &= \sum_{x \in T_1^* - \{u\}} (f(x) - d_{G_1 - S_1^*}(x)) + (f(u) - d_{G_1 - S_1^*}(u)) \\
&\geq \sum_{x \in T_1^* - \{u\}} (f(x) - d_{G_1 - S_1}(x)) - 2 \\
&= \sum_{x \in T_1} (f(x) - d_{G_1 - S_1}(x)) - 2.
\end{aligned}$$

Thus by (2.4),

$$\sum_{x \in T_1^*} (f(x) - d_{G_1 - S_1^*}(x)) > \sum_{x \in S_1} f(x) - 2 = \sum_{x \in S_1^*} f(x),$$

contradicting the minimality of S with respect to (2.4). Hence, Claim 1 holds. □

So, by proving Claim 1, we have actually obtained that $V(G_1) - V(G) \subseteq T_1$, since $d_{G_1-S_1}(x) \le 2 = f(x)$ for $x \in V(G_1) - V(G)$. Define

$$\begin{aligned}
X_1 &= \{x \in V(G_1) - V(G) : N_{G_1}(x) \subseteq S_1\},\\
X_2 &= \{x \in V(G_1) - V(G) : N_{G_1}(x) \subseteq V(G_1) - S_1\},\\
X_3 &= \{x \in V(G_1) - V(G) : |N_{G_1}(x) \cap S_1| = 1\},\\
X_4 &= \{x \in V(G_1) - V(G) : |N_{G_1}(x) \cap T_1| = 1\},\\
X_{3,1} &= X_3 \cap X_4,\\
X_{3,2} &= X_3 - X_{3,1}.
\end{aligned}$$

Note that $V(G_1) - V(G) = X_1 \cup X_2 \cup X_3$, $X_3 = X_{3,1} \cup X_{3,2}$ and

$$d_{G_1-S_1}(x) = \begin{cases} 0, & x \in X_1,\\ 2, & x \in X_2,\\ 1, & x \in X_3. \end{cases}$$

In addition, we define $S = S_1$, $T = T_1 - (V(G_1) - V(G))$ and $T_0 = \{x \in V(G) - S : d_{G-S}(x) \le f(x)\}$ for use with Lemma 2.1. Clearly $T \subseteq T_0$, since $d_{G-S}(x) \le d_{G_1-S}(x)$ for every $x \in T$. More precisely, for every $x \in T$ we have $d_{G_1-S_1}(x) = d_{G-S}(x) + |N_{G_1}(x) \cap X_{3,1}|$, so that

$$\begin{aligned}
\sum_{x\in T_1} f(x) - \sum_{x \in T_1} d_{G_1-S_1}(x) &= 2|X_1| + |X_3| + \sum_{x\in T} f(x) - \sum_{x\in T} d_{G_1-S_1}(x)\\
&= 2|X_1| + |X_{3,1}| + |X_{3,2}| + \sum_{x\in T} f(x)\\
&\quad - \sum_{x\in T} d_{G-S}(x) - \sum_{x\in T} |N_{G_1}(x) \cap X_{3,1}|\\
&= 2|X_1| + |X_{3,2}| + \sum_{x\in T} f(x) - \sum_{x\in T} d_{G-S}(x)
\end{aligned}$$

by using the fact that $\sum_{x\in T} |N_{G_1}(x) \cap X_{3,1}| = |X_{3,1}|$. Thus by (2.4),

$$\sum_{x\in T} f(x) - \sum_{x\in T} d_{G-S}(x) > \sum_{x\in S} f(x) - 2|X_1| - |X_{3,2}|. \tag{2.5}$$

But $|X_{3,2}| = |E_1 \cap E_G(S, V(G) - (S \cup T))|$, $|X_1| = |E_1 \cap E_G(S,S)|$ and $f(x) = \ell$ for $x \in S \cup T$. Thus (2.5) yields

$$\ell|T| - \sum_{x\in T} d_{G-S}(x) > \ell|S| - 2|E_1 \cap E_G(S,S)| - |E_1 \cap E_G(S, V(G) - (S \cup T))|$$

contradicting Lemma 2.1. Therefore Theorem 1.2 holds. □

Proof of Theorem 1.3

If we define $G_1 = G - E_1$, then it suffices to show that G_1 possesses a fractional ℓ-factor. Suppose that this does not hold. Then by Theorem 1.1, there exists a subset S of $V(G_1)$ such that

$$\ell|T| - \sum_{x\in T} d_{G_1-S}(x) > \ell|S|, \tag{2.6}$$

where $T = \{x : x \in V(G_1) - S, d_{G_1-S}(x) \le \ell\}$. We also have

$$k|T| - \sum_{x\in T} d_{G-S}(x) = |E_G(S,T)| \le k|S| \tag{2.7}$$

by using the fact that G is k-regular. But

$$\begin{aligned}\sum_{x\in T} d_{G_1-S}(x) &\ge \sum_{x\in T} d_{G-S}(x) - 2|E_1| \\ &\ge \sum_{x\in T} d_{G-S}(x) - 2\left\lfloor \frac{k-\ell}{2} \right\rfloor,\end{aligned}$$

so 2.6 implies

$$\ell|T| - \sum_{x\in T} d_{G-S}(x) + 2\left\lfloor \frac{k-\ell}{2} \right\rfloor > \ell|S|. \tag{2.8}$$

Furthermore, (2.6) yields $|T| \ge |S| + 1$. Hence we can obtain from (2.7)

$$\ell|T| - \sum_{x\in T} d_{G-S}(x) + (k-\ell) \le \ell|S|$$

contradicting (2.8). Therefore G_1 has a fractional ℓ-factor and thus Theorem 1.3 holds. □

3 Remarks on the Sharpness of the Results

We will show in this section that Theorems 1.2 and 1.3 are in some sense best possible. More precisely, we will show that the number of edges of a k-regular graph to which the indicator function assigns integral values, either 1 or 0, cannot be increased. We will first describe a family of graphs G which constitutes a counterexample to an opposite claim for Theorem 1.2. Let ℓ, k be positive integers such that k is even and $\ell \le k-1$. We start from a k-regular bipartite graph H with bipartition (X,Y) where $|X| = |Y| = r$. Let $u \in Y$, $N_H(u) = \{u_1, u_2, \ldots, u_k\}$ and define G to be the graph obtained from H after the deletion of vertex u and the addition of the independent edges $u_1u_2, u_3u_4, \ldots, u_{k-1}u_k$. Clearly, G is also a k-regular graph. Define $M = E(G) - E(H)$ and let $Q \subseteq M$ such that $|Q| = \lfloor \frac{\ell}{2} \rfloor + 1$. We will show that the family of graphs G does not contain a fractional ℓ-factor with indicator function h such that $h(e) = 1$ for every $e \in Q$. For the proof of the above claim, we work as follows. Let G_1 be the graph obtained from G after the insertion of a vertex of degree 2 to every edge belonging to Q. Clearly, G possesses a fractional ℓ-factor with indicator function h such that $h(e) = 1$ for every $e \in Q$ if and only if G_1 has a fractional f-factor satisfying $f(x) = \ell$ for every $x \in V(G)$ and $f(x) = 2$

for every $x \in V(G_1) - V(G)$. But the latter does not hold, because if we let $S = X$, $T = \{x \in V(G_1) - S : d_{G_1-S} \le f(x)\} = (Y - \{u\}) \cup (V(G_1) - V(G))$, then

$$\sum_{x\in T} f(x) - \sum_{x\in T} d_{G_1-S}(x) > \sum_{x\in S} f(x)$$

since $\sum_{x\in T} f(x) = 2(\lfloor \frac{\ell}{2} \rfloor + 1) + \ell(r-1)$, $\sum_{x\in T} d_{G_1-S}(x) = 0$ and $\sum_{x\in S} f(x) = \ell r$.

Hence the family of graphs G, we have just described, shows that Theorem 1.2 will not hold if the number of edges to which the indicator function assigns the value 1 is slightly higher.

We will next show that the number of edges to which the indicator function assigns the value 0 in Theorem 1.3 also cannot be increased. The family of graphs G which constitutes a counterexample to an opposite claim are constructed as follows. Let ℓ, k be positive integers such that k is even and $\ell \le k$. We start from a k-regular graph H with bipartition (X, Y) where $|X| = |Y| = r$. Let $u \in Y, N_H(u) = \{u_1, u_2, \ldots, u_k\}$ and define G to be the graph obtained from H after the deletion of vertex u and the addition of the independent edges $u_1u_2, u_3u_4, \ldots, u_{k-1}u_k$. It is obvious that G is also a k-regular graph. Define $M = E(G) - E(H)$, let $Q \subseteq M$ such that $|Q| = \lfloor \frac{k-\ell}{2} \rfloor + 1$ and let $G_1 = G - Q$. We will show that G_1 does not possess a fractional ℓ-factor. Let $S = Y - \{u\}$ and $T = \{x \in V(G_1) - S : d_{G_1-S}(x) \le \ell\} = X$. Then

$$\ell|T| - \sum_{x\in T} d_{G_1-S}(x) > \ell|S|$$

since $|T| = r$, $|S| = r - 1$ and $\sum_{x\in T} d_{G_1-S}(x) = k - 2(\lfloor \frac{k-\ell}{2} \rfloor + 1)$. Thus G_1 does not contain a fractional ℓ-factor. But G_1 possesses a fractional ℓ-factor if and only if G contains a fractional ℓ-factor with indicator function which assigns the value 0 to all the elements of Q. Hence the family of graphs G, we have just described, shows that Theorem 1.3 is also best possible.

Finally, a natural question that may arise is whether we can obtain a sufficient condition for the existence of a fractional factor in a regular graph having indicator function assigning to some prescribed edges the value 1 and to some others the value 0. We will describe a family of graphs G in order to show that this is not the case. Let H be a k-regular bipartite graph with bipartition (X, Y) where $|X| = |Y| = r$, $X = \{u_1, u_2, \ldots, u_r\}$, $Y = \{v_1, v_2, \ldots, v_r\}$ and let e_1, e_2 be two independent edges of H having $\{u_d, v_m\}$,$\{u_t, v_z\}$ as sets of end-vertices respectively. Let also G be the graph obtained from H by deleting the edges e_1, e_2 and by adding the edges u_du_t and v_mv_z. Clearly, G is also a k-regular graph. Furthermore, G does not contain a fractional ℓ-factor with indicator function h such that $h(u_du_t) = 1$ and $h(v_mv_z) = 0$. For the proof of the above claim, we work as previously. Let G_1 be the graph obtained from G by deleting the edge v_mv_z and by inserting a vertex of degree 2 to the edge u_du_t and let u be this new vertex. We have that graph G contains a fractional ℓ-factor with indicator function h such that $h(u_du_t) = 1$ and $h(v_mv_z) = 0$ if and only if G_1 possesses a fractional f-factor such that $f(u) = 2$ and $f(x) = \ell$ for every $x \in V(G)$. But, G_1 does not contain such a fractional f-factor because if $X = S$ and

$T = \{x \in V(G_1) - S : d_{G_1-S}(x) \leq f(x)\} = Y \cup \{u\}$, then

$$\sum_{x \in T} f(x) - \sum_{x \in T} d_{G_1-S}(x) > \sum_{x \in S} f(x)$$

since $\sum_{x \in T} f(x) = r\ell + 2$, $\sum_{x \in T} d_{G_1-S}(x) = 0$ and $\sum_{x \in S} f(x) = r\ell$.

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