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A characterization of trees having a minimum vertex cover which is also a minimum total dominating set*

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Abstract

A vertex cover of a graph $G = (V, E)$ is a set $X \subseteq V$ such that each edge of G is incident to at least one vertex of X. A dominating set $D \subseteq V$ is a total dominating set of G if the subgraph induced by D has no isolated vertices. A $(\gamma_t - \tau)$-set of G is a minimum vertex cover which is also a minimum total dominating set. In this article we give a constructive characterization of trees having a $(\gamma_t - \tau)$-set.

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1 Introduction

Throughout this paper $G = (V, E)$ will be a finite, undirected, simple and connected graph of order n. The *neighborhood* of a vertex $v \in V$ is the set $N(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, the *open neighborhood*, $N(X)$, is defined to be $\bigcup_{v \in X} N(v)$ and *the closed neighborhood* of X is defined as $N[X] = N(X) \cup X$. The degree of a vertex $v \in V$ is $d(v) = |N(v)|$. A vertex $v \in V$ is an *end vertex* if $d(v) = 1$. A *support vertex*, or *support*, is the neighbor of an end vertex; a *strong support vertex* is the neighbor of at least two end vertices. For a set $S \subseteq V$, and $v \in S$, the *private neighborhood* $\mathrm{pn}(v, S)$ of $v \in S$ is defined by $\mathrm{pn}(v, S) = \{u \in V\colon\ N(u) \cap S = \{v\}\}$. Each vertex in $\mathrm{pn}(v, S)$ is called a *private neighbor* of v.

A *vertex cover* of G is a set $X \subseteq V$ such that each edge of G is incident to at least one vertex of X. A minimum vertex cover is a vertex cover of smallest possible cardinality. The *vertex cover number* of G, $\tau(G)$, is the cardinality of a minimum vertex cover of G. A vertex cover of cardinality $\tau(G)$ is called a $\tau(G)$-set.

The minimum vertex cover problem arises in various important applications, including multiple sequence alignments in computational biochemistry (see for example [15]). In computational biochemistry there are many situations where conflicts between sequences in a sample can be resolved by excluding some of the sequences. Of course, exactly what constitutes a conflict must be precisely defined in the biochemical context. It is possible to define a conflict graph where the vertices represent the sequences in the sample and there is an edge between two vertices if and only if there is a conflict between the corresponding sequences. The aim is to remove the fewest possible sequences that will eliminate all conflicts, which is equivalent to finding a minimum vertex cover in the conflict graph G. Several approaches, such as the use of a parameterized algorithm [4] and the use of a simulated annealing algorithm [17], have been developed to deal with this problem.

A subset D of V is *dominating* in G if $N[D] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G. A dominating set D is a *total dominating set* of G if the subgraph $G[D]$ induced by D has no isolates. In [2], Cockayne et al. defined the *total domination number* $\gamma_t(G)$ of a graph G to be the minimum cardinality among all total dominating sets of G. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set.

A total vertex cover is a set which is both a total dominating set and vertex cover. In [5], Dutton studies total vertex covers of minimum size. He proved that, in general, the associated decision problem is $\mathcal{NP}$-complete, and gives some bounds of the size of a minimum total vertex cover of a graph G in terms of $\gamma_t(G)$ and $\tau(G)$; this parameter has received some attention in recent years [6, 13]. In this work, we explore a particular case of total vertex covers. A $(\gamma_t - \tau)$-*set* of G is a total vertex cover which is both a $\gamma_t(G)$-set and a $\tau(G)$-set. While every graph has a total vertex cover, by considering K_2, it is trivial to observe that not every graph has a $(\gamma_t - \tau)$-set. So, it is natural to ask for a characterization of graphs having a $(\gamma_t - \tau)$-set.

Clearly, a graph G having a $(\gamma_t - \tau)$-set also satisfies $\gamma_t(G) = \tau(G)$; a graph satisfying this equation will be called a $(\gamma_t - \tau)$-*graph*. Again, K_2 is an example of a graph which is not a $(\gamma_t - \tau)$-graph, and so, the following question arises: Does every $(\gamma_t - \tau)$-graph contain a $(\gamma_t - \tau)$-set? Unfortuately, the answer is no (consider the path on 8 vertices, P_8). So, another natural problem to consider is to find a characterization of $(\gamma_t - \tau)$-graphs.

Total domination in graphs is well described in [9] and more recently in [11] and [12]. Among the different variants of domination, total domination is probably the best known and the most widely studied. Total domination has been successfully related to many graph theoretic parameters [12]; in particular, an additional motivation for this work is the following observation. It is known that for every graph G, $\gamma(G) \leq \alpha'(G)$, where $\alpha'(G)$ is the matching number of G. Nonetheless, neither $\alpha'(G)$ nor $\gamma_t(G)$ bounds the other one, and it is an interesting problem to find families of graphs G such that $\gamma_t(G) \leq \alpha'(G)$, [12]. On the other hand, in [7], Hartnell and Rall characterized all the graphs G such that $\gamma(G) = \tau(G)$. Recalling that for every bipartite graph G we have $\tau(G) = \alpha'(G)$, it is natural to consider the problem of characterizing bipartite graphs G such that $\gamma_t(G) = \tau(G)$.

A usual approach in the literature for characterizing families of trees with a certain property is to consider a constructive characterization. First, a family B of trees having the property P (where it is usually trivial to verify it) is chosen as a (recursive) base, and then, some operations preserving P are introduced. Finally, it is proved that the family of trees having the property P are precisely those trees that can be constructed from a tree in B by recursive applications of the proposed operations. This approach has been used extensively, to characterize, for example, Roman trees [10], trees with equal independent domination and restrained domination numbers, trees with equal independent domination and weak domination numbers [8], trees with equal independent domination and secure domination numbers [14], trees with at least k disjoint maximum matchings [16], trees with equal 2-domination and 2-independence numbers [1], trees with equal domination and independent domination numbers, trees with equal domination and total domination numbers [3], etc. In [3], a general framework for studying constructive characterizations of trees having an equality between two parameters is discussed.

The main goal of this article is to provide a constructive characterization of the trees having a $(\gamma_t - \tau)$-set. For unexplained terms and symbols we refer the reader to [9]. The rest of the paper is structured as follows. In Section 2 we present some basic results that will be used in the rest of the paper; it is also proved that the difference between $\gamma_t(G)$ and $\tau(G)$ can be arbitrarily large. Section 3 is devoted to proving our main result; showing that the family of trees T having a $(\gamma_t - \tau)$-set can be constructed through four simple operations starting from P_4. In the final section some related problems are proposed.

2 Basic results relating $\gamma_t(G)$ and $\tau(G)$

In Section 3 we will define four operations which will be used to construct all the trees having a $(\gamma_t - \tau)$-set. Such operations will be defined using the following definition.

Definition 2.1. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two disjoint graphs, and let u and v be vertices in $V(G)$ and $V(H)$, respectively. The sum of G with H via the edge uv, $G +_{uv} H$, is defined as $V(G +_{uv} H) = V(G) \cup V(H)$ and $E(G +_{uv} H) = E(G) \cup E(H) \cup \{uv\}$.

Moreover, if $H = K_1 = \{v\}$, we say that we add v to G supported by u.

Let G and H be two graphs with $u \in G$ and $v \in H$. Notice that, regardless of the choice of u and v, the following inequalities are always satisfied:

$$\max\{\gamma_t(G), \gamma_t(H)\} \leq \gamma_t(G +_{uv} H) \leq \gamma_t(G) + \gamma_t(H),$$

$$\max\{\tau(G), \tau(H)\} \leq \tau(G +_{uv} H) \leq \tau(G) + \tau(H) + 1.$$

It is also worth noticing that, for each of the previous four inequalities, there are examples where they are strict, and examples where they are equalities; we will come across them in the following sections.

We will now use the previously defined sum to prove that the difference between γ_t and τ can be arbitrarily large, even for trees.

Proposition 2.1. *For any positive integer k there exists a tree $T_{(k)}$ such that $\tau(T_{(k)}) - \gamma_t(T_{(k)}) = k$.*

Proof. Let $P_{4k+2} = (v_1, v_2, \dots, v_{4k+2})$ be a path. Add $2k+2$ new vertices to P_{4k+2}, each supported by a different one of the $2k+2$ vertices $\{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k+1}, v_{4k+2}\}$. The graph that we obtain is a tree $T_{(k)}$ such that $\gamma_t(T_{(k)}) = 2k+2$, and $\tau(T_{(k)}) = 3k+2$. Thus, we have $\tau(T_{(k)}) - \gamma_t(T_{(k)}) = k$. See Figure 1. □

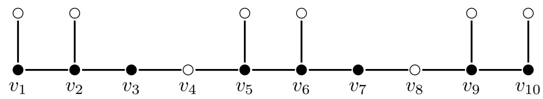

Figure 1: Example of $T_{(k)}$ with $k = 2$.

Proposition 2.2. *For every positive integer k there exists a tree $T'_{(k)}$ such that $\gamma_t(T'_{(k)}) - \tau(T'_{(k)}) = k$.*

Proof. Let $P_{4k-1} = (v_1, v_2, \dots, v_{4k-1})$ be a path. Add $2k$ new vertices to P_{4k-1} each supported by one of the vertices with an odd index. The graph that we obtain is a tree $T'_{(k)}$ such that $\gamma_t(T'_{(k)}) = 3k, \tau(T'_{(k)}) = 2k$. Hence, $\gamma_t(T'_{(k)}) - \tau(T'_{(k)}) = k$. See Figure 2.

□

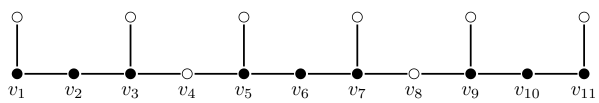

Figure 2: Example of $T'_{(k)}$ with $k = 3$.

The following simple remark will be useful in the proof of our main result.

Remark 2.3. *Let G be a graph with at least three vertices. If G is not a star, then there exists a minimum total dominating set $D \subseteq V(G)$ such that D contains no end vertex of G.*

Proof. Let D be a $\gamma_t(G)$-set and x belonging to D be an end vertex of G such that $N(x) = \{y\}$. Then $(D - \{x\}) \cup \{z\}$ is a total dominating set of G, where $z \in N(y)$ is not an end vertex of G. □

Our next result will also be very useful in the following section.

Lemma 2.4. *If $\gamma_t(G) = \tau(G)$ and D is a $(\gamma_t - \tau)$-set of G, then D contains no end vertex of G.*

Proof. Let $D \subseteq V(G)$ be a $(\gamma_t - \tau)$-set of G. If D contains an end vertex x, then, since D is a total dominating set, it follows that there exists a vertex $y \in D \cap N_G(x)$. This implies that $D - \{x\}$ is a vertex cover of G, a contradiction to the assumption that $\gamma_t(G) = \tau(G)$. □

As we mentioned in the introduction, not every tree contains a $(\gamma_t - \tau)$-set. The smallest tree having a $(\gamma_t - \tau)$-set is P_4, which also happens to be the smallest $(\gamma_t - \tau)$-tree. But not every $(\gamma_t - \tau)$-tree contains a $(\gamma_t - \tau)$-set. Actually, it is not hard to find an infinite class of $(\gamma_t - \tau)$-trees not having a $(\gamma_t - \tau)$-set, the most simple one is the family of paths P_{4k}, for $k \geq 2$. Thus, the class of trees having a $(\gamma_t - \tau)$-set is properly contained in the class of $(\gamma_t - \tau)$-trees.

Given a class of graphs, it is common in graph theory to aim for a characterization in terms of a set of forbidden induced subgraphs, because such a characterization directly implies polynomial time recognition for the class. Unfortunately, neither $(\gamma_t - \tau)$-trees, nor trees having a $(\gamma_t - \tau)$-set, admit a characterization of this kind. To prove this fact, consider the following construction.

Recall that the corona of a graph G is the graph obtained from G by adding a new vertex v' to G supported by v, for every vertex $v \in V(G)$. If H is the corona of the graph G, then clearly $V(G)$ is a $(\gamma_t - \tau)$-set of H. Hence, any graph G is an induced subgraph of a $(\gamma_t - \tau)$-graph (of a graph having a $(\gamma_t - \tau)$-set), and thus, there exists no forbidden subgraph characterization of $(\gamma_t - \tau)$-graphs (of graphs having a $(\gamma_t - \tau)$-set).

In our next section, we will obtain a constructive characterization of trees having a $(\gamma_t - \tau)$-set. Towards this end, we finish this section introducing a definition and proving a simple technical result.

Definition 2.2. Let G be a graph and S a $\gamma_t(G)$-set. A vertex v is S-*quasi-isolated* if there exists $u \in S$ such that $\mathrm{pn}(u, S) = \{v\}$. A vertex v is *quasi-isolated* if it is S-quasi-isolated for some $\gamma_t(G)$-set S.

A vertex v is a 2-*support* if it is at distance two from an end vertex. The next proposition shows that if a vertex is a 2-support, then it is not quasi-isolated.

Proposition 2.5. *Let G be a graph and $v \in V$ a 2-support. Then the vertex v is non-quasi-isolated.*

Proof. Let $x, y, v \in V$ be an end vertex, a support and a 2-support of G, respectively, such that $y \in N(x) \cap N(v)$. For every $\gamma_t(G)$-set S, $y \in S$, $v \in N(y)$ and $x \in \text{pn}(y, S)$, therefore for any $u \in S$, $\text{pn}(u, S) \neq \{v\}$. Hence, v is not quasi-isolated. □

3 Trees having a $(\gamma_t - \tau)$-set

We define the family $\mathcal{T}$ of trees to consist of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that T_1 is the path P_4, $T = T_k$ and, if $k \geq 2$, then for $1 \leq i \leq k-1$, the tree T_{i+1} can be obtained from T_i by one of the following operations.

- **Operation $\mathcal{O}_1$**: Consider $u \in V(T)$ such that u belongs to some $(\gamma_t - \tau)$-set. Let v be an end vertex of a path P_4. Then do the sum of T with P_4 via the edge uv.
- **Operation $\mathcal{O}_2$**: Let $u \in V(T)$ such that u belongs to some $(\gamma_t - \tau)$-set. Then add a new vertex v to T supported by u.
- **Operation $\mathcal{O}_3$**: Let $u \in V(T)$ such that u belongs to some $(\gamma_t - \tau)$-set and u is not a quasi-isolated vertex. Let $P_2 = (v, w)$ be a path with two vertices. Then do the sum of T with P_2 via the edge uv.
- **Operation $\mathcal{O}_4$**: Let $u \in V(T)$ such that u is not a quasi-isolated vertex of T. Let v be a support vertex of a path P_4. Then do the sum of T with P_4 via the edge uv.

Our next lemma is valid for any tree, not necessarily a tree in $\mathcal{T}$.

Lemma 3.1. *Let T be a tree. If T_i is a tree obtained from T by an operation $\mathcal{O}_i$, $1 \leq i \leq 4$, then:*

1. $\gamma_t(T_1) = \gamma_t(T) + 2$ *and* $\tau(T_1) = \tau(T) + 2$;
2. $\gamma_t(T_2) = \gamma_t(T)$ *and* $\tau(T_2) = \tau(T)$;
3. $\gamma_t(T_3) = \gamma_t(T) + 1$ *and* $\tau(T_3) = \tau(T) + 1$;
4. $\gamma_t(T_4) = \gamma_t(T) + 2$ *and* $\tau(T_4) = \tau(T) + 2$;

and hence, $\gamma_t(T) - \tau(T) = \gamma_t(T_i) - \tau(T_i)$, *for* $1 \leq i \leq 4$. *In particular* $\gamma_t(T) = \tau(T)$ *if and only if* $\gamma_t(T_i) = \tau(T_i)$, *for* $1 \leq i \leq 4$.

Proof. Observe that for $1 \leq i \leq 4$, $\gamma_t(T_i) \geq \gamma_t(T)$ and $\tau(T_i) \geq \tau(T)$. We consider four cases.

- Suppose $i = 1$, $P_4 = (v, x, y, z)$ and $T_1 = T +_{uv} P_4$. Let S be a $\gamma_t(T)$-set (a $\tau(T)$-set, respectively). Then, $S' = S \cup \{x, y\}$ ($S' = S \cup \{v, y\}$, resp.), is a total dominating set (vertex cover, resp.) of T_1. Thus, $\gamma_t(T_1) \leq \gamma_t(T) + 2$ and $\tau(T_1) \leq \tau(T) + 2$.

For purposes of contradiction, let D be a $\gamma_t(T_1)$-set such that $|D| \leq \gamma_t(T)+1$. Define $S = D \cap V(P_4)$, then $2 \leq |S| \leq 3$. Suppose $|S| = 2$, then $v \notin D$ and $D - S$ is a total dominating set of T with cardinality less than or equal to $\gamma_t(T) - 1$. If $|S| = 3$, then $(D - S) \cup \{w\}$ for $w \in N_T(u)$ is a total dominating set of T with cardinality less than or equal to $\gamma_t(T) - 1$. Therefore, $\gamma_t(T_1) = \gamma_t(T) + 2$.

For purposes of contradiction, let D be a $\tau(T_1)$-set such that $|D| \leq \tau(T) + 1$. Define $S = D \cap V(P_4)$, then $|S| = 2$. Suppose $S = \{x, y\}$, or $S = \{x, z\}$ or $S = \{v, y\}$, then $D - S$ is a vertex cover of T with cardinality less than or equal to $\tau(T) - 1$. Hence, $\tau(T_1) = \tau(T) + 2$.

- For $i = 2$ the proof is straightforward.

- Suppose $i = 3$, $P_2 = (v, w)$ and $T_3 = T +_{uv} P_2$. Let S be a $\gamma_t(T)$-set such that $u \in S$, then $S' = S \cup \{v\}$ is a total dominating set of T_3. Similarly, if S is a $\tau(T)$-set then $S' = S \cup \{v\}$ is a vertex cover of T_3. Thus, $\gamma_t(T_3) \leq \gamma_t(T) + 1$ and $\tau(T_3) \leq \tau(T) + 1$.

  We will show that $\gamma_t(T_3) = \gamma_t(T)+1$, so, for purposes of contradiction, let D be a $\gamma_t(T_3)$-set such that $|D| < \gamma_t(T) + 1$. It suffices to assume that $|D| = \gamma_t(T)$, and there is no end vertex in D (such a set exists by Remark 2.3). Then $D \cap V(P_2) = \{v\}$ and $u \in D$. Since $|D - \{v\}| < \gamma_t(T)$, the set $D - \{v\}$ is not a total dominating set of T. But, for all $z \in N_T(u)$, the set $D' = (D - \{v\}) \cup \{z\}$ is a $\gamma_t(T)$-set such that u is D'-quasi-isolated, a contradiction. So, $\gamma_t(T_3) = \gamma_t(T) + 1$.

  By definition of vertex cover, it is not possible that $\tau(T_3) = \tau(T)$, so $\tau(T_3) = \tau(T) + 1$.

- Suppose $i = 4$, $P_4 = (x, v, y, z)$ and $T_4 = T +_{uv} P_4$. Let S be a $\gamma_t(T)$-set (a $\tau(T)$-set, respectively). Then, $S' = S \cup \{v, y\}$ is a total dominating set (vertex cover, resp.) of T_4. Thus, $\gamma_t(T_4) \leq \gamma_t(T) + 2$ and $\tau(T_4) \leq \tau(T) + 2$.

  For purposes of contradiction, let D be a $\gamma_t(T_4)$-set such that $|D| \leq \gamma_t(T) + 1$. Then $D \cap V(P_4) = \{v, y\}$. Since $|D - \{v, y\}| \leq \gamma_t(T) - 1$, the set $D - \{v, y\}$ is not a total dominating set of T. But, for all $w \in N_T(u)$, the set $D' = (D - \{v, y\}) \cup \{w\}$ is a $\gamma_t(T)$-set such that u is D'-quasi-isolated, a contradiction. So, $\gamma_t(T_4) = \gamma_t(T) + 2$.

  By definition of vertex cover, it is not possible that $\tau(T_4) \leq \tau(T) + 1$, so $\tau(T_4) = \tau(T) + 2$.

□

Corollary 3.2. *Suppose T is a tree with D a $(\gamma_t - \tau)$-set of T. If T_i is a tree obtained from T by an operation $\mathcal{O}_i$, $1 \leq i \leq 4$, then T_i has a $(\gamma_t - \tau)$-set D_i.*

Proof. Let D be a $(\gamma_t - \tau)$-set of T. With the notation of the above lemma, we have:

- If $i = 1$ then $D_1 = D \cup \{x, y\}$.

- If $i = 2$ then $D_2 = D$.
- If $i = 3$ then $D_3 = D \cup \{v\}$.
- If $i = 4$ then $D_4 = D \cup \{v, y\}$.

□

Theorem 3.3. *If $T \in \mathcal{T}$, then T is a $(\gamma_t - \tau)$-tree.*

Proof. Let $T = P_4$; then $\gamma_t(T) = \tau(T) = 2$. By Lemma 3.1 and Corollary 3.2, the proof is straightforward. □

As mentioned at the end of Section 2, there exist $(\gamma_t - \tau)$-trees that are not in family $\mathcal{T}$.

Lemma 3.4. *Let T be a tree and u a vertex in T.*

1. *Let $P_2 = (v, w)$ be a path of order two. Suppose that u belongs to some $\gamma_t(T)$-set D of T and define T' to be the sum of T with P_2 via the edge uv. If u is D-quasi-isolated, then $\gamma_t(T) = \gamma_t(T')$.*
2. *Let v and w be the support vertices of a path P_4. Define T' to be the sum of T with P_4 via the edge uv. If u is a quasi-isolated vertex, then $\gamma_t(T) = \gamma_t(T') + 1$.*

Proof. Let D be a $\gamma_t(T)$-set such that u is D-quasi-isolated. There exists $z \in D$ such that $\text{pn}(z, D) = \{u\}$. It is easy to verify that $D' = (D - \{z\}) \cup \{v\}$, is a $\gamma_t(T')$-set, in the first case, and $D' = (D - \{z\}) \cup \{v, w\}$ is a $\gamma_t(T')$-set for the second case. □

Our main result is the following.

Theorem 3.5. *Let T be a tree. If T has a $(\gamma_t - \tau)$-set, then $T \in \mathcal{T}$.*

Proof. By induction on $n = |V(T)|$. Since $\gamma_t(T) = \tau(T)$, we have $n \geq 4$. The only tree T with four vertices and equality $\gamma_t(T) = \tau(T)$ is P_4, and $P_4 \in \mathcal{T}$.

Let T be a tree with $n > 4$ and let D be a $(\gamma_t - \tau)$-set of T. If T has a strong support vertex v with an end vertex u, then D is a $(\gamma_t - \tau)$-set of $T' = T - \{u\}$. By the induction hypothesis $T' \in \mathcal{T}$ and, using operation $\mathcal{O}_2$ we have that $T \in \mathcal{T}$. Therefore we can assume that there are no strong support vertices in T.

Let $P = (v_0, \ldots, v_l)$ be a longest path in T. Then $d_T(v_1) = 2$ and by Lemma 2.4 the vertices $v_1, v_2 \in D$. The proof of the theorem follows from the next two claims.

Claim 1. If there exists a vertex $x \in N_T(v_2) \cap D$ such that $x \neq v_1$ then $T \in \mathcal{T}$.

Proof of Claim 1. Observe that $d_T(v_2) > 2$. Otherwise, $d_T(v_2) = 2$ and hence $x = v_3$ and $D - \{v_2\}$ is a vertex cover of T, contradicting $\gamma_t(T) = \tau(T)$. Notice that x is not an end vertex of T, otherwise $D - \{x\}$ would be a vertex cover of T, a contradiction, thus, x is a support vertex of T. Let T' be the tree $T' = T - \{v_0, v_1\}$. Since $v_2 \in D$, the set $D - \{v_1\}$ is a vertex cover of T', and $D - \{v_1\}$ is a total dominating set of T'. This implies that $\tau(T') \leq \tau(T) - 1$ and that $\gamma_t(T') \leq \gamma_t(T) - 1$.

Let D' be a $\gamma_t(T')$-set. Since x is a support vertex in T', $x \in D'$ and we may thus assume that $v_2 \in D'$. It now follows that $D' \cup \{v_1\}$ is a total dominating set of T. Thus

$$\gamma_t(T) \leq |D' \cup \{v_1\}| = \gamma_t(T') + 1 \leq (\gamma_t(T) - 1) + 1 = \gamma_t(T),$$

which implies that $\gamma_t(T') = \gamma_t(T) - 1$. On the other hand, if A is a $\tau(T')$-set, then $A \cup \{v_1\}$ is a vertex cover of T. This implies that

$$\tau(T) \leq |A \cup \{v_1\}| = \tau(T') + 1 \leq (\tau(T) - 1) + 1 = \tau(T).$$

Therefore, $\tau(T') = \tau(T) - 1$. Combining all this and using the fact that $\gamma_t(T) = \tau(T)$ we get that $\tau(T') = \gamma_t(T')$. Since $D - \{v_1\}$ is a vertex cover of T' (of cardinality $\tau(T')$), which is also a total dominating set of T' (of cardinality $\gamma_t(T')$), it now follows that T' has a $(\gamma_t - \tau)$-set. Now, it follows from the induction hypothesis that $T' \in \mathcal{T}$. We have already noticed that v_2 is a 2-support of T', so it follows from Lemma 2.5 that v_2 is not quasi-isolated in T', and using operation $\mathcal{O}_3$, we have that $T \in \mathcal{T}$.

Claim 2. If $N_T(v_2) \cap D = \{v_1\}$, then $T \in \mathcal{T}$.

Proof of Claim 2. If $N_T(v_2) \cap D = \{v_1\}$, then v_2 is a support vertex or $d_T(v_2) = 2$.

Observe that if $d_T(v_3) = 1$, since T does not have strong support vertices, then $T = P_4$. Therefore, $d_T(v_3) \geq 2$. Since D is a vertex cover of T and $v_3 \notin D$, $|N_T(v_3) \cap D| \geq 2$.

Suppose v_2 is a support vertex with end vertex neighbor x, and let T' be the tree $T' = T - \{v_0, v_1, v_2, x\}$. Since $v_3 \notin D$, every edge incident to v_3 in T must be covered by some of its neighbors. Thus, the set $D - \{v_1, v_2\}$ is a vertex cover of T'. Also, since $|N_T(v_3) \cap D| \geq 2$, vertex v_3 is dominated by some vertex in $D - \{v_1, v_2\}$; clearly vertices (other than v_3) in $V(T')$ are not dominated in T by v_1 or v_2, so they must be dominated by some vertex in $D - \{v_1, v_2\}$, and therefore $D - \{v_1, v_2\}$ is a total dominating set of T'. Hence, $\gamma_t(T') \leq \gamma_t(T) - 2$ and $\tau(T') \leq \tau(T) - 2$. Now, let D' be a $\gamma_t(T')$-set. Since $\{v_1, v_2\}$ is a total dominating set of $T[\{v_0, v_1, v_2, x\}]$, it is clear that $D' \cup \{v_1, v_2\}$ is a total dominating set of T. Hence,

$$\gamma_t(T) \leq |D' \cup \{v_1, v_2\}| = \gamma_t(T') + 2 \leq (\gamma_t(T) - 2) + 2 = \gamma_t(T),$$

which implies $\gamma_t(T') = \gamma_t(T) - 2$. On the other hand, let A be a $\tau(T')$-set, hence, since $\{v_1, v_2\}$ covers all the edges in the graph $T[\{v_0, v_1, v_2, v_3, x\}]$, it is clear that $A \cup \{v_1, v_2\}$ is a vertex cover of T, and thus,

$$\tau(T) \leq |A \cup \{v_1, v_2\}| = \tau(T') + 2 \leq (\tau(T) - 2) + 2 = \tau(T).$$

Hence $\tau(T') = \tau(T) - 2$, and thus, following a reasoning analogous to that of the previous claim, we obtain that $D - \{v_1, v_2\}$ is a $(\gamma_t - \tau)$-set of T'. Now, we can apply the induction hypothesis to obtain $T' \in \mathcal{T}$. Notice that v_3 is not a quasi-isolated vertex of T', otherwise Lemma 3.4 would imply $\gamma_t(T) = \gamma_t(T') + 1$, but $\gamma_t(T') = \gamma_t(T) - 2$. Hence, we can obtain T from T' using operation $\mathcal{O}_4$, therefore $T \in \mathcal{T}$.

Now we may assume that $d_T(v_2) = 2$. For purposes of contradiction, suppose that $d_T(v_3) > 2$. Hence, there is a path $P_3 = (a, b, c)$ which is attached to v_3 by the edge cv_3. Since D is a $\gamma_t(T)$-set, we have $b, c \in D$. But then $(D \cup \{v_3\}) - \{v_2, c\}$ is a vertex cover of T, a contradiction. Thus $d_T(v_3) = 2$.

Since D is a vertex cover of T and $v_3 \notin D$, we have $v_4 \in D$. If $d_T(v_4) = 1$, then $T = P_5$, and D is not a $\gamma_t(T)$-set. If $d_T(v_4) = 2$, then $v_5 \in D$, in this case $(D - \{v_2, v_4\}) \cup \{v_3\}$ is a vertex cover of T, a contradiction. Hence, $d_T(v_4) > 2$.

Define T' as $T' = T - \{v_0, v_1, v_2, v_3\}$, we will show that the set $D' = D - \{v_1, v_2\}$ is a $(\gamma_t - \tau)$-set of T' containing v_4. Notice first that v_1 dominates exactly v_0 and v_2 in T, and v_2 dominates exactly v_1 and v_3 in T. Hence, no vertex in $V(T')$ is dominated in T by v_1 or v_2, and hence every vertex of T' is dominated in T' by some vertex in $D - \{v_1, v_2\}$. Thus, $D - \{v_1, v_2\}$ is a total dominating set of T', and we have $\gamma_t(T') \leq \gamma_t(T) - 2$. If D' is a $\gamma_t(T')$-set, and using the fact that $T[\{v_0, v_1, v_2, v_3\}]$ is isomorphic to P_4, then it is easy to conclude that $D' \cup \{v_1, v_2\}$ is a total dominating set of D. Therefore

$$\gamma_t(T) \leq |D' \cup \{v_1, v_2\}| = \gamma_t(T') + 2 \leq (\gamma_t(T) - 2) + 2 = \gamma_t(T),$$

which implies that $\gamma_t(T) = \gamma_t(T') - 2$. Also, since v_1 and v_2 do not cover any edges in T', it is clear that in T, all the edges of T' are covered by $D - \{v_1, v_2\}$. Thus, $D - \{v_1, v_2\}$ is a vertex cover of T', so we have $\tau(T') \leq \tau(T) - 2$. Let A be a vertex cover of T', if $v_4 \in A$, then $A \cup \{v_1, v_2\}$ is a vertex cover of T. Therefore

$$\tau(T) \leq |A \cup \{v_1, v_2\}| = \tau(T') + 2 \leq (\tau(T) - 2) + 2 = \tau(T)$$

which implies that $\tau(T) = \tau(T') + 2$. If $v_4 \notin A$ for every $\tau(T')$-set, then, since we have already noticed that $D - \{v_1, v_2\}$ is a vertex cover of T', it must be the case that $\tau(T') \leq \tau(T) - 3$. But then, $A \cup \{v_1, v_3\}$ would be a $\tau(T)$-set with $\tau(T) - 1$ vertices, a contradiction. Thus, there is at least one $\tau(T')$-set containing v_4, and, as mentioned above, $\tau(T) = \tau(T') + 2$. Therefore, $D - \{v_1, v_2\}$ is a $(\gamma_t - \tau)$-set of T', and the induction hypothesis impiles $T' \in \mathcal{T}$. Finally, we can obtain T from T' using operation $\mathcal{O}_1$.

□

Therefore, we have proved the following theorem.

Theorem 3.6. *It T is a tree, then $T \in \mathcal{T}$ if and only if T has a $(\gamma_t - \tau)$-set.*

4 Further work and open problems

Once we have characterized the trees having a $(\gamma_t - \tau)$-set, the following natural step is to consider the following problem.

Problem 4.1. *Find a characterization for the $(\gamma_t - \tau)$-trees.*

If we let $\mathcal{T}'$ be the family of all $(\gamma_t - \tau)$-trees, it is clear that the family $\mathcal{T}$, of all trees having a $(\gamma_t - \tau)$-set, is contained in $\mathcal{T}'$. We have already observed in Section 2, that this containment is proper. Moreover, we can slightly modify the operations

$\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_3$ to preserve the equality $\gamma_t = \tau$, but not necessarily preserving the existence of a $(\gamma_t - \tau)$-set, thus obtaining a larger infinite family of trees, say $\mathcal{S}$, such that $\mathcal{T} \subset \mathcal{S} \subset \mathcal{T}'$. The modified operations for a tree T are the following (notice the relaxation of the choice of u, cf. Section 2).

- **Operation $\mathcal{O}_1'$**: Let u be a vertex in T, and let v be an end vertex of a path P_4. Then do the sum of T with P_4 via the edge uv.
- **Operation $\mathcal{O}_2'$**: Let $u \in V(T)$ such that u belongs to some $\gamma_t(T)$-set and also belongs to some $\tau(T)$-set. Then add a new vertex v to T supported by u.
- **Operation $\mathcal{O}_3'$**: Let $u \in V(T)$ such that u belongs to some $\gamma_t(T)$-set and u it is not a quasi-isolated vertex. Let $P_2 = (v, w)$ be a path with two vertices. Then do the sum of T with P_2 via the edge uv.

Notice that the family of paths of length $4k$, $k \geq 2$, mentioned in Section 2 as an example of an infinite family of $(\gamma_t - \tau)$-graphs not having a $(\gamma_t - \tau)$-set, can be obtained from P_4 by recursively applying operation $\mathcal{O}_1'$; this shows that the inclusion $\mathcal{T} \subset \mathcal{T}'$ is proper. Similarly, examples can be found of a tree T' obtained from a tree T by applying operation $\mathcal{O}_i$, $i \in \{2, 3\}$, such that T has a $(\gamma_t - \tau)$-set, but T' does not.

From the computational point of view, for any tree T, both $\gamma_t(T)$ and $\tau(T)$ can be determined in polynomial time. Hence, the problem of determining if $\gamma_t(T) = \tau(T)$, for a tree T, is polynomial time solvable. For the case of trees having a $(\gamma_t - \tau)$-set, Theorem 3.6 does not trivially imply a polynomial algorithm to determine the existence of a $(\gamma_t - \tau)$-set in a tree, so the following problem seems to be interesting.

Problem 4.2. *Find the complexity of determining the existence of a $(\gamma_t - \tau)$-set in a tree.*

Of course, it is also interesting to ask both problems for general graphs.

Problem 4.3. *For a given graph G:*

- *Find the complexity of determining whether $\gamma_t(G) = \tau(G)$.*
- *Find the complexity of determining the existence of a $(\gamma_t - \tau)$-set in G.*

Our intuition says that the existence of a $(\gamma_t - \tau)$-set is so restrictive in the structure of G that the second problem might be solvable in polynomial time.

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