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Connected odd factors of graphs

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Abstract

An odd factor of a graph is a spanning subgraph in which every vertex has odd degree. Catlin [*J. Graph Theory* **12** (1988), 29–44] proved that every 4-edge-connected graph of even order has a connected odd factor. In this paper, we consider graphs of odd order, and show that for every 4-edge-connected graph G of odd order, there exists a vertex w such that $G - w$ has a connected odd factor. Moreover, we show that the condition on 4-edge-connectedness in the above theorem is best possible.

1 Introduction

In this paper, we mainly deal with *multigraphs*, which may have multiple edges but have no loops. A graph without multiple edges or loops is called a *simple graph*. Let

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G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is called its *order* and denoted by $|G|$, and the number of edges in G is called its *size* and denoted by $e(G)$. The degree of a vertex v in G is denoted by $\deg_G(v)$.

An *odd subgraph* (respectively, *even subgraph*) of G is a subgraph in which every vertex has odd degree (resp. positive even degree). A spanning odd subgraph of G is called its *odd factor*, and a spanning even subgraph of G is called its *even factor*. It follows immediately from the handshaking lemma that a connected multigraph containing an odd factor has even order. This condition is also sufficient as shown in Theorem 1 and Proposition 7. For a graph G, let $odd(G)$ denote the number of odd components (i.e., components of odd order) of G, and for a set $\mathbb{S}$ of integers, an $\mathbb{S}$-*factor* of G is a spanning subgraph F satisfying $\deg_F(v) \in \mathbb{S}$ for all $v \in V(F)$.

Theorem 1 (Amahashi [1]). *Let n be a positive odd integer. Then a multigraph G has a $\{1, 3, \ldots, n\}$-factor if and only if*

$$odd(G - S) \leq n|S| \quad \textit{for all} \quad S \subset V(G).$$

In particular, every connected multigraph of even order has an odd factor (i.e., a $\{1, 3, 5, \ldots\}$-factor).

A multigraph having a connected even factor is called a *supereulerian multigraph*. A survey on supereulerian multigraphs is found in Catlin [3] and Kouider and Vestergaad [8]. The following theorem gives a sufficient condition for a graph to have a connected even factor, which was shown by using a well-known result on two edge-disjoint spanning trees [10, 12].

Theorem 2 (Jaeger [7]). *Every 4-edge-connected multigraph has a connected even factor.*

There are infinitely many 3-edge-connected cubic graphs (i.e., 3-regular graphs) which have no Hamiltonian cycles. Since a connected even factor of a cubic graph is a Hamiltonian cycle, the above fact says that there exist infinitely many 3-edge-connected simple graphs which have no connected even factors.

Analogously, we focus on a connected odd factor in this paper. Catlin [2] proved the following. In fact, he proved a stronger statement in terms of *collapsible* subgraphs.

Theorem 3 (Catlin [2], Theorem 2). *Every 4-edge-connected multigraph of even order has a connected odd factor.*

We show that we cannot lower the edge-connectivity condition in Theorem 3 as follows.

Proposition 4. *There exist infinitely many 3-edge-connected simple graphs of even order which have no connected odd factors.*

By the handshaking lemma, it is clear that every connected graph of odd order has no odd factor, so when we deal with a connected graph G of odd order, we might consider an odd factor in $G - w$ for some vertex w. This motivates us to show our main theorems.

Theorem 5. *For every 4-edge-connected multigraph G of odd order, there exists a vertex w such that $G - w$ has a connected odd factor.*

Theorem 6. *There exist infinitely many 3-edge-connected simple graphs G of odd order such that for every vertex v of G, $G - v$ has no connected odd factor.*

2 Proofs of Theorems

We begin with some other notations. Let G be a multigraph. Then let $V_{\text{even}}(G)$ and $V_{\text{odd}}(G)$ denote the set of vertices of even degree and that of odd degree, respectively. For a vertex set X of G, the subgraph of G induced by X is denoted by $\langle X \rangle_G$. For two disjoint vertex sets X and Y of G, the set of edges of G joining X to Y is denoted by $E_G(X, Y)$, and the number of edges of G joining X to Y is denoted by $e_G(X, Y)$. Thus $e_G(X, Y) = |E_G(X, Y)|$.

For a positive integer k, a spanning k-regular subgraph of G is called a *k-regular factor* or briefly a *k-factor*. For a vertex set T of G, a subgraph J of G is called a *T-join* if $V_{\text{odd}}(J) = T$. The following is a well-known fact. As far as we know, it was first proved in [6], but it appeared in several literatures, such as [2, Lemma 1].

Proposition 7. *Let G be a connected multigraph and $T \subseteq V(G)$. Then there exists a T-join in G if and only if $|T|$ is even.*

We prove Proposition 4. It is known that the Petersen graph of order 10, denoted by PG_{10}, is a 3-edge-connected simple graph and does not have a Hamiltonian cycle (see (1) of Figure 1). Let M be a simple graph of even order which has the following property: M has three specified vertices v_1, v_2 and v_3 such that the new graph $M + u$ obtained from M by adding a new vertex u together with three new edges uv_1, uv_2 and uv_3 is 3-edge-connected. For example, every complete graph with even order and every graph obtained from 3-edge-connected graph with odd order by removing a vertex of degree 3 can be M. Two examples of graphs M are shown in (2) of Figure 1.

Proof of Proposition 4. For every vertex x of the Petersen graph PG_{10}, we replace x with a graph M, that is, we delete x and add a graph M keeping the edges incident to x in PG_{10} with new ends v_1, v_2 and v_3. Note that such a graph M is denoted by M_x since we can choose a graph M depending on x as shown in (3) of Figure 1. We denote the resulting graph by G^*. Then G^* has even order since every M_x has even order, and G^* is 3-edge-connected since both $M_x + u$ and PG_{10} is 3-edge-connected. Moreover, it is obvious that there are infinitely many such graphs G^* since there are infinitely many graphs M.

We now show that G^* has no connected odd factors. Suppose that G^* has a connected odd factor F. Then for every vertex x of PG_{10}, we have

$$\sum_{v \in V(M_x)} \deg_F(v) = e_F\Big(V(M_x), V(G^*) - V(M_x)\Big) + 2e\big(\langle V(M_x)\rangle_F\big).$$

Since F is an odd factor of G^* and M_x has even order, it follows from the above equality that $\eta := e_F\big(V(M_x), V(G^*) - V(M_x)\big)$ is even. Since F is a connected factor, η is positive. We know that every edge of F joining $V(M_x)$ to $V(G^*) - V(M_x)$ corresponds to an edge of the basis Petersen graph PG_{10}. Hence $\eta = 2$ since PG_{10} is a cubic graph. Thus the set of edges of F joining $V(M_x)$ to $V(G^*) - V(M_x)$ for all $x \in V(PG_{10})$ forms a connected 2-factor of PG_{10}, which is a Hamiltonian cycle of PG_{10}, but this contradicts the fact that PG_{10} has no Hamiltonian cycle. Consequently Proposition 4 is proved. □

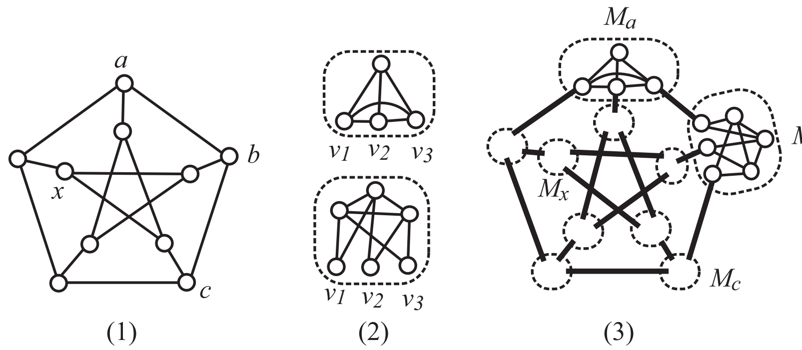

Figure 1: (1) The Petersen graph PG_{10} of order 10. (2) Two examples of graphs M. (3) A graph G^*, which is obtained from PG_{10} by replacing each vertex x by a graph M_x.

In order to prove Theorem 5, we use the following two theorems. The first theorem was shown by using the result on two edge-disjoint spanning trees [10, 12].

Theorem 8 (Catlin [4], see [5]). *Let $k \geq 1$ be an integer and let G be a multigraph. Then G is $2k$-edge-connected if and only if for all $X \subseteq E(G)$ with $|X| \leq k$, $G - X$ has k edge-disjoint spanning trees.*

A k-edge-connected multigraph G is said to be *minimally k-edge-connected* if for every edge e of G, $G - e$ is not k-edge-connected. Then the following holds.

Theorem 9 (Mader [9], Problem 49 of §6 in [11]). *Let $k \geq 1$ be an integer. Then every minimally k-edge-connected graph has a vertex of degree k. In particular, every k-edge-connected multigraph G has a k-edge-connected spanning subgraph H that has a vertex of degree k in H.*

We prove Theorem 5 by the similar arguments to those of Theorems 2 and 3, using Theorems 8 and 9 efficiently.

Proof of Theorem 5. By Theorem 9, G has a 4-edge-connected spanning subgraph H that has a vertex w of degree 4 in H. Let X be a set of two edges incident with w. By Theorem 8, $H - X$ has 2 edge-disjoint spanning trees T_1' and T_2'. Since w has degree two in $H - X$, w is a leaf in both T_1' and T_2'. Thus, $T_1 = T_1' - w$ and $T_2 = T_2' - w$ are edge-disjoint spanning trees in $H - w$.

Then $|V_{\text{even}}(T_1)|$ is even (possibly $V(T_1) = \emptyset$) since $|T_1| = |H - w|$ and $|V_{\text{odd}}(T_1)| = |T_1| - |V_{\text{even}}(T_1)|$ are both even. By Proposition 7, T_2 has a subgraph J such that $\deg_J(x)$ is odd for all $x \in V_{\text{even}}(T_1)$ and $\deg_J(y)$ is even for every $y \in V(J) - V_{\text{even}}(T_1)$. Then $T_1 \cup J$ is a connected odd factor of $H - w$, which is obviously the desired connected odd factor of $G - w$. □

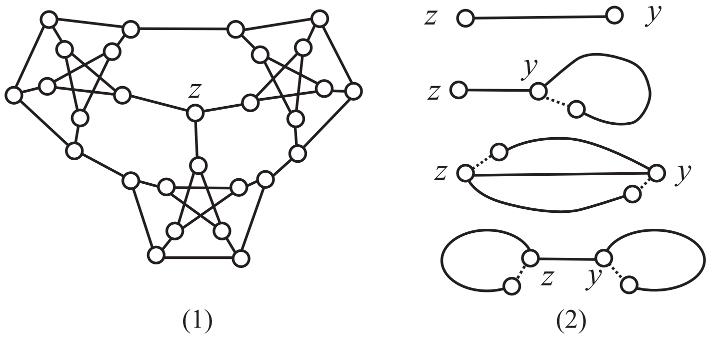

Figure 2: (1) A 3-edge-connected cubic graph G_{28} of order 28 with a specified vertex z, which has no Hamiltonian path. (2) A connected spanning subgraph F of G^* such that $\deg_F(z), \deg_F(y) \in \{1, 3\}$ and $\deg_F(x) = 2$ for all $x \in V(G_{28}) - \{y, z\}$.

We then prove Theorem 6, whose proof is similar to that of Proposition 4. Let G_{28} be the cubic graph of order 28 shown in Figure 2. Since the Petersen graph is 3-edge-connected, so is G_{28}. Then G_{28} has no Hamiltonian path.

Proof of Theorem 6. Let M be a graph of even order defined in the proof of Proposition 4 (see (2) of Figure 1). Let z be the central vertex of G_{28} shown in Figure 2. For every vertex x of G_{28} with $x \neq z$, we replace x by a graph M, that is, we delete x and add a graph M keeping the edges incident to x in G_{28} with new ends v_1, v_2 and v_3. Note that such a graph M is denoted by M_x since we can choose a graph M depending on x (see (3) of Figure 1). We denote the resulting graph by G^{**}. Then G^{**} has odd order since every M_x has even order, and G^{**} is 3-edge-connected since G_{28} and M_x are 3-edge-connected. It is obvious that there are infinitely many such graphs G^{**}.

We now show that for every vertex w of G^{**}, $G^{**} - w$ has no connected odd factor. Suppose that $G^{**} - w$ has a connected odd factor F for some vertex w. We first assume that w is contained in some M_y, $y \in V(G_{28}) - \{z\}$. For every

$x \in V(G_{28}) - \{y, z\}$, we have

$$\sum_{v \in V(M_x)} \deg_F(v) = e_F\Big(V(M_x), V(G^{**}) - V(M_x)\Big) + 2e\big(\langle V(M_x)\rangle_F\big).$$

Since every vertex of M_x has odd degree in F and M_x has even order, it follows from the above equality that $\eta_x := e_F\big(V(M_x), V(G^{**}) - V(M_x)\big)$ is even. Since F is a connected factor, η_x is positive. Thus $\eta_x = 2$ since G_{28} is a cubic graph. Note that $\deg_F(z) = 1$ or 3. For M_y with $w \in V(M_y)$, we have

$$\sum_{v \in V(M_y)-\{w\}} \deg_F(v) = e_F\Big(V(M_y - w), V(G^{**}) - V(M_y - w)\Big) + 2e(\langle V(M_y - w)\rangle_F).$$

Since every vertex of $M_y - w$ has odd degree in F, and $M_y - w$ has odd order, it follows from the above equality that $\eta_y := e_F\big(V(M_y), V(G^{**}) - V(M_y)\big)$ is odd. Thus η_y is 1 or 3.

We know that each edge of F joining $V(M_x)$ to $V(G^{**}) - V(M_x)$ for $x \in V(G_{28}) - \{z\}$ or joining z to $V(G^{**}) - \{z\}$ corresponds to an edge of G_{28}. Thus, the set of edges of F joining $V(M_x)$ to $V(G^{**}) - V(M_x)$ for $x \in V(G_{28}) - \{z\}$, and joining z to $V(G^{**}) - \{z\}$ forms a connected spanning subgraph $\widetilde{F}$ of G_{28} such that $\deg_{\widetilde{F}}(z), \deg_{\widetilde{F}}(y) \in \{1, 3\}$ and $\deg_{\widetilde{F}}(x) = 2$ for all $x \in V(G_{28}) - \{y, z\}$.

If $\deg_{\widetilde{F}}(z) = 1$ and $\deg_{\widetilde{F}}(y) = 1$, then $\widetilde{F}$ must be a Hamiltonian path of G_{28}, which contradicts the fact that G_{28} has no Hamiltonian path. If $\deg_{\widetilde{F}}(z) = 1$ and $\deg_{\widetilde{F}}(y) = 3$, then by removing one edge of $\widetilde{F}$ incident with y not contained in the path in $\widetilde{F}$ connecting y and z, we obtain a Hamiltonian path of G_{28}, a contradiction (see the second graph of Figure 2 (2)). The same situation occurs when $\deg_{\widetilde{F}}(z) = 3$ and $\deg_{\widetilde{F}}(y) = 1$. Suppose that $\deg_{\widetilde{F}}(z) = 3$ and $\deg_{\widetilde{F}}(y) = 3$. Then $\widetilde{F}$ is either a spanning subgraph consisting of three edge disjoint paths connecting z and y, or consisting of two disjoint cycles and a path internally disjoint from the cycles such that one cycle contains z, the other contains y and the path connects z and y. We choose two edges of $\widetilde{F}$ so that one is incident with z, the other is incident with y, and furthermore, the chosen edges are not contained in a same path in $\widetilde{F}$ connecting z and y in the former case, and the chosen edges are not contained in a path in $\widetilde{F}$ connecting z and y in the latter case. By removing the chosen edges from $\widetilde{F}$, we obtain a Hamiltonian path of G_{28}, a contradiction (see the third graph and fourth graph of Figure 2 (2)).

We then assume that $w = z$. In this case, by the same argument to show that $\eta_x = 2$ in above, we obtain a Hamiltonian cycle $\widetilde{F}$ of $G_{28} - z$ by $\deg_{\widetilde{F}}(x) = 2$ for all $x \in V(G_{28} - z)$. This implies that G_{28} has a Hamiltonian path starting at z. This is a contradiction. Consequently Theorem 6 is proved. □

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