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p^2-Cycle decompositions of the tensor product of complete graphs

R. S. Manikandan

Department of Mathematics
Bharathidasan University Constituent College
Lalgudi 621 601
India
manirs2004@yahoo.co.in

P. Paulraja   T. Sivakaran

Department of Mathematics
Kalasalingam Academy of Research and Education
Anand Nagar, Krishnankoil 626 126
India
ppraja56@gmail.com   shivaganesh1431991@gmail.com

Abstract

In this paper, we consider C_{p^2}-decompositions of the tensor product of complete graphs, $K_m \times K_n$, where $p \geq 2$ is a prime. It is proved that for any prime $p \geq 2$, with $m, n \geq 3$, the cycle C_{p^2} decomposes $K_m \times K_n$ if and only if (1) either m or n is odd, (2) $p^2 \mid \binom{m}{2} n(n-1)$, and (3) $p^2 \leq mn$. This is a companion result of C_{p^2}-decompositions of $K_m \circ \overline{K}_n$, considered by B.R. Smith [*J. Combin. Des.* 18 (2010), 401–414].

1 Introduction and Definitions

All graphs considered here are simple and finite. For a graph G, the graph G^* denotes the *symmetric digraph* of G; that is, the digraph obtained from G by replacing each of its edges by a symmetric pair of arcs. For a graph G and a positive integer λ, the graph $G(\lambda)$ is the graph obtained from G by replacing each of its edges by λ parallel edges. For a graph G and n a positive integer, nG denotes n vertex disjoint copies of G. Also we denote by P_k a path on k vertices and we let C_k (respectively, $\overrightarrow{C_k}$) denote a cycle (respectively, directed cycle) of length k. If S is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle S \rangle$ of G *induced by* S is the graph having vertex set S and whose edge

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set consists of those edges of G incident with two vertices of S. Similarly, if E' is a nonempty subset of $E(G)$, then the subgraph $\langle E' \rangle$ of G *induced by* E' is the graph whose vertex set consists of those vertices of G incident with at least one edge of E' and whose edge set is E'. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. For a digraph D, its arc set is denoted by $A(D)$. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of the graph G such that $E(G) = \bigcup_{i=1}^{k} E(H_i)$, then we say that $H_1, H_2, \ldots, H_k$ decompose G and we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. If each $H_i \cong H$, then we say that H decomposes G and we write $H \,|\, G$. A graph G has a C_k*-decomposition* or a k*-cycle decomposition* whenever $C_k \,|\, G$. A 2*-factor* of G is a spanning 2-regular subgraph. A 2*-factorization* of G is a decomposition of G into 2-factors. If G has a 2-factorization and each 2-factor of it contains only cycles of length k as its components, then we say that G has a C_k*-factorization* and we write $C_k \,||\, G$. A k-regular graph G is called *Hamilton cycle decomposable* if G is decomposable into $\frac{k}{2}$ Hamilton cycles when k is even and into $\frac{k-1}{2}$ Hamilton cycles together with a perfect matching when k is odd.

For two graphs G and H their *wreath product*, $G \circ H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ or, $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* of the graphs G and H, has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$; see Fig. 1. Clearly, the tensor product is commutative and distributive over edge-disjoint union of graphs; that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \cdots \oplus (H_k \times H)$. It can be observed that $K_m \circ \overline{K}_n$ is isomorphic to the complete m-partite graph in which each partite set has n vertices and $(K_m \circ \overline{K}_n) - E(nK_m) \cong K_m \times K_n$.

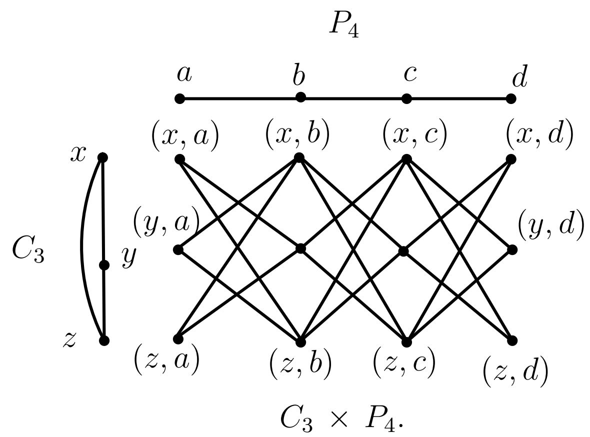


Fig. 1.

A graph G with m edges is called *graceful* if it is possible to label the vertices of G with distinct element from the set $\{0, 1, 2, \ldots, m\}$ in such a way that the induced edge labeling, which prescribes the integer $|i - j|$ to the edge joining vertices labeled i and j, assigns the labels $1, 2, \ldots, m$ to the edges of G. Such a labeling is called a *graceful labeling.* Thus, a *graceful graph* is a graph that admits a graceful labeling.

Let the vertices of K_n be $\{1, 2, \ldots, n\}$; then the edge ij of K_n is said to be of *distance* $\min\{|i-j|, n-|i-j|\}$. Hence there are exactly two edges of distance k, $k \leq (n-1)/2$, incident with each of its vertices and exactly one edge of distance $n/2$ incident with each of its vertices, if n is even. Let $V(K_n^*) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, and let n be an odd integer. The distance of the arc (v_i, v_j) in K_n^* is the integer d with $-\lfloor \frac{n}{2} \rfloor \leq d \leq \lfloor \frac{n}{2} \rfloor$ such that $j - i \equiv d \pmod{n}$. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$; if G contains the set of edges $F_i(X, Y) = \{x_j y_{j+i} \mid 0 \leq j \leq n-1$, where addition in the subscript is taken modulo $n\}$, $0 \leq i \leq n-1$, then we say that G has the 1*-factor of distance* i *from* X *to* Y. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. It is important to note that $F_i(X, Y) = F_{n-i}(Y, X)$, $0 \leq i \leq n-1$.

Finding a C_k-decomposition of K_{2n+1} or $K_{2n} - F$, where F is a perfect matching of K_{2n}, is completely settled by Alspach et al. [4] and Šajna [32]. An alternate proof for a C_{2k+1}-decomposition of K_{2n+1} is obtained by Buratti [14]. A similar problem is also considered for complete multipartite graphs; in [18], Hanani proved that the necessary conditions are sufficient for the existence of a C_3-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$; Billington et al. [13] proved that the necessary conditions are sufficient for the existence of a C_5-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$. Further, Cavenagh [17] solved the C_{2k}-decomposition problem of complete multipartitie graphs, where $k = 2, 3, 4$. Manikandan and Paulraja obtained a necessary and sufficient condition for the existence of a C_p-decomposition of $K_m \circ \overline{K}_n$, $p \geq 5$, is a prime; see [22, 23]. In [33, 34, 35], it is proved that the necessary conditions for the existence of C_{2p}, C_{3p} and C_{p^2} decompositions of $K_m \circ \overline{K}_n$ are sufficient. Further, in [28], Muthusamy and Shanmuga Vadivu proved the existence of a C_k-decomposition of $K_m \circ \overline{K}_n$ whenever k is even. Very recently, irrespective of the pairity of k, Buratti et al. [15] actually solved the existence problem for a k-cycle decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. Decompositions of $(K_m \circ \overline{K}_n)(\lambda)$ into cycles of variable lengths are considered in [9].

A similar problem of decomposing $K_m \times K_n$ into cycles of length k is considered here. In a group divisible design the edge set of $K_m \circ \overline{K}_n$ is partitioned into complete subgraphs whereas in a modified group divisible design the edge set of $K_m \times K_n$ is partitioned into complete subgraphs; see [1, 2, 6, 7, 8]. Hence the graph $K_m \times K_n$ is an important subgraph of $K_m \circ \overline{K}_n$. A necessary and sufficient condition for the existence of a C_p-decomposition of $K_m \times K_n$, $p \geq 5$, p a prime, is obtained by Manikandan and Paulraja [22, 23, 26]; it is pertinent to mention that a p-cycle decomposition of $K_m \times K_n$ is effectively used to obtain a p-cycle decomposition of $K_m \circ \overline{K}_n$; see [22, 23]. Hamilton cycle decompositions of $K_m \times K_n$ are completely settled by Balakrishnan et al.; see [10]. For related developments of the study of hamilton cycle decompositions in tensor products of complete multipartite graphs or of a complete graph and a complete bipartite graph, or a complete bipartite graph and a complete multipartite graph, see [21, 24, 25]. Existence of a resolvable even cycle decomposition of $K_m \times K_n$ can be found in [29].

In this paper we prove that the obvious necessary conditions for $K_m \times K_n$,

$m, n \geq 3$ to admit a C_{p^2}-decomposition are sufficient, where $p \geq 2$ is a prime. In fact, we have proved the following result:

Theorem 1.1. *For any prime $p \geq 2$, $m, n \geq 3$, $C_{p^2} \mid K_m \times K_n$ if and only if* (1) *either m or n is odd,* (2) $p^2 \mid \binom{m}{2} n(n-1)$, *and* (3) $p^2 \leq mn$.

This is a companion result of Smith [35] who proved the existence of a C_{p^2}-decomposition of $K_m \circ \overline{K}_n$ whenever the necessary conditions are satisfied.

For our future reference we list below some known results.

Theorem 1.2. [4]. *For any odd integer $t \geq 3$, if $n \equiv 1$ or t (mod $2t$), then $C_t \mid K_n$.*

Theorem 1.3. (see [20]). *Let $m \geq 3$ be an odd integer.*

(1) If $m \equiv 1$ or $3 \pmod 6$, then $C_3 \mid K_m$.
(2) If $m \equiv 5 \pmod 6$, then K_m can be decomposed into $(m(m-1)-20)/6$ 3-cycles and a K_5.

Theorem 1.4 is proven in [3] when m is an odd prime, but one can easily see that the same proof works for any odd integer m.

Theorem 1.4. [3]. *If m and k are odd integers and $3 \leq k \leq m$, then $C_m \,||\, C_k \circ \overline{K}_m$.*

Theorem 1.5. [27]. *The graph $K_{2m} \times C_{2n+1}$ has a Hamilton cycle decomposition.*

Theorem 1.6. [13, 18, 22, 26]. *For any prime p, $3 \leq p \leq mn$, and $m \geq 3$, we have $C_p \mid K_m \circ \overline{K}_n$ if and only if (1) $(m-1)n$ is even and (2) $p \mid m(m-1)n^2$.*

Theorem 1.7. [11]. *The graph $C_r \times C_s$ can be decomposed into two Hamilton cycles if and only if at least one of r and s is odd.*

Theorem 1.8. [23]. *For $m \geq 3$ and $k \geq 1$, $C_{2k+1} \mid C_{2k+1} \times K_m$.*

Theorem 1.9. [27]. *If $C_r \,||\, G$ and $s \mid m$, then $C_{rs} \,||\, G \times K_m$, except possibly for $m \equiv 2 \pmod 4$ when r is odd.*

The proof of Theorem 1.10 follows by proceeding as in the proof of Theorem 1.9 (see [27], Theorem 3.10).

Theorem 1.10. *If $C_r \mid G$ and $s \mid m$, then $C_{rs} \mid G \times K_m$, except possibly for $m \equiv 2 \pmod 4$ when r is odd.*

Theorem 1.11. [31]. *If H is a graceful graph of size m, then K_{2m+1} is H-decomposable.*

A *circulant graph* $G = \mathrm{Circ}(n; L)$ is graph with vertex set $V(G) = \{v_0, v_1, v_2, v_3, \ldots, v_{n-1}\}$ and edge set $E(G) = \{v_i v_{i+\ell} \mid i \in \mathbb{Z}_n, \ell \in L\}$, where $L \subseteq \{1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The elements of L are called *distances* of the circulant graph and L is called the *set of distances*.

Theorem 1.12. [12]. *Any connected circulant of degree* 4 *can be decomposed into Hamilton cycles.*

Theorem 1.13. [16, 19, 34]. *Suppose* G *admits a decomposition into cycles of length* k *. Then for each positive integer* ℓ *the graph* $G \circ \overline{K_\ell}$ *admits a* C_k *-decomposition and also a* $C_{k\ell}$ *-decomposition.*

2 C_{p^2}-Decomposition of $C_n \times K_m$

Lemma 2.1. *For* $n \in \{3, 5\}$ *,* $C_9 \,|\, C_n \times K_{10}$ *.*

Proof. Let $G = C_n \times K_{10}$ ($\cong K_{10} \times C_n$). Let H be the graph of Fig. 2. Let the vertex sets of K_{10} and C_n be $\{u_0, u_1, u_2, u_3, u_4, v_0, v_1, v_2, v_3, v_4\}$ and $\{1, 2, 3, \ldots, n\}$, respectively. Let $\sigma = (u_0\, u_1\, u_2\, u_3\, u_4)\,(v_0\, v_1\, v_2\, v_3\, v_4)$ be the permutation on the vertex set of K_{10}. Then $\sigma^i(H), 0 \le i \le 4$ decomposes K_{10}; see Fig. 3, that is, $H \,|\, K_{10}$.

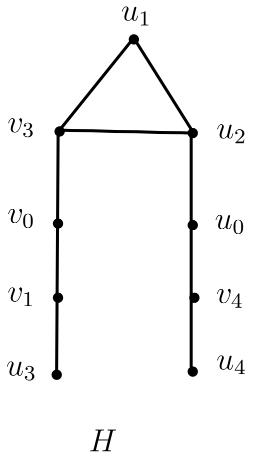
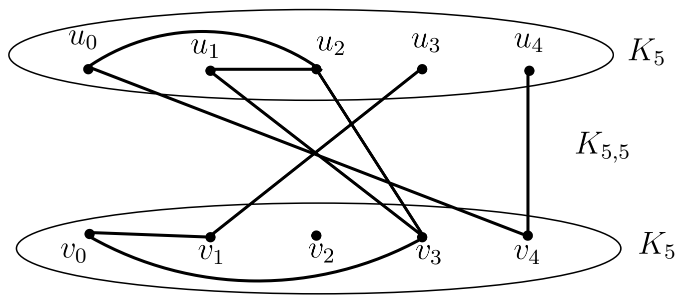
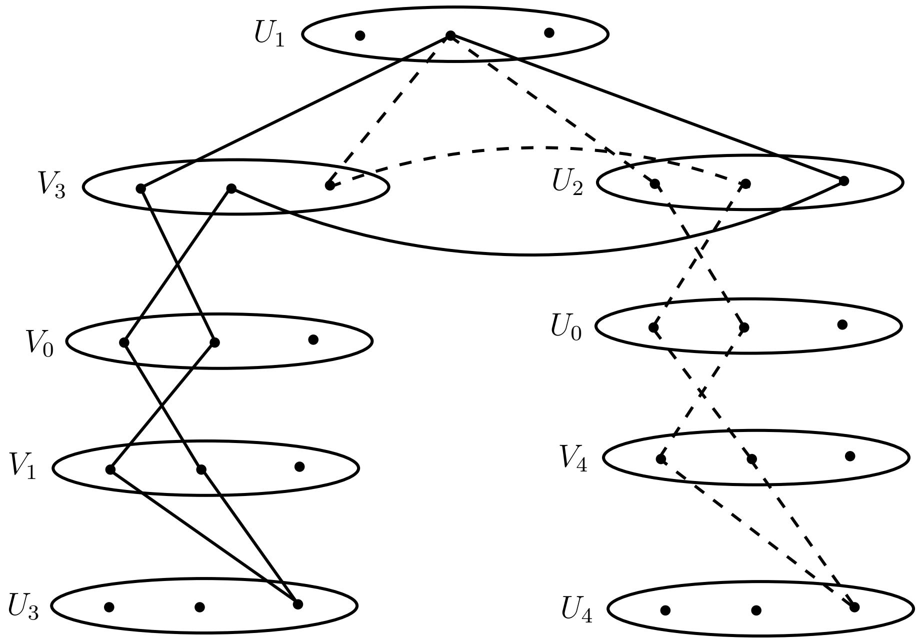


Fig. 2.



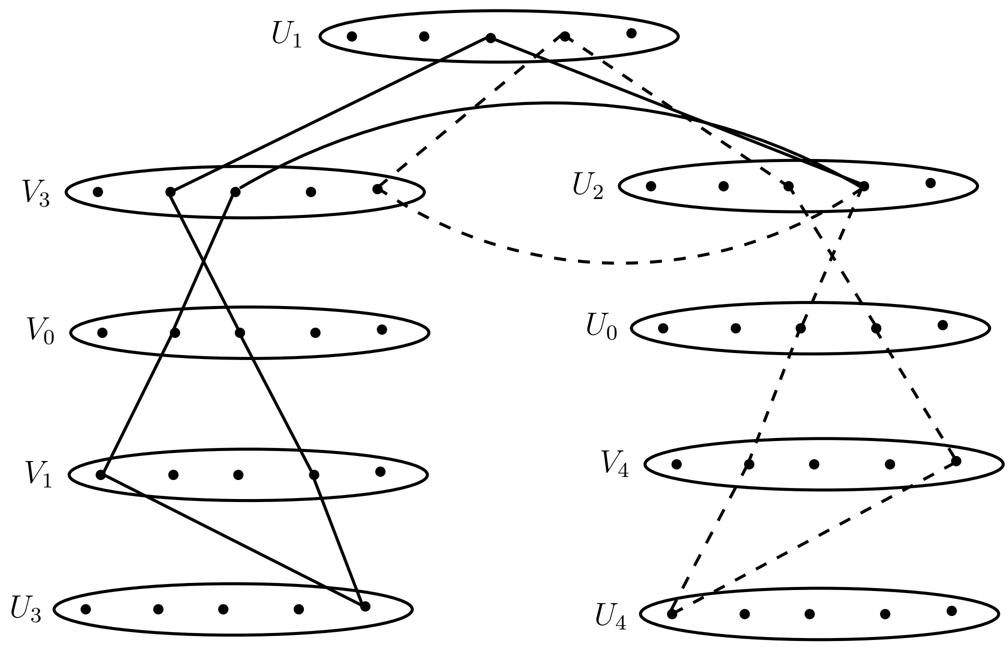
The K_{10} is shown as $K_5 \oplus K_5 \oplus K_{5,5}$.
A base copy of H in K_{10} is shown explicitly.
Fig. 3.

Since $K_{10} \times C_n = (H \oplus H \oplus \cdots \oplus H) \times C_n = (H \times C_n) \oplus (H \times C_n) \oplus \cdots \oplus (H \times C_n)$, it is enough to prove that $C_9 \,|\, H \times C_n$. For $n = 3$ (respectively, $n = 5$), two base cycles, each of length 9, in $H \times C_3$ (respectively, $H \times C_5$), are shown in Fig. 4 (respectively, Fig. 5), where the sets of vertices $\{u_i\} \times V(C_n)$ and $\{v_i\} \times V(C_n)$, $0 \le i \le 4$, are shown as U_i and V_i, $0 \le i \le 4$, respectively. Rotating each of these base cycles (of the graphs of Figures 4 and 5) successively, to the right $n - 1$ times, we get a desired C_9-decomposition of $H \times C_n \cong C_n \times H$. $\square$



Two base cycles, each of length 9, of $H \times C_3$ are given in solid and broken edges.

Fig. 4.



Two base cycles, each of length 9, of $H \times C_5$ are given in solid and broken edges.

Fig. 5.

The following lemma is proved in [22].

Lemma 2.2. [22]. *For any odd integer* $t \geq 11$ *, we have* $C_t \,|\, C_3 \times K_{t+1}$ *.*

Combining Lemmas 2.1 and 2.2, we obtain the following lemma:

Lemma 2.3. *For any odd integer* $t \geq 9$ *, we have* $C_t \,|\, C_3 \times K_{t+1}$.

The following lemma is proved in [22].

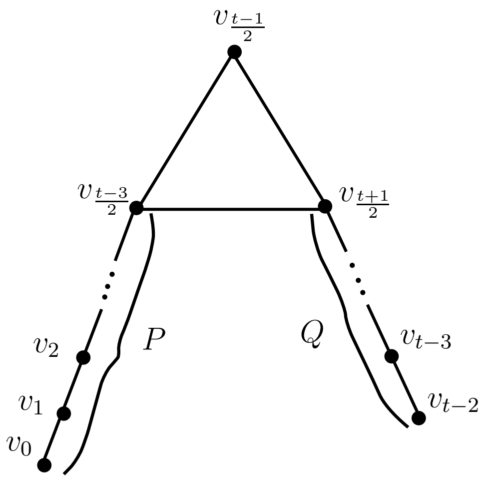
Lemma 2.4. [22]. *For any odd integer* $t \geq 11$ *,* $C_t \,|\, C_5 \times K_{t+1}$.

Combining Lemmas 2.1 and 2.4, we obtain the following lemma:

Lemma 2.5. *For any odd integer* $t \geq 9$ *,* $C_t \,|\, C_5 \times K_{t+1}$.

Lemma 2.6. *For any odd integer* $t \geq 5$ *,* $C_t \,|\, C_3 \times K_{2t}$ *, and for any odd integer* $t \geq 7$ *,* $C_t \,|\, C_5 \times K_{2t}$.

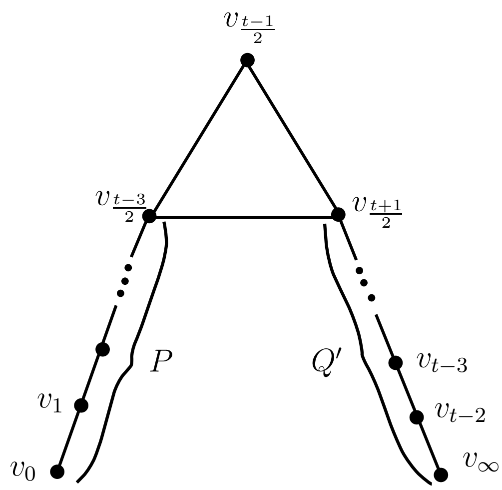
Proof. Let the graph $G = C_n \times K_{2t}$ $(\cong K_{2t} \times C_n)$. Denote the vertex set of K_{2t} by $\{v_\infty, v_0, v_1, v_2, \ldots, v_{2t-2}\}$. Let H_1 be the graph of Fig. 6 with $t-1$ vertices and $t-1$ edges with $V(H_1) = \{v_0, v_1, v_2, \ldots, v_{t-2}\}$.



Length of $P = (t-3)/2$ and length of $Q = (t-5)/2$.

H_1

Fig. 6.



Length of $P =$ length of $Q' = (t-3)/2$.

$H = H_1 \cup \{v_{t-2}v_\infty\}$.

Fig. 7.

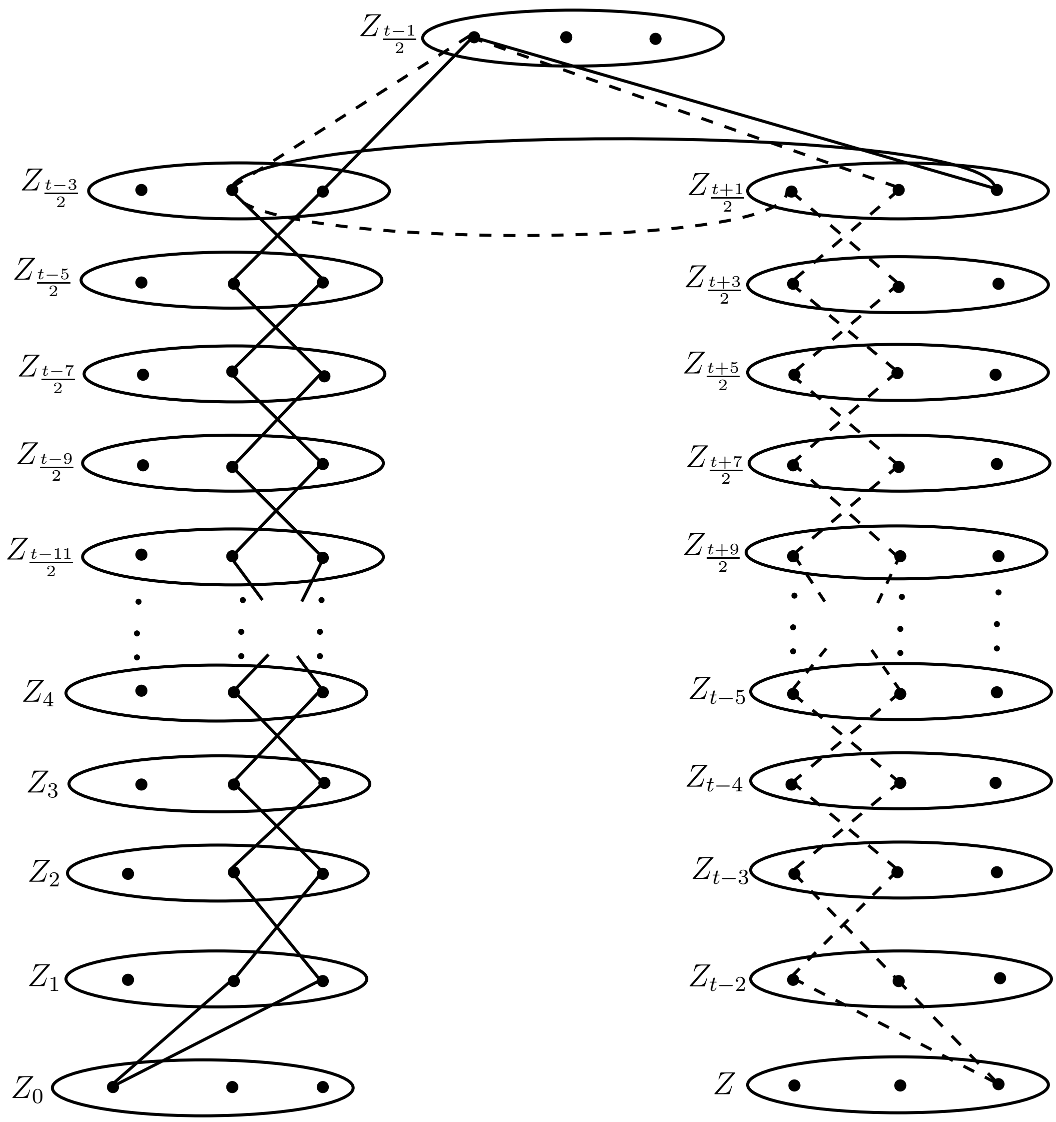
H_1 is a graceful graph with the following labeling:

Label the vertex v_i with the label a_i , where $a_i = \begin{cases} i/2 & \text{if } i \text{ is even,} \\ t - (\frac{i+1}{2}) & \text{if } i \text{ is odd.} \end{cases}$

The graph H_1 is graceful, and so $H_1 \,|\, K_{2t-1}$, by Theorem 1.11. Let H be a graph obtained from H_1 by adding the edge $v_{t-2}v_\infty$; see Fig. 7. Since $H_1 \,|\, K_{2t-1}$, we have $H \,|\, K_{2t}$.

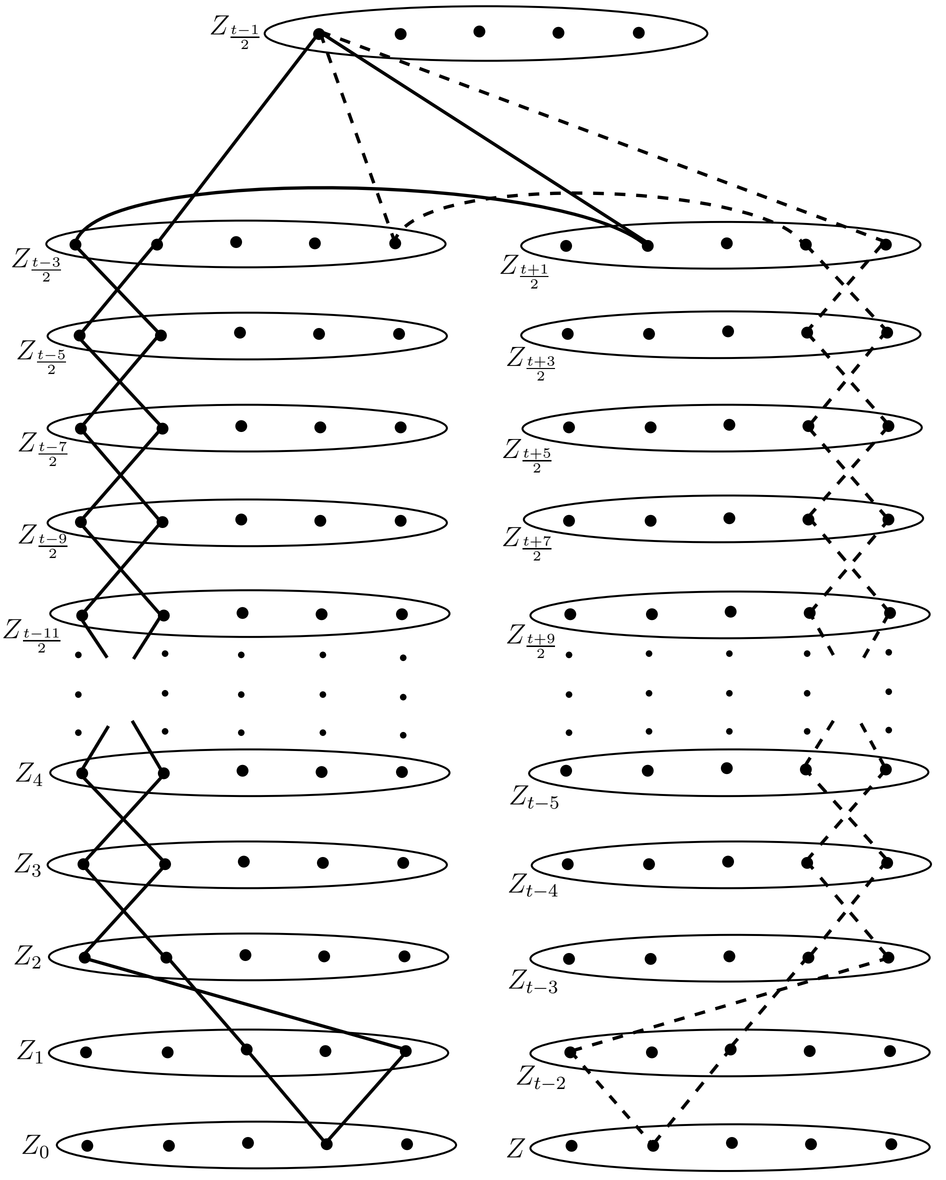
Since $K_{2t} \times C_n = (H \oplus H \oplus \cdots \oplus H) \times C_n = (H \times C_n) \oplus (H \times C_n) \oplus \cdots \oplus (H \times C_n)$, it is enough to prove that $C_t \,|\, H \times C_n$. Let $V(C_n) = \{1, 2, 3, \ldots, n\}$ and let $V(H) =$

$\{v_\infty, v_0, v_1, v_2, \ldots, v_{t-2}\}$; see Fig. 7. For $n = 3$ (respectively, $n = 5$), two base cycles, each of length t in $H \times C_3$ (respectively, $H \times C_5$), are shown in Fig. 8 (respectively, Fig. 9), where $\{v_\infty\} \times V(C_n)$ and $\{v_i\} \times V(C_n)$ are shown as Z and Z_i, $0 \le i \le t-2$, respectively. Rotating each of these base cycles (of the graph of Figs. 8 and 9) successively, to the right $n-1$ times, we obtain a C_t-decomposition of $H \times C_n \cong C_n \times H$. □



Two base cycles, each of length t, of $H \times C_3$ are given in solid and broken edges.

Fig. 8.



Two base cycles, each of length t, of $H \times C_5$ are given in solid and broken edges.

Fig. 9.

Theorem 2.1. *If $p \geq 3$ is a prime and $m \equiv 0$ or $1 \pmod{p^2}$, then $C_{p^2} \mid C_3 \times K_m$.*

Proof. We prove this theorem in two cases.

Case 1. $m \equiv 1 \pmod{p^2}$.
Let $m = kp^2 + 1$. If $k = 1$, then the result follows by Lemma 2.3. If $k = 2$, then $C_{p^2} \mid K_{2p^2+1}$, by Theorem 1.2. Consequently, $C_3 \times K_{2p^2+1} = (C_3 \times C_{p^2}) \oplus (C_3 \times C_{p^2}) \oplus \cdots \oplus (C_3 \times C_{p^2})$ and $C_{p^2} \mid C_3 \times C_{p^2} \cong C_{p^2} \times C_3$, by Theorem 1.8. Next assume that

$k \geq 3$. Consider the graph

$$\begin{aligned} C_3 \times K_{kp^2+1} &= C_3 \times \big(\underbrace{K_{p^2+1} \oplus K_{p^2+1} \oplus \cdots \oplus K_{p^2+1}}_{k \text{ times}} \oplus K_k \circ \overline{K}_{p^2}\big) \\ &= \underbrace{C_3 \times K_{p^2+1} \oplus C_3 \times K_{p^2+1} \oplus \cdots \oplus C_3 \times K_{p^2+1}}_{k \text{ times}} \oplus C_3 \times (K_k \circ \overline{K}_{p^2}) \\ &= \underbrace{C_3 \times K_{p^2+1} \oplus C_3 \times K_{p^2+1} \oplus \cdots \oplus C_3 \times K_{p^2+1}}_{k \text{ times}} \oplus (C_3 \times K_k) \circ \overline{K}_{p^2} \end{aligned}$$

By Lemma 2.3, $C_{p^2} \,|\, C_3 \times K_{p^2+1}$. Since $C_3 \,|\, C_3 \times K_k$, by Theorem 1.8, we have $(C_3 \times K_k) \circ \overline{K}_{p^2} = (C_3 \oplus C_3 \oplus \cdots \oplus C_3) \circ \overline{K}_{p^2} = (C_3 \circ \overline{K}_{p^2}) \oplus (C_3 \circ \overline{K}_{p^2}) \oplus \cdots \oplus (C_3 \circ \overline{K}_{p^2})$. By Theorem 1.4, $C_{p^2} \,|\, C_3 \circ \overline{K}_{p^2}$.

Case 2. $m \equiv 0 \pmod{p^2}$.
Let $m = kp^2$. If $k = 1$, then $C_3 \times K_{p^2} = (C_3 \times C_{p^2}) \oplus (C_3 \times C_{p^2}) \oplus \cdots \oplus (C_3 \times C_{p^2})$ and $C_{p^2} \,|\, C_3 \times C_{p^2}$, by Theorem 1.8. If $k = 2$, then the result follows by Lemma 2.6. Assume that $k \geq 3$. The graph $C_3 \times K_{kp^2} = C_3 \times \big(kK_{p^2} \oplus (K_k \circ \overline{K}_{p^2})\big) = k(C_3 \times K_{p^2}) \oplus C_3 \times (K_k \circ \overline{K}_{p^2}) = k(C_3 \times K_{p^2}) \oplus (C_3 \times K_k) \circ \overline{K}_{p^2}$. Clearly $C_3 \times K_{p^2} = (C_3 \times C_{p^2}) \oplus (C_3 \times C_{p^2}) \oplus \cdots \oplus (C_3 \times C_{p^2})$ and $C_{p^2} \,|\, C_3 \times C_{p^2}$, by Theorem 1.8. The graph $(C_3 \times K_k) \circ \overline{K}_{p^2} = (C_3 \circ \overline{K}_{p^2}) \oplus (C_3 \circ \overline{K}_{p^2}) \oplus \cdots \oplus (C_3 \circ \overline{K}_{p^2})$, by Theorem 1.8 and $C_{p^2} \,|\, C_3 \circ \overline{K}_{p^2}$, by Theorem 1.4. □

The proof of the following lemma is similar to the proof of Lemma 3.1 of [22].

Lemma 2.7. *For any odd integer $t \geq 5$, $C_t \,||\, C_5 \times C_t$.*

Theorem 2.2. *If $p \geq 3$ is a prime and $m \equiv 0$ or $1 \pmod{p^2}$, then $C_{p^2} \,|\, C_5 \times K_m$.*

Proof. We prove the theorem in two cases.

Case 1. $m \equiv 1 \pmod{p^2}$.
Let $m = kp^2 + 1$. If $k = 1$, then the result follows by Lemma 2.5. If $k = 2$, then $C_{p^2} \,|\, K_{2p^2+1}$, by Theorem 1.2. Consequently, $C_5 \times K_{2p^2+1} = (C_5 \times C_{p^2}) \oplus (C_5 \times C_{p^2}) \oplus \cdots \oplus (C_5 \times C_{p^2})$. Now $C_{p^2} \,|\, C_5 \times C_{p^2}$ by Lemma 2.7. Next assume that $k \geq 3$. Consider the graph

$$\begin{aligned} C_5 \times K_{kp^2+1} &= C_5 \times \big(\underbrace{K_{p^2+1} \oplus K_{p^2+1} \oplus \cdots \oplus K_{p^2+1}}_{k \text{ times}} \oplus K_k \circ \overline{K}_{p^2}\big) \\ &= \underbrace{C_5 \times K_{p^2+1} \oplus C_5 \times K_{p^2+1} \oplus \cdots \oplus C_5 \times K_{p^2+1}}_{k \text{ times}} \oplus (C_5 \times K_k) \circ \overline{K}_{p^2}. \end{aligned}$$

By Lemma 2.5, $C_{p^2} \,|\, C_5 \times K_{p^2+1}$. Since $C_5 \,|\, C_5 \times K_k$, by Theorem 1.8, $(C_5 \times K_k) \circ \overline{K}_{p^2} = (C_5 \oplus C_5 \oplus \cdots \oplus C_5) \circ \overline{K}_{p^2} = (C_5 \circ \overline{K}_{p^2}) \oplus (C_5 \circ \overline{K}_{p^2}) \oplus \cdots \oplus (C_5 \circ \overline{K}_{p^2})$. By Theorem 1.4, $C_{p^2} \,|\, C_5 \circ \overline{K}_{p^2}$.

Case 2. $m \equiv 0 \pmod{p^2}$.
Let $m = kp^2$. If $k = 1$, then $C_5 \times K_{p^2} = (C_5 \times C_{p^2}) \oplus (C_5 \times C_{p^2}) \oplus \cdots \oplus (C_5 \times C_{p^2})$ and $C_{p^2} \,|\, C_5 \times C_{p^2}$, by Lemma 2.7. If $k = 2$, then the result follows by Lemma 2.6.

Assume that $k \geq 3$. The graph $C_5 \times K_{kp^2} = C_5 \times \big(kK_{p^2} \oplus (K_k \circ \overline{K}_{p^2})\big) = k(C_5 \times K_{p^2}) \oplus (C_5 \times K_k) \circ \overline{K}_{p^2}$. Clearly $C_5 \times K_{p^2} = (C_5 \times C_{p^2}) \oplus (C_5 \times C_{p^2}) \oplus \cdots \oplus (C_5 \times C_{p^2})$ and $C_{p^2} \mid C_5 \times C_{p^2}$, by Lemma 2.7. The graph $(C_5 \times K_k) \circ \overline{K}_{p^2} = (C_5 \circ \overline{K}_{p^2}) \oplus (C_5 \circ \overline{K}_{p^2}) \oplus \cdots \oplus (C_5 \circ \overline{K}_{p^2})$, by Theorem 1.8 and $C_{p^2} \mid C_5 \circ \overline{K}_{p^2}$, by Theorem 1.4. □

3 C_{rs}-Decomposition of $G \times K_m$

Lemma 3.1. *Let $s \geq 7$ be an odd integer and $r \geq 3$. Then there exists a directed s-cycle decomposition $\mathscr{C}$ of K_{2s}^* such that each arc of the directed cycles of $\mathscr{C}$ can be assigned the label 1 or $r-1$, so that both the symmetric pair of arcs (a_i, a_j) and (a_j, a_i) receive the same label 1 or $r-1$ and the sum of the labels of the arcs on each of the directed cycles of $\mathscr{C}$ is relatively prime to r.*

Proof. Let $s = 2k+1$. Let $A = \{a_0, a_1, \ldots, a_{s-1}\}$ and $B = \{b_0, b_1, \ldots, b_{s-1}\}$ be a partition of the vertex set of K_{2s}^*. Let $\sigma = (a_0\, a_1 \ldots a_{s-1})(b_0\, b_1 \ldots b_{s-1})$ be a permutation of the vertex set of $V(K_{2s}^*)$. We obtain a $\overrightarrow{C_s}$-decomposition $\mathscr{C}$ of K_{2s}^* as follows: let $\overrightarrow{C} = a_0\, b_{2k}\, a_1\, b_{2k-1}\, a_2\, b_{2k-2} \ldots b_{k+1}\, a_k\, a_0$ be a (base) directed cycle of length s, which has exactly one arc, namely (a_k, a_0), of distance $-k$ in $\langle A \rangle$ and all of its other arcs join vertices of A and B. Consider the following s directed s-cycles $\mathscr{C}' = \{\overrightarrow{C}, \sigma(\overrightarrow{C}), \sigma^2(\overrightarrow{C}), \ldots, \sigma^{s-1}(\overrightarrow{C})\}$ of K_{2s}^*. Let $\mathscr{C}_0'$ be the set of reverse directed cycles of $\mathscr{C}'$. Let $\mathscr{C}''$ be the $2s$ directed cycles in $\mathscr{C}' \cup \mathscr{C}_0'$. The above directed base cycle $\overrightarrow{C}$ is constructed based on a technique in [5].

Now assign the label 1 or $r-1$ to the arcs of the directed cycles of $\mathscr{C}''$ as follows: alternately assign the labels 1 and $r-1$, beginning from the first arc of each of the directed cycles in $\mathscr{C}' \subseteq \mathscr{C}''$, so that the first and last arcs of the directed s-cycle get the label 1. If the arc (x, y) of a directed s-cycle in $\mathscr{C}'$ is assigned the label 1 (respectively, $r-1$), its corresponding arc (y, x) in a directed s-cycle of $\mathscr{C}_0'$ is also assigned the same label 1 (respectively, $r-1$). Thus every symmetric pair of arcs (x, y) and (y, x) on the directed cycles of $\mathscr{C}''$ receive the same label. It can be easily verified that sum of the labels of the arcs of each of the directed cycles in $\mathscr{C}''$ is $r\lfloor \frac{s}{2} \rfloor + 1$, which is relatively prime to r.

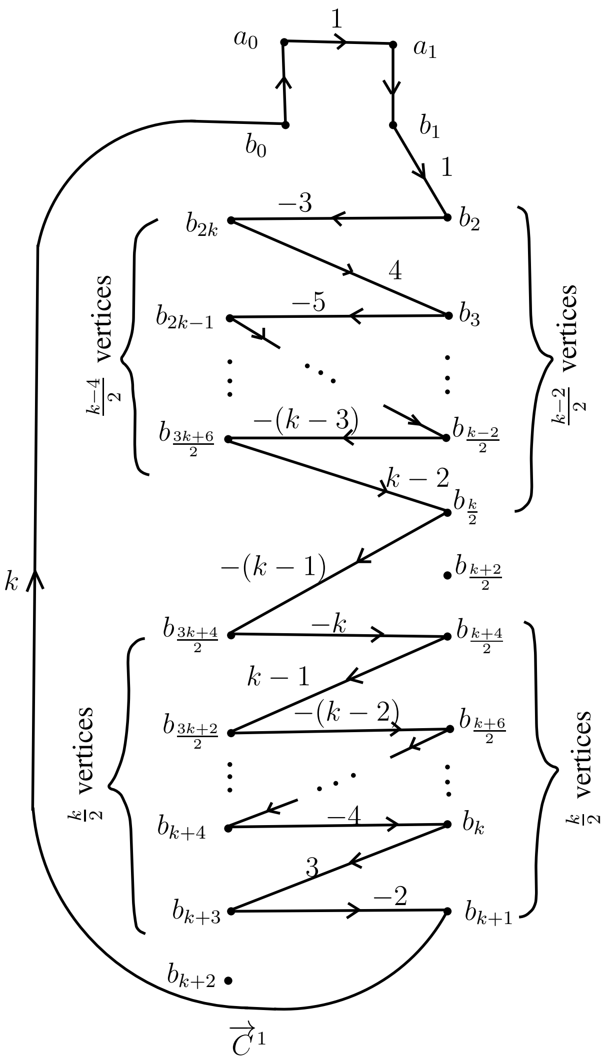
Clearly, $K_{2s}^* - A(\mathscr{C}'')$, where $A(\mathscr{C}'')$ denotes the arc set of $\mathscr{C}''$, is the subdigraph of K_{2s}^* having the arc set $X = \{(a_i, b_i), (b_i, a_i) \mid 0 \leq i \leq s-1\} \cup \{$all arcs of distances $\pm 1, \pm 2, \ldots, \pm(k-1)$ in $\langle A \rangle$ and all arcs of distances $\pm 1, \pm 2, \ldots, \pm k,$ in $\langle B \rangle\}$. (Note that arcs of distances $\pm k$ of A have already been used by the directed cycles of $\mathscr{C}''$.)

Next, we obtain a $\overrightarrow{C_s}$-decomposition of $K = K_{2s}^* - A(\mathscr{C}'')$ and assign the label 1 or $r-1$ to the arcs of the directed cycles of the decomposition.

Case 1. k is even.
In $K = K_{2s}^* - A(\mathscr{C}'')$, consider the (base) directed s-cycle

$$\overrightarrow{C}^1 : b_0\, a_0\, a_1\, b_1\, b_2\, b_{2k}\, b_3\, b_{2k-1} \ldots b_{\frac{3k+4}{2}}\, b_{\frac{k+4}{2}}\, b_{\frac{3k+2}{2}}\, b_{\frac{k+6}{2}} \ldots b_k\, b_{k+3}\, b_{k+1}\, b_0;$$

see Fig. 10 and the directed s-cycles $\mathscr{D} = \{\overrightarrow{C}^1, \sigma(\overrightarrow{C}^1), \sigma^2(\overrightarrow{C}^1), \ldots, \sigma^{s-1}(\overrightarrow{C}^1)\}$, where $\sigma = (a_0\, a_1\, a_2 \ldots a_{s-1})(b_0\, b_1\, b_2 \ldots b_{s-1})$; observe that $\overrightarrow{C}^1$ misses exactly two of the b_j, namely, $b_{(k+2)/2}$ and b_{k+2}. Each of the directed cycles of $\mathscr{D}$ contains exactly two arcs from the set $\{(a_i, b_i), (b_i, a_i) \mid 0 \leq i \leq s-1\}$ and the remaining $s-2$ arcs have the distances $\pm 3, \pm 4, \ldots, \pm k$, 1, -2 in $\langle B \rangle$ and the distance 1 in $\langle A \rangle$; see Fig. 10.



A (base) directed cycle $\overrightarrow{C}^1$ in the graph K.

Fig. 10.

Next we assign the label 1 or $r-1$ to the arcs of the directed cycles in $\mathscr{D}$ as follows: we partition the distances $\{\pm 3, \pm 4, \ldots, \pm k\}$ of the arcs of $\sigma^i(\overrightarrow{C}^1)$, $0 \leq i \leq s-1$ into two sets $X = \{\pm 3, \pm 5, \pm 7 \ldots, \pm(k-1)\}$ and $Y = \{\pm 4, \pm 6, \pm 8 \ldots, \pm k\}$ and assign the label 1 (respectively, $r-1$) to the arcs with distances in X (respectively, Y). We have yet to label the five arcs of $\sigma^i(\overrightarrow{C}^1)$, $0 \leq i \leq s-1$, namely, $\big(\sigma^i(b_0), \sigma^i(a_0)\big)$, $\big(\sigma^i(a_0), \sigma^i(a_1)\big)$, $\big(\sigma^i(a_1), \sigma^i(b_1)\big)$, $\big(\sigma^i(b_1), \sigma^i(b_2)\big)$ and $\big(\sigma^i(b_{k+3}), \sigma^i(b_{k+1})\big)$. Other than these five arcs in each of the directed cycles of $\mathscr{D}$, the sum of the labels of the arcs is a multiple of r, as both 1 and $r-1$ occur an equal number of times. Now we label the three arcs $\big(\sigma^i(b_0), \sigma^i(a_0)\big)$, $\big(\sigma^i(a_0), \sigma^i(a_1)\big)$, $\big(\sigma^i(a_1), \sigma^i(b_1)\big)$, $0 \le i \le s-1$, of $\sigma^i(\overrightarrow{C}^1)$ as shown in Fig. 11; assign the label of the other two arcs, $\big(\sigma^i(b_1), \sigma^i(b_2)\big)$, $\big(\sigma^i(b_{k+3}), \sigma^i(b_{k+1})\big)$, $0 \le i \le s-1$, of $\sigma^i(\overrightarrow{C}^1)$ as follows: for $0 \le \ell \le (s-1)/2$, the arc $\big(\sigma^{2\ell}(b_1), \sigma^{2\ell}(b_2)\big)$ is assigned the label 1 and the arc $\big(\sigma^{2\ell}(b_{k+3}), \sigma^{2\ell}(b_{k+1})\big)$ is assigned the label $r-1$, and for $0 \le \ell \le (s-3)/2$, let the arcs $\big(\sigma^{2\ell+1}(b_1), \sigma^{2\ell+1}(b_2)\big)$ and $\big(\sigma^{2\ell+1}(b_{k+3}), \sigma^{2\ell+1}(b_{k+1})\big)$ be assigned the labels $r-1$ and 1, respectively. On these five arcs the sum of the labels is relatively prime to r. The sum of the labels of each directed s-cycle in $\mathscr{D}$ is either $r\lceil \frac{s}{2} \rceil - 1$ or $r\lfloor \frac{s}{2} \rfloor + 1$, which is relatively prime to r.

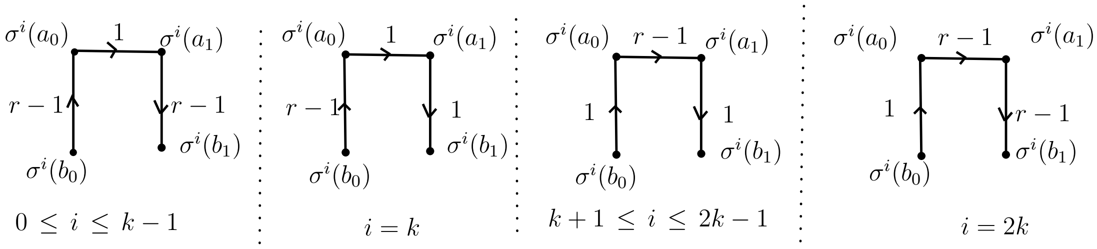


Fig. 11.

The directed s-cycles in $\mathscr{D}$ do not contain all the arcs of $K = K_{2s}^* - A(\mathscr{C}'')$. Now, consider the subdigraph $R = K_{2s}^* - A(\mathscr{C}'' \cup \mathscr{D})$, consisting of all arcs of distances $\pm 2, \pm 3, \ldots, \pm(k-1), -1$, in $\langle A \rangle$, and the arcs of distances -1, 2 in $\langle B \rangle$. The subdigraph R will be decomposed into directed cycles of the required type after considering the next case.

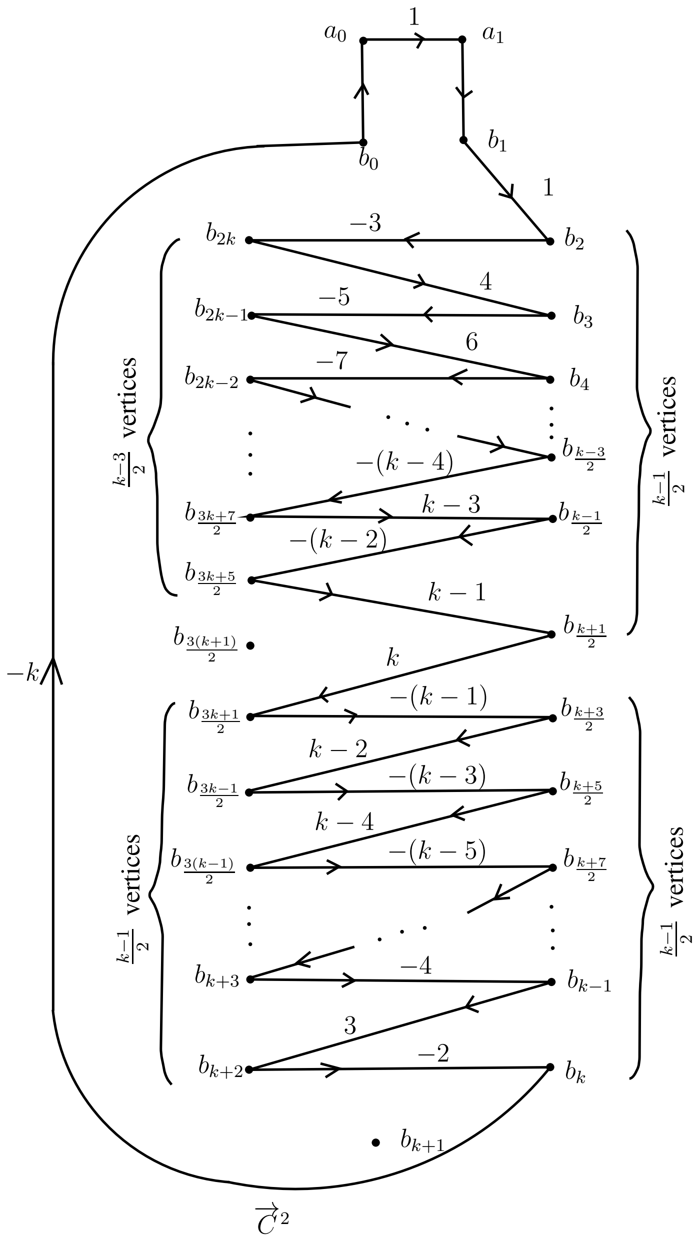
Case 2. k is odd.

In $K = K_{2s}^* - A(\mathscr{C}'')$, consider the (base) directed s-cycle

$$\overrightarrow{C}^2 : b_0 a_0 a_1 b_1 b_2 b_{2k} b_3 b_{2k-1} \ldots b_{\frac{k+1}{2}} b_{\frac{3k+1}{2}} b_{\frac{k+3}{2}} b_{\frac{3k-1}{2}} \ldots b_{k+2} b_k b_0;$$

see Fig. 12 and the directed s-cycles $\mathscr{D} = \{\overrightarrow{C}^2, \sigma(\overrightarrow{C}^2), \sigma^2(\overrightarrow{C}^2), \ldots, \sigma^{s-1}(\overrightarrow{C}^2)\}$, where $\sigma = (a_0\, a_1 \ldots a_{s-1})(b_0\, b_1 \ldots b_{s-1})$; observe that $\overrightarrow{C}^2$ misses exactly two of the b_j, namely, $b_{3(k+1)/2}$ and b_{k+1}. Each of the directed cycles of $\mathscr{D}$ contains exactly two arcs from the set $\{(a_i, b_i), (b_i, a_i) \,|\, 0 \le i \le s-1\}$ and the remaining $s-2$ arcs have the distances $\pm 3, \pm 4, \ldots, \pm k, 1, -2$ in $\langle B \rangle$ and the distance 1 in $\langle A \rangle$; see Fig. 12.

Next we assign the label 1 or $r-1$ to the arcs of the directed cycles in $\mathscr{D}$ as follows: we partition the distances $\{\pm 3, \pm 4, \ldots, \pm k\}$ of the arcs of $\sigma^i(\overrightarrow{C}^2)$, $0 \le i \le s-1$, into two sets $L = \{\pm 3, \pm 5, \pm 7 \ldots, \pm k\}$ and $M = \{\pm 4, \pm 6, \pm 8 \ldots, \pm(k-1)\}$ and assign the label 1 (respectively, $r-1$) to the arcs of $\sigma^i(\overrightarrow{C}^2)$ with distances in L (respectively, M) and assign the label $r-1$ to the arcs $\sigma^i(b_0, a_0)$ and $\sigma^i(a_1, b_1)$, $0 \le i \le s-1$. We have yet to label the three arcs of $\sigma^i(\overrightarrow{C}^2)$, $0 \le i \le s-1$, namely, $\big(\sigma^i(a_0), \sigma^i(a_1)\big)$, $\big(\sigma^i(b_1), \sigma^i(b_2)\big)$ and $\big(\sigma^i(b_{k+2}), \sigma^i(b_k)\big)$, $0 \le i \le s-1$. Apart from these three arcs of $\sigma^i(\overrightarrow{C}^2)$, the sum of the labels of arcs of each of the directed cycles of $\mathscr{D}$ is a multiple of r, because both 1 and $r-1$ occur an equal number of times.



A (base) directed cycle $\overrightarrow{C}^2$ in the graph K.

Fig. 12.

We label these three arcs as follows: for $0 \le \ell \le (s-1)/2$, assign the label 1 to the arcs $\big(\sigma^{2\ell}(b_1), \sigma^{2\ell}(b_2)\big)$ and $\big(\sigma^{2\ell}(b_{k+2}), \sigma^{2\ell}(b_k)\big)$ and the label $r-1$ to the arc $\big(\sigma^{2\ell}(a_0), \sigma^{2\ell}(a_1)\big)$, and for $0 \le \ell \le (s-3)/2$, assign the label $r-1$ to the arcs $\big(\sigma^{2\ell+1}(b_1), \sigma^{2\ell+1}(b_2)\big)$ and $\big(\sigma^{2\ell+1}(b_{k+2}), \sigma^{2\ell+1}(b_k)\big)$ and the label 1 to the arc $\big(\sigma^{2\ell+1}(a_0), \sigma^{2\ell+1}(a_1)\big)$; on these three arcs the sum of the labels is relatively prime to r.

Observe that the sum of the labels of each of the directed s-cycles in $\mathscr{D}$ is either $r\lceil\frac{s}{2}\rceil - 1$ or $r\lfloor\frac{s}{2}\rfloor + 1$, which is relatively prime to r. The directed s-cycles in $\mathscr{D}$ do not contain all the arcs of $K = K^*_{2s} - A(\mathscr{C}'')$. Now, consider the subdigraph $R = K^*_{2s} - A(\mathscr{C}'' \cup \mathscr{D})$, consisting of all arcs of distances $\pm 2, \pm 3, \ldots, \pm(k-1), -1$, in $\langle A\rangle$, and the arcs of distances $-1, 2$ in $\langle B\rangle$.

Next, irrespective of the parity of k, we show that $R = K^*_{2s} - A(\mathscr{C}'' \cup \mathscr{D})$ can be decomposed into $\overrightarrow{C}_s$. Clearly, the arc set of R is all arcs of distances $\{\pm 2, \pm 3, \ldots, \pm(k-1), -1$, in $\langle A\rangle\} \cup \{$all arcs of distances $-1, 2$ in $\langle B\rangle\}$; the arcs of R are contained in $\langle A\rangle \cup \langle B\rangle$. Let $\overrightarrow{C}^3$ and $\overrightarrow{C}^4$ be two directed s-cycles of $\langle B\rangle$ induced by the arcs of distances -1 and 2, in $\langle B\rangle$, respectively, and $\overrightarrow{C}^5$ be the directed s-cycle induced by the arcs of distance 1 in $\langle A\rangle$. Each arc in $A(\overrightarrow{C}^3) \cup A(\overrightarrow{C}^4) \cup A(\overrightarrow{C}^5)$ is assigned the same label of its symmetric arc on the directed cycles of $\mathscr{D}$. In each of the directed s-cycles we have constructed so far, including $\mathscr{D}$, the number of arcs receiving the label 1 and $r-1$ differs by one. Hence the sum of labels of the arcs of each of these directed s-cycles is relatively prime to r. The remaining arc set of the digraph $R' = R - \big(A(\overrightarrow{C}^3) \cup A(\overrightarrow{C}^4) \cup A(\overrightarrow{C}^5)\big)$ is $\{$all arcs of distances $\pm 2, \pm 3, \ldots, \pm(k-1)$ in $\langle A\rangle\}$.

If k is even, then consider the pairs of distances $\{2,3\}, \{4,5\}, \ldots, \{k-2, k-1\}$ in $\langle A\rangle$. Clearly, each pair of distances gives rise to a circulant graph in $U(\langle A\rangle)$, the underlying graph of $\langle A\rangle$, that can be decomposed into two hamilton cycles (of length s), by Theorem 1.12. Each of these hamilton cycles of $U(\langle A\rangle)$ is oriented into two directed hamilton cycles of R', one in the clockwise direction and the other in the anticlockwise direction. The reverse cycles of these two directed cycles give another two directed hamilton cycles, taking care of the distances $-i$ and $-(i+1)$ in $\langle A\rangle$. Then each pair $(i, i+1)$ gives rise to four directed s-cycles of R'. We assign the labels 1 and $r-1$ alternately to the directed cycles so that the symmetric pair of arcs, that is, the arcs (a_i, a_j) and (a_j, a_i), receive the same label 1 or $r-1$. Clearly, the sum of the labels of each of the directed cycles is $r\lfloor\frac{s}{2}\rfloor + 1$, which is relatively prime to r.

If k is odd, then as above, pair the distances, except the distances ± 2, as follows: $\{3,4\}, \{5,6\}, \ldots, \{k-2, k-1\}$. Repeating the same argument as in the last paragraph, we obtain arc-disjoint labeled directed s-cycles of R'. The arcs of distance 2 (respectively, -2) induce a directed s-cycle. Let $\overrightarrow{C}^6$ (respectively, $\overrightarrow{C}^7$) be the directed s-cycle whose arcs have the distance 2 (respectively, -2) in $\langle A\rangle$. In fact, $\overrightarrow{C}^7$ is the reverse cycle of $\overrightarrow{C}^6$. The arcs of $\overrightarrow{C}^6$ are alternately labeled 1 and $r-1$ so that the first and last arcs of the directed s-cycle get the label 1 and each arc in $A(\overrightarrow{C}^7)$ is assigned the same label of its symmetric arc in $\overrightarrow{C}^6$. Thus, the sum of the labels of each of the directed cycles in R' is $r\lfloor\frac{s}{2}\rfloor + 1$, which is relatively prime to r. □

Lemma 3.2. *If $s \geq 3$ is an odd integer and $r \geq 3$, then $C_{rs} \mid C_r \times K_{2s}$.*

Proof. We complete the proof in three cases.

Case 1. $s \geq 7$.

Let $V(K_{2s}^*) = \{a_0, a_1, a_2, \ldots a_{s-1}, b_0, b_1, b_2, \ldots, b_{s-1}\}$. Obtain a directed s-cycle decomposition $\mathscr{C}$ of K_{2s}^* with arcs labeled 1 or $r-1$ as in the proof of Lemma 3.1. We assume that $V(K_{2s}) = V(K_{2s}^*)$.

To each directed s-cycle in $\mathscr{C}$, we construct an rs-cycle in $C_r \times K_{2s} \cong K_{2s} \times C_r$ as follows: let $\overrightarrow{C_s} \in \mathscr{C}$. For our convenience, let $\overrightarrow{C_s} = (w_0, w_1, \ldots, w_{s-1})$ and $C_r = (v_1, v_2, \ldots, v_r)$. Then $V(\overrightarrow{C_s}) \times V(C_r) = \bigcup_{i=0}^{s-1}\big(\{w_i\} \times V(C_r)\big) \subset V(K_{2s} \times C_r)$. We denote $\{w_i\} \times V(C_r)$ by W_i, $0 \leq i \leq s-1$. If the label of the arc (w_i, w_{i+1}) is $r-1$ (respectively, 1) in $\overrightarrow{C_s}$, then consider the set of edges $F_{r-1}(W_i, W_{i+1})$ (respectively, $F_1(W_i, W_{i+1})$) from W_i to W_{i+1} in $\langle W_i \cup W_{i+1}\rangle \subseteq C_s \times C_r$; in fact, the label of an arc, namely, 1 or $r-1$ of the directed s-cycle is used to decide the distance of the 1-factor from W_i to W_{i+1} in $\langle W_i \cup W_{i+1}\rangle$; recall that $F_1(W_i, W_{i+1}) = F_{r-1}(W_{i+1}, W_i)$. Then $\bigcup_{i=0}^{s-1} F_{\ell_i}(W_i, W_{i+1})$, where ℓ_i is the label of the arc (w_i, w_{i+1}) in $\overrightarrow{C_s}$, is a cycle of length rs, as the sum $\sum_{i=0}^{s-1} \ell_i$ is relatively prime to r. Thus each directed cycle $\overrightarrow{C_s}$ of K_{2s}^* yields a cycle of length rs in the graph $K_{2s} \times C_r \cong C_r \times K_{2s}$.

Case 2. $s = 5$.

Let $V(K_{10}) = \{a_0, a_1, \ldots, a_4, b_0, b_1, \ldots, b_4\}$, $A_i = \{a_i\} \times V(C_r)$, $0 \leq i \leq 4$, and $B_i = \{b_i\} \times V(C_r)$, $0 \leq i \leq 4$. Let $\sigma = (A_0A_1A_2A_3A_4)(B_0B_1B_2B_3B_4)$ be a permutation, where A_i (respectively, B_j) is the partite set of $K_{10} \times C_r$ that corresponds to the vertex a_i (respectively, b_j) of K_{10}. Then the eighteen $5r$-cycles which decompose the graph $K_{10} \times C_r$ are listed below; (recall that $F_1(A_i, B_j) = F_{r-1}(B_j, A_i)$).

$C_1^i = F_1\big(\sigma^i(A_0), \sigma^i(B_4)\big) \oplus F_{r-1}\big(\sigma^i(B_4), \sigma^i(A_1)\big) \oplus F_1\big(\sigma^i(A_1), \sigma^i(B_3)\big) \oplus$
$\quad F_{r-1}\big(\sigma^i(B_3), \sigma^i(A_2)\big) \oplus F_1\big(\sigma^i(A_2), \sigma^i(A_0)\big)$, $0 \leq i \leq 4$;
$C_2^i = F_1\big(\sigma^i(A_0), \sigma^i(A_2)\big) \oplus F_{r-1}\big(\sigma^i(A_2), \sigma^i(B_3)\big) \oplus F_1\big(\sigma^i(B_3), \sigma^i(A_1)\big) \oplus$
$\quad F_{r-1}\big(\sigma^i(A_1), \sigma^i(B_4)\big) \oplus F_1\big(\sigma^i(B_4), \sigma^i(A_0)\big)$, $0 \leq i \leq 4$;
$C^{11} = F_1(A_0, A_1) \oplus F_{r-1}(A_1, B_1) \oplus F_1(B_1, B_2) \oplus F_{r-1}(B_2, B_0) \oplus F_{r-1}(B_0, A_0)$;
$C^{12} = F_1(A_1, A_2) \oplus F_{r-1}(A_2, B_2) \oplus F_{r-1}(B_2, B_3) \oplus F_1(B_3, B_1) \oplus F_{r-1}(B_1, A_1)$;
$C^{13} = F_1(A_2, A_3) \oplus F_1(A_3, B_3) \oplus F_1(B_3, B_4) \oplus F_{r-1}(B_4, B_2) \oplus F_{r-1}(B_2, A_2)$;
$C^{14} = F_{r-1}(A_3, A_4) \oplus F_1(A_4, B_4) \oplus F_{r-1}(B_4, B_0) \oplus F_1(B_0, B_3) \oplus F_1(B_3, A_3)$;
$C^{15} = F_{r-1}(A_4, A_0) \oplus F_{r-1}(A_0, B_0) \oplus F_1(B_0, B_1) \oplus F_{r-1}(B_1, B_4) \oplus F_1(B_4, A_4)$;
$C^{16} = F_{r-1}(A_0, A_4) \oplus F_{r-1}(A_4, A_3) \oplus F_1(A_3, A_2) \oplus F_1(A_2, A_1) \oplus F_1(A_1, A_0)$;
$C^{17} = F_{r-1}(B_0, B_2) \oplus F_{r-1}(B_2, B_4) \oplus F_{r-1}(B_4, B_1) \oplus F_1(B_1, B_3) \oplus F_1(B_3, B_0)$;
$C^{18} = F_1(B_1, B_0) \oplus F_{r-1}(B_0, B_4) \oplus F_1(B_4, B_3) \oplus F_{r-1}(B_3, B_2) \oplus F_1(B_2, B_1)$.

Case 3. $s = 3$.

Let $V(K_6) = \{w_1, w_2, \ldots, w_6\}$. Let $V(K_6 \times C_r) = V(K_6) \times V(C_r) = \bigcup_{i=1}^{6}\big(\{w_i\} \times V(C_r)\big)$. We denote $\{w_i\} \times V(C_r)$ by W_i, $1 \leq i \leq 6$. Then the ten $3r$-cycles which decompose the graph $K_6 \times C_r$ are listed below;
$C^1 = F_1(W_1, W_4) \oplus F_1(W_4, W_2) \oplus F_{r-1}(W_2, W_1)$;
$C^2 = F_1(W_2, W_5) \oplus F_{r-1}(W_5, W_3) \oplus F_1(W_3, W_2)$;
$C^3 = F_1(W_3, W_4) \oplus F_1(W_4, W_6) \oplus F_{r-1}(W_6, W_3)$;
$C^4 = F_1(W_4, W_1) \oplus F_{r-1}(W_1, W_5) \oplus F_1(W_5, W_4)$;

$C^5 = F_1(W_5, W_2) \oplus F_{r-1}(W_2, W_6) \oplus F_{r-1}(W_6, W_5)$;
$C^6 = F_1(W_6, W_1) \oplus F_{r-1}(W_1, W_3) \oplus F_{r-1}(W_3, W_6)$;
$C^7 = F_{r-1}(W_1, W_2) \oplus F_1(W_2, W_3) \oplus F_{r-1}(W_3, W_1)$;
$C^8 = F_1(W_4, W_3) \oplus F_{r-1}(W_3, W_5) \oplus F_{r-1}(W_5, W_4)$;
$C^9 = F_{r-1}(W_5, W_1) \oplus F_1(W_1, W_6) \oplus F_1(W_6, W_5)$;
$C^{10} = F_{r-1}(W_6, W_2) \oplus F_1(W_2, W_4) \oplus F_{r-1}(W_4, W_6)$. □

Corollary 3.1. *If $s \geq 3$ is an odd integer, $r \geq 3$ and $C_r \mid G$, then $C_{rs} \mid G \times K_{2s}$.*

Proof. As $C_r \mid G$, $G \times K_{2s} = (C_r \times K_{2s}) \oplus (C_r \times K_{2s}) \oplus \cdots \oplus (C_r \times K_{2s})$. Now apply Lemma 3.2 to the graph $C_r \times K_{2s}$. □

Lemma 3.3. *Let $(s, m) \neq (1, 2)$ and let $r \geq 3$ be an odd integer with $m \equiv 2 \pmod 4$. If $s \mid m$, then $C_{rs} \mid C_r \times K_m$.*

Proof. Since $m \equiv 2 \pmod 4$, let $m = 2\ell$, where ℓ is an odd integer.

Case 1. s is odd.

Subcase 1.1. $s = 1$.
By Theorem 1.8, $C_r \mid C_r \times K_m$.

Subcase 1.2. $s \geq 3$.
Since $s \mid m$, $s \mid \ell$. Let $m = 2st$, where t is an odd integer. If $t = 1$, then $s = \ell$ and hence $C_{rs} \mid C_r \times K_{2s}$, by Lemma 3.2. So we assume that $t \geq 3$. The graph $C_r \times K_m = C_r \times K_{2st} = C_r \times \big(tK_{2s} \oplus (K_t \circ \overline{K}_{2s})\big) = t(C_r \times K_{2s}) \oplus C_r \times (K_t \circ \overline{K}_{2s})$. By Lemma 3.2, $C_{rs} \mid (C_r \times K_{2s})$. Further, the graph

$$\begin{aligned} C_r \times (K_t \circ \overline{K}_{2s}) &= (C_r \times K_t) \circ \overline{K}_{2s} \\ &= (C_r \oplus C_r \oplus \cdots \oplus C_r) \circ \overline{K}_{2s}, \text{by Theorem 1.8} \\ &= \big((C_r \circ \overline{K}_s) \oplus (C_r \circ \overline{K}_s) \oplus \cdots \oplus (C_r \circ \overline{K}_s)\big) \circ \overline{K}_2 \\ &= (C_{rs} \oplus C_{rs} \oplus \cdots \oplus C_{rs}) \circ \overline{K}_2, \text{by Theorem 1.13} \\ &= (C_{rs} \circ \overline{K}_2) \oplus (C_{rs} \circ \overline{K}_2) \oplus \cdots \oplus (C_{rs} \circ \overline{K}_2) \\ &= C_{rs} \oplus C_{rs} \oplus \cdots \oplus C_{rs}, \text{ by Theorem 1.13.} \end{aligned}$$

Case 2. $s \geq 2$ is even.
Since $s \mid m$, $s = 2q$, where q is an odd integer.

Subcase 2.1. $q = 1$.

Now, $C_r \times K_m = (C_r \times K_2) \oplus (C_r \times K_2) \oplus \cdots \oplus (C_r \times K_2) = C_{2r} \oplus C_{2r} \oplus \cdots \oplus C_{2r}$, since r is odd.

Subcase 2.2. $q \geq 3$.

Let $m = 2qa = sa$, where a is an odd integer. If $a = 1$, then $s = m$. By Theorem 1.5, $C_{rs} \mid C_r \times K_{2q}$. So we assume that $a \geq 3$. Since $C_r \times K_{sa} = C_r \times \big(aK_s \oplus$

$(K_a \circ \overline{K}_s)) = a(C_r \times K_s) \oplus C_r \times (K_a \circ \overline{K}_s)$, by Theorem 1.5, $C_{rs} \,|\, C_r \times K_s$; further,

$$\begin{aligned} C_r \times (K_a \circ \overline{K}_s) &= (C_r \times K_a) \circ \overline{K}_s \\ &= (C_r \oplus C_r \oplus \cdots \oplus C_r) \circ \overline{K}_s, \text{by Theorem 1.8} \\ &= (C_r \circ \overline{K}_s) \oplus (C_r \circ \overline{K}9_s) \oplus \cdots \oplus (C_r \circ \overline{K}_s) \\ &= C_{rs} \oplus C_{rs} \oplus \cdots \oplus C_{rs}, \text{by Theorem 1.13.} \end{aligned}$$

□

Corollary 3.2. *Let $(s, m) \neq (1, 2)$ and let $r \geq 3$ be an odd integer with $m \equiv 2 \,(\text{mod}\, 4)$. If $C_r \,|\, G$ and $s \,|\, m$, then $C_{rs} \,|\, G \times K_m$.*

Proof. Since $C_r \,|\, G$, we have $G \times K_m = (C_r \times K_m) \oplus (C_r \times K_m) \oplus \cdots \oplus (C_r \times K_m)$. Now apply Lemma 3.3 to the graph $C_r \times K_m$. □

Theorem 1.10 can be improved as follows using Corollary 3.2.

Theorem 3.1. *Let $(s, m) \neq (1, 2)$ and let $r \geq 3$ be an odd integer. If $C_r \,|\, G$ and $s \,|\, m$, then $C_{rs} \,|\, G \times K_m$.*

In the proof of the following lemma we obtain some decompositions based on [5].

Lemma 3.4. *Let $s \geq 5$ be an odd integer and $r \geq 3$. Then there exists a directed s-cycle decomposition $\mathscr{C}$ of K^*_{s+1} such that each arc of the directed cycles of $\mathscr{C}$ can be assigned the label 1 or $r-1$, so that both the symmetric pair of arcs (a_i, a_j) and (a_j, a_i) receive the same label and the sum of the labels of the arcs on each of the directed cycles of $\mathscr{C}$ is relatively prime to r.*

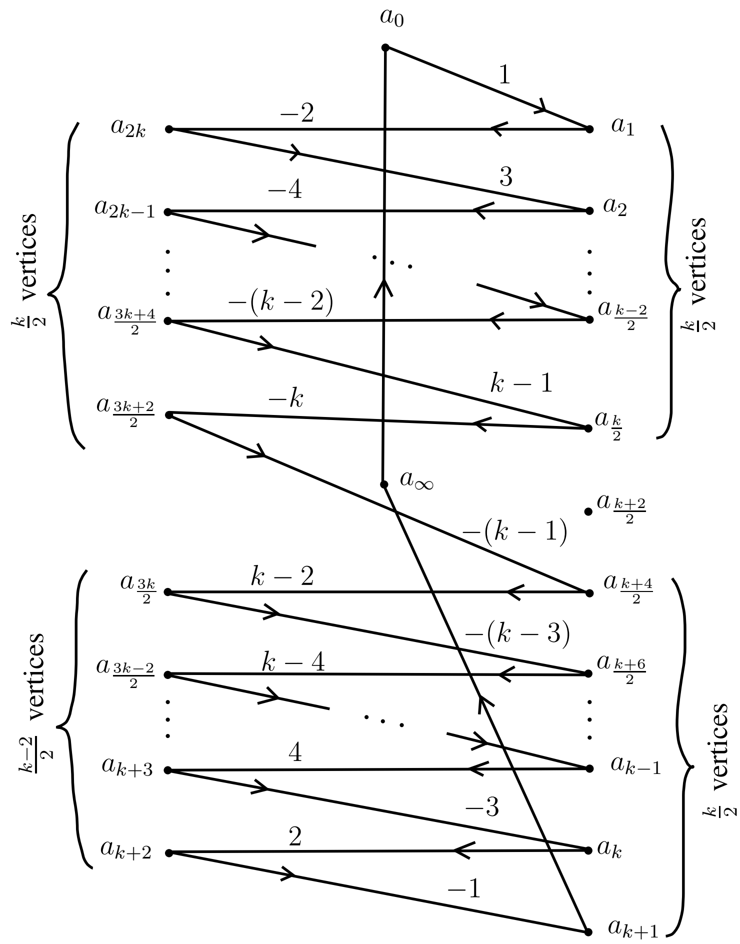
Proof. Let $s = 2k+1$ and $V(K^*_{s+1}) = \{a_\infty, a_0, a_1, \ldots, a_{s-1}\}$. Let $\sigma = (a_\infty)(a_0 a_1 a_2 \ldots a_{s-1})$ be a permutation in $V(K^*_{s+1})$. We obtain a $\overrightarrow{C_s}$-decomposition $\mathscr{C}$ of K^*_{s+1} as follows.

Case 1. k is even.
Let $\overrightarrow{C} : a_0\, a_1\, a_{2k}\, a_2\, a_{2k-1} \ldots a_{\frac{k}{2}}\, a_{\frac{3k+2}{2}}\, a_{\frac{k+4}{2}}\, a_{\frac{3k}{2}} \ldots a_{k+2}\, a_{k+1}\, a_\infty\, a_0$ be a (base) directed cycle of length s. Now $\mathscr{C}' = \{\overrightarrow{C}, \sigma(\overrightarrow{C}), \sigma^2(\overrightarrow{C}), \ldots, \sigma^{s-1}(\overrightarrow{C})\}$ is a decomposition of K^*_{s+1} into directed s-cycles; observe that $\overrightarrow{C}$ misses exactly one vertex, namely, $a_{(k+2)/2}$ of K^*_{s+1}. Each of the distances $\pm 1, \pm 2, \pm 3, \pm 4, \ldots, \pm(k-1), -k$ appears exactly once in each of its directed s-cycles of $\mathscr{C}'$, except the distance k; see Fig. 13.

Assign the label 1 (respectively, $r-1$) to the arcs of distances $\pm(2i+1)$, $0 \leq i \leq (s-5)/4$ (respectively, $\pm 2i, 1 \leq i \leq (s-5)/4$) to each of the directed s-cycles of $\mathscr{C}'$; assign the label $r-1$ to the arcs $\big(\sigma^i(a_{k+1}), \sigma^i(a_\infty)\big)$ and $\big(\sigma^i(a_\infty), \sigma^i(a_0)\big)$, $0 \leq i \leq s-1$ of $\sigma^i(\overrightarrow{C})$. We label the arcs $(\sigma^i(a_{\frac{k}{2}}), \sigma^i(a_{\frac{3k+2}{2}}))$, $0 \leq i \leq s-1$ as follows: for $0 \leq \ell \leq (s-1)/2$, the arc $\big(\sigma^{2\ell}(a_{\frac{k}{2}}), \sigma^{2\ell}(a_{\frac{3k+2}{2}})\big)$ is assigned the label 1 and, $0 \leq \ell \leq (s-3)/2$, the arc $\big(\sigma^{2\ell+1}(a_{\frac{k}{2}}), \sigma^{2\ell+1}(a_{\frac{3k+2}{2}})\big)$ is assigned the label

$r-1$. It can easily be verified that the sum of the labels of the arcs of each of the directed s-cycles of $\mathscr{C}'$ is relatively prime to r.

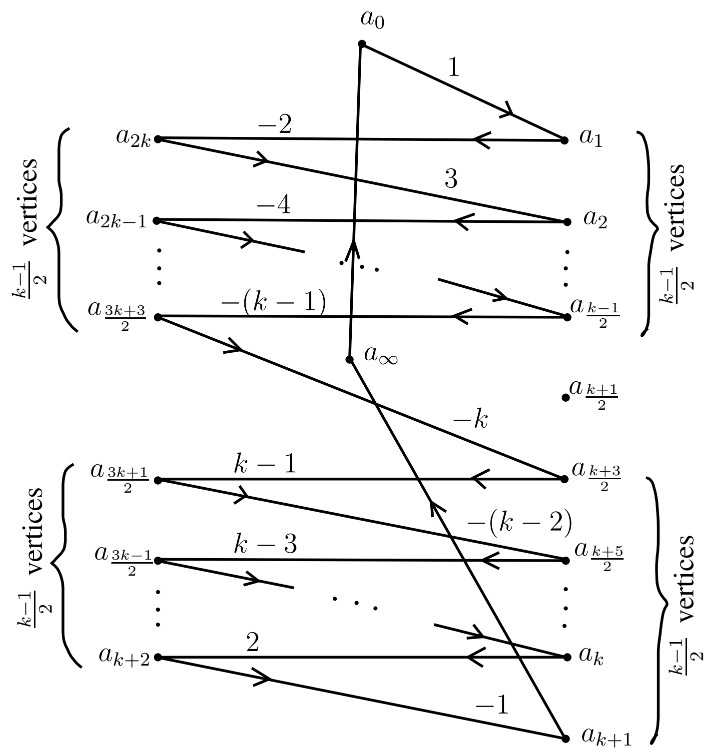
Case 2. k is odd.
Let $\overrightarrow{C}^1 : a_0a_1a_{2k}a_2a_{2k-1}\ldots a_{\frac{k-1}{2}}a_{\frac{3k+3}{2}}a_{\frac{k+3}{2}}a_{\frac{3k+1}{2}}\ldots a_{k+2}a_{k+1}a_\infty a_0$ be a (base) directed cycle of length s. Let $\sigma=(a_\infty)(a_0a_1\ldots a_{s-1})$ be a permutation of $V(K^*_{s+1})$. A set of s directed s-cycles $\mathscr{C}'$ is given by $\{\overrightarrow{C}^1,\sigma(\overrightarrow{C}^1),\sigma^2(\overrightarrow{C}^1),\ldots,\sigma^{s-1}(\overrightarrow{C}^1)\}$; observe that $\overrightarrow{C}^1$ misses exactly one vertex, namely, $a_{(k+1)/2}$ of K^*_{s+1}. It can be seen that each of the distances $\pm1,\pm2,\pm3,\pm4,\ldots,\pm(k-1),-k$ appears exactly once in each of its directed s-cycles of $\mathscr{C}'$, except the distance k; see Fig. 14.



A (base) directed cycle $\overrightarrow{C}$ with the distances marked on the arcs of the directed cycle.

Fig. 13.

Assign the label 1 (respectively, $r-1$) to the arcs of distances $\pm(2i+1)$, $0\le i\le(s-7)/4$ (respectively, $\pm 2i$, $1\le i\le(s-3)/4$) to the directed s-cycles of $\mathscr{C}'$. We have yet to label the three arcs of $\sigma^i(\overrightarrow{C}^1)$, namely, $\big(\sigma^i(a_{k+1}),\sigma^i(a_\infty)\big)$, $\big(\sigma^i(a_\infty),\sigma^i(a_0)\big)$, $\big(\sigma^i(a_{\frac{3k+3}{2}}),\sigma^i(a_{\frac{k+3}{2}})\big)$, $0\le i\le s-1$. Other than these three arcs of $\sigma^i(\overrightarrow{C}^1)$, the sum of the labels of the arcs of each of the directed cycles of $\mathscr{C}'$ is a multiples of r, since both 1 and $r-1$ occur an equal number of times. These three arcs with labels are shown in Fig. 15; on these three arcs the sum of the labels is relatively prime to r.



A (base) directed cycle $\overrightarrow{C}^1$ with the distances marked on the arcs of the cycle.

Fig. 14.

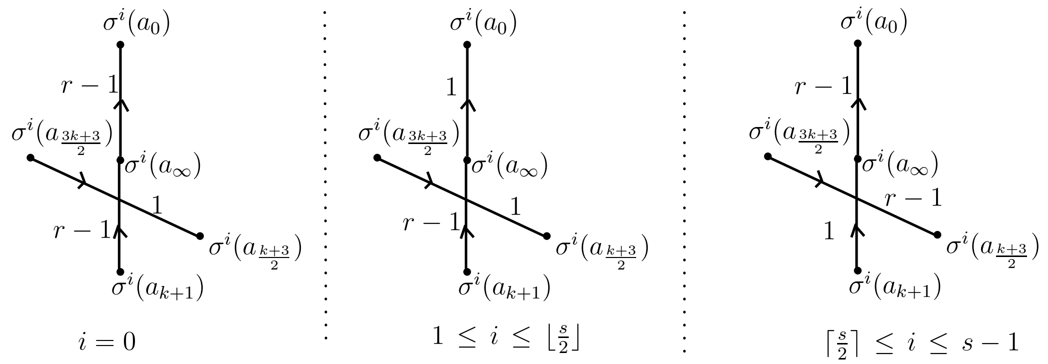


Fig. 15.

Regardless of the parity of k, we prove that $K = K^*_{s+1} - A(\mathscr{C}')$ can be decomposed into $\overrightarrow{C}_s$. Clearly, the arc set of K is all arcs of distance k in K^*_{s+1}; these arcs induce a directed s-cycle $\overrightarrow{C}^2 = a_0 a_k a_{2k} a_{k-1} \ldots a_{k+1} a_0$. In $\overrightarrow{C}^2$, assign the label to the arc (a_i, a_{i+k}) of distance k, the same label of the arc (a_{i+k}, a_i) of distance $-k$ in the cycles of $\mathscr{C}'$. When k is even, the arcs of distance $-k$ in the directed cycles of $\mathscr{C}'$ are exactly $\big(\sigma^i(a_{\frac{k}{2}}), \sigma^i(a_{\frac{3k+2}{2}})\big)$, $0 \le i \le s-1$, and these arcs are assigned the label 1 or $r-1$ according as i is even or odd, respectively. Hence the sum of the labels of the arcs of $\overrightarrow{C}^2$ is relatively prime to r , since s is odd. When k is odd, the arcs of distance $-k$ in the directed cycles of $\mathscr{C}'$ are exactly $\big(\sigma^i(a_{\frac{3k+3}{2}}), \sigma^i(a_{\frac{k+3}{2}})\big)$, $0 \le i \le s-1$, and these arcs are assigned the label 1 for $0 \le i \le \lfloor s/2 \rfloor$, and $r-1$ for $\lceil s/2 \rceil \le i \le s-1$. Hence the sum of the labels of the arcs of $\overrightarrow{C}^2$ is relatively prime to r , because s is odd. □

Theorem 3.2. *If* $s \ge 3$ *is an odd integer and* $r \ge 3$ *, then* $C_{rs} \,|\, C_r \times K_{s+1}$ *.*

Proof. We consider two cases.

Case 1. $s \ge 5$.

Let $V(K^*_{s+1}) = \{a_\infty, a_0, a_1, a_2, \dots a_{s-1}\}$. Obtain a directed s-cycle decomposition $\mathscr{C}$ of K^*_{s+1} with arcs labeled 1 or $r-1$ as in the proof of Lemma 3.4. We assume that $V(K_{s+1}) = V(K^*_{s+1})$.

To each directed s-cycle in $\mathscr{C}$, we obtain an rs-cycle in $C_r \times K_{s+1} \cong K_{s+1} \times C_r$ as follows: let $\overrightarrow{C_s} \in \mathscr{C}$. For our convenience, let $\overrightarrow{C_s} = (a_1, a_2, \dots, a_s)$ and $C_r = (v_1, v_2, \dots, v_r)$. Then $V(\overrightarrow{C_s}) \times V(C_r) = \bigcup_{i=1}^{s} \big(\{a_i\} \times V(C_r)\big) \subset V(K_{s+1} \times C_r)$. We denote $\{a_i\} \times V(C_r)$ by A_i , $1 \le i \le s$. If the label of the arc (a_i, a_{i+1}) is $r-1$ (respectively, 1) in $\overrightarrow{C_s}$, then consider the set of edges $F_{r-1}(A_i, A_{i+1})$ (respectively, $F_1(A_i, A_{i+1})$) from A_i to A_{i+1} in $\langle A_i \cup A_{i+1} \rangle \subseteq C_s \times C_r$; in fact, the label of an arc, namely, 1 or $r-1$ of the directed s-cycle, is used to decide the distance of the 1-factor from A_i to A_{i+1} in $\langle A_i \cup A_{i+1} \rangle$. Then $\bigcup_{i=1}^{s} F_{\ell_i}(A_i, A_{i+1})$, where ℓ_i is the label of the arc (a_i, a_{i+1}) in $\overrightarrow{C_s}$, is a cycle of length rs , since the sum $\sum_{i=0}^{s-1} \ell_i$ is relatively prime to r . Thus each directed cycle $\overrightarrow{C_s}$ of K^*_{s+1} yields a cycle of length rs in the graph $K_{s+1} \times C_r \cong C_r \times K_{s+1}$.

Case 2. $s = 3$.

Let u_1, u_2, u_3, u_4 be the vertices of K_4 . Then $V(K_4 \times C_r) = V(K_4) \times V(C_r) = \bigcup_{i=1}^{4} \big(\{u_i\} \times V(C_r)\big)$. We denote $\{u_i\} \times V(C_r)$ by U_i , $1 \le i \le 4$. Then the four $3r$-cycles which decompose the graph $K_4 \times C_r$ are listed below;

$$\begin{aligned} C^1 &= F_1(U_1, U_2) \oplus F_{r-1}(U_2, U_3) \oplus F_1(U_3, U_1); \\ C^2 &= F_{r-1}(U_1, U_4) \oplus F_{r-1}(U_4, U_2) \oplus F_1(U_2, U_1); \\ C^3 &= F_1(U_1, U_3) \oplus F_1(U_3, U_4) \oplus F_{r-1}(U_4, U_1); \\ C^4 &= F_{r-1}(U_2, U_4) \oplus F_1(U_4, U_3) \oplus F_{r-1}(U_3, U_2). \end{aligned}$$

□

Corollary 3.3. *If* $s \ge 3$ *is an odd integer,* $r \ge 3$ *and* $C_r \,|\, G$ *, then* $C_{rs} \,|\, G \times K_{s+1}$ *.*

Proof. Since $C_r \,|\, G$, $G \times K_{s+1} = (C_r \times K_{s+1}) \oplus (C_r \times K_{s+1}) \oplus \cdots \oplus (C_r \times K_{s+1})$. Now apply Theorem 3.2 to the graph $C_r \times K_{s+1}$. □

4 Proof of the main result

We use the following theorem in the proof of Theorem 1.1.

Theorem 4.1. [30]. *If* $m, n \geq 3$, *then* $C_4 \,|\, (K_m \times K_n)(\lambda)$ *if and only if* $4 \,|\, \lambda\binom{m}{2} n(n-1)$ *and* $(K_m \times K_n)(\lambda)$ *is an even regular graph.*

Proof of Theorem 1.1. The proof of necessity is obvious, and we prove the sufficiency in two cases. Because of Theorem 4.1, we suppose that $p \geq 3$. Since the tensor product is commutative, we assume that m is odd; so $m \equiv 1, 3$ or $5 \pmod 6$.

Case 1. $n \equiv 0$ or $1 \pmod{p^2}$.

Subcase 1.1. $m \equiv 1$ or $3 \pmod 6$.

By Theorem 1.3, $C_3 \,|\, K_m$ and hence $K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)$. Apply Theorem 2.1 to $C_3 \times K_n$.

Subcase 1.2. $m \equiv 5 \pmod 6$.

By Theorem 1.3, $K_m = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_5 \oplus C_5$. The graph $K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n) \oplus (C_5 \times K_n) \oplus (C_5 \times K_n)$. Since $C_{p^2} \,|\, C_3 \times K_n$ and $C_{p^2} \,|\, C_5 \times K_n$, by Theorems 2.1 and 2.2, $C_{p^2} \,|\, K_m \times K_n$.

Case 2. $n \not\equiv 0 \pmod{p^2}$ and $n \not\equiv 1 \pmod{p^2}$.

Subcase 2.1. $p^2 \,|\, \binom{m}{2}$.

Clearly, $m \equiv 0$ or $1 \pmod{p^2}$. Since m is odd, $m \equiv 1$ or $p^2 \pmod{2p^2}$. Since $C_{p^2} \,|\, K_m$, by Theorem 1.2, $K_m \times K_n = (C_{p^2} \times K_n) \oplus (C_{p^2} \times K_n) \oplus \cdots \oplus (C_{p^2} \times K_n)$. Now apply Theorem 1.8 to $C_{p^2} \times K_n$.

Subcase 2.2. $p \,|\, \binom{m}{2}$ and $p \,|\, n(n-1)$.

Since $C_p \,|\, K_m$, by Theorem 1.2, it suffices to show that $C_{p^2} \,|\, C_p \times K_n$, where $n \equiv 0$ or $1 \pmod p$.

Subcase 2.2.1. n is odd.

Since $C_p \,|\, K_n$, by Theorem 1.2, $C_p \times K_n = (C_p \times C_p) \oplus (C_p \times C_p) \oplus \cdots \oplus (C_p \times C_p)$ and $C_{p^2} \,|\, C_p \times C_p$, by Theorem 1.7.

Subcase 2.2.2. n is even.

If $n \equiv 0 \pmod p$, then $C_{p^2} \,|\, C_p \times K_n$, by Theorem 3.1. If $n \equiv 1 \pmod p$, then let $n = kp + 1$, k odd (since n is even). If $k = 1$, the result follows by Theorem 3.2 and so we assume that $k \geq 3$. The graph

$$\begin{aligned} C_p \times K_{kp+1} &= C_p \times \big(\underbrace{K_{p+1} \oplus K_{p+1} \oplus \cdots \oplus K_{p+1}}_{k \text{ times}} \oplus K_k \circ \overline{K}_p\big) \\ &= \underbrace{C_p \times K_{p+1} \oplus C_p \times K_{p+1} \oplus \cdots \oplus C_p \times K_{p+1}}_{k \text{ times}} \oplus C_p \times (K_k \circ \overline{K}_p); \end{aligned}$$

and $C_{p^2} \,|\, (C_p \times K_{p+1})$, by Theorem 3.2 and $C_p \,|\, K_k \circ \overline{K}_p$, by Theorem 1.6. Consequently, $C_p \times (K_k \circ \overline{K}_p) = C_p \times (C_p \oplus C_p \oplus \cdots \oplus C_p)$ and $C_{p^2} \,|\, C_p \times C_p$, by Theorem 1.7. This completes the proof of the theorem. □

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