AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(1)** (2019), Pages 1–24

On the Ramsey numbers for stars versus connected graphs of order six

Roland Lortz

Technische Universität Braunschweig
Institut Computational Mathematics
AG Algebra und Diskrete Mathematik, 38092 Braunschweig
Germany
r.lortz@tu-braunschweig.de

Ingrid Mengersen

Moorhüttenweg 2d
38104 Braunschweig
Germany
ingrid.mengersen@t-online.de

Abstract

We investigate the Ramsey number $r(S_n, G)$ where S_n denotes the star of order n and G is a connected graph of order six. The values of $r(S_n, G)$ are determined for any $G \neq K_{2,2,2}$ with chromatic number $\chi(G) \geq 3$ with but a few exceptions for some G with $\chi(G) = 3$ in case of some small n. Partial results on $r(S_n, G)$ are obtained if $\chi(G) = 2$. In any case, $r(S_n, G)$ is evaluated for $n \leq 5$. With our results, $r(T_n, G)$ is completely known for every tree T_n of order n and every connected graph of order six with $\chi(G) \geq 4$.

1 Introduction

The Ramsey number $r(T_n, G)$, where T_n denotes a tree of order n and G is a graph of order m, has been intensively studied. Chvátal [5] proved that

$$r(T_n, K_m) = (n-1)(m-1) + 1 \tag{1}$$

for any tree T_n. Moreover, the values of $r(T_n, G)$ are almost completely known for nearly complete graphs G. Chartrand, Gould and Polimeni [4] showed that

$$r(T_n, G) = (n-1)(m-2) + 1 \tag{2}$$

ISSN: 2202-3518 ©The author(s). Released under the CC BY-ND 4.0 International License

for $n \geq 4$ and every graph G of order $m \geq 4$ and clique number $cl(G) = m - 1$. Gould and Jacobson [12] proved that

$$r(T_n, G) = (n-1)(m-3) + 1 \tag{3}$$

for $n \geq 4$ and all graphs G of order $m \geq 6$ and $cl(G) = m - 2$, where $T_n \neq S_n$ in case of $m = 6$. Furthermore, $r(T_n, G)$ has been studied for special graphs G such as books, cycles or bipartite graphs. Here we just want to mention some results important in connection with our paper, a survey can be found in [32]. Rousseau and Sheehan [34] and Erdős, Faudree, Rousseau and Schelp [8] investigated $r(T_n, B_m)$ for the book graph $B_m = K_{1,1,m}$ and obtained the following result:

$$r(T_n, B_m) = 2n - 1 \quad \text{for } n \geq 3m - 3. \tag{4}$$

Faudree, Schelp and Rousseau [11] considered $G = K_m - K_t$ and showed that, for $n \geq 2$, $m \geq 2$, $t \geq 1$ and $m \geq 2t - \lfloor (t-1)/(n-1) \rfloor (n-1)$,

$$r(T_n, K_m - K_t) = (n-1)(m - t + \lfloor (t-1)/(n-1) \rfloor) + 1, \tag{5}$$

except for $(T_n, K_m - K_t) = (S_4, K_6 - K_3)$. Some effort has been made to evaluate $r(S_n, G)$ for bipartite graphs G, especially for trees, cycles of even length and complete bipartite graphs. These cases are not completely settled, not even the values of $r(S_n, C_4)$ are entirely known. Parsons [31] proved that

$$r(S_n, C_4) \leq n + \lceil \sqrt{n-1} \rceil \quad \text{for } n \geq 3, \tag{6}$$

and, for any prime power q,

$$r(S_{q^2+1}, C_4) = q^2 + q + 1 \quad \text{and} \quad r(S_{q^2+2}, C_4) = q^2 + q + 2. \tag{7}$$

Moreover, Burr, Erdős, Faudree, Rousseau and Schelp [3] showed that

$$r(S_n, C_4) > n - 1 + \lfloor \sqrt{n-1} - 6(n-1)^{11/40} \rfloor \tag{8}$$

if n is sufficiently large. Recently, some progress in evaluating $r(S_n, C_4)$ has been made by Wu, Sun, Zhang and Radziszowski [35]. Faudree, Rousseau and Schelp [10] systematically studied $r(T_n, G)$ for all connected graphs G of order at most five. In particular they proved that, for $n \geq 4$ and every connected graph G on five vertices with chromatic number $\chi(G) = 3$,

$$r(T_n, G) = 2n - 1 + \epsilon, \tag{9}$$

with $\epsilon = 2$ if $(T_n, G) = (S_n, K_5 - 2K_2)$ where n is even, $\epsilon = 1$ if $(T_n, G) = (S_n, K_5 - P_4)$ where n is even or if $(T_n, G) = (S_4, K_5 - K_3)$ and $\epsilon = 0$ otherwise. For non-tree graphs G with $\chi(G) = 2$, $r(T_n, G)$ has not been completely evaluated. The main reason is the lack of knowledge about $r(S_n, C_4)$ and $r(S_n, K_{2,3})$.

In this paper we will begin to extend the results obtained in [10] to connected graphs of order six. The list of all 112 such graphs given in Table 1 is taken from

[15], more detailed information about these graphs can be found in [26]. A formula to compute $r(T_n, G)$ for $n = 3$, the first nontrivial case, and every graph G of order m is given in [6]. Thus, we may always assume that $n \geq 4$. Moreover, we will make use of the well-known lower bound

$$r(F, G) \geq (n-1)(\chi(G) - 1) + s(G) \tag{10}$$

for any connected graph F of order n and any graph G with chromatic surplus $s(G) \leq n$ (see [8] or [10]). Only a few values of $r(T_n, G)$ are missing for connected graphs G of order six with $\chi(G) \geq 4$ because of (1), (2) and (3). We close this gap and show that $r(T_n, G)$ attains the lower bound given in (10) with only one exception. For $\chi(G) \leq 3$, different methods seem to be required to evaluate $r(T_n, G)$ depending on whether T_n is or is not a star. Here we focus on $T_n = S_n$, the case $T_n \neq S_n$ is treated in [28]. With a few exceptions for small n, the values of $r(S_n, G)$ are determined for every connected graph $G \neq K_{2,2,2}$ of order six with $\chi(G) = 3$. For $n \geq 5$ the values differ by at most 2 from the lower bound given in (10), whereas it is shown in [27] that $r(S_n, K_{2,2,2})$ can be significantly larger. Partial results on $r(S_n, G)$ are obtained for the connected graphs G of order six with $\chi(G) = 2$. As could be expected, problems arise in case of non-tree graphs. These graphs contain a cycle C_4 or C_6, and for any $G \neq K_{2,4}$ not containing a cycle C_6 we obtain that $r(S_n, G)$ matches $r(S_n, C_4)$ or $r(S_n, K_{2,3})$ if n is sufficiently large. A complete evaluation fails because of the missing values of $r(S_n, C_4)$ and $r(S_n, K_{2,3})$.

This paper also makes a contribution to evaluate $r(F, G)$ for small graphs F and G. If F and G both have at most five vertices, $r(F, G)$ is almost completely known (see [6], [7], [17], also cf. [32]). Some effort has been made to determine $r(F, G)$ for graphs F of order at most five and graphs G of order six (see [1, 9, 13, 18, 20, 21, 22, 23, 25, 26, 29, 33]). The results in this paper together with $r(S_4, K_{2,2,2}) = 10$ (see [27]), $r(S_5, K_{2,2,2}) = 11$ (see [13] and [27]) and the results on $r(F, G)$ for disconnected graphs G of order six obtained in [25] yield all values of $r(S_n, G)$ for $n \leq 5$ and any graph G of order six.

Some specialized notation will be used. A coloring of a graph always means a 2-coloring of its edges with colors red and green. An (F_1, F_2)-coloring is a coloring containing neither a red copy of F_1 nor a green copy of F_2. We use V to denote the vertex set of K_n and define $d_r(v)$ to be the number of red edges incident to $v \in V$ in a coloring of K_n. Moreover, $\Delta_r = \max_{v \in V} d_r(v)$. The set of vertices joined red to v is denoted by $N_r(v)$. Similarly we define $d_g(v)$, Δ_g and $N_g(v)$. For $U \subseteq V(K_n)$, the subgraph induced by U is denoted by $[U]$. Furthermore, $[U]_r$ and $[U]_g$ denote the red and the green subgraph induced by U. We write $G' \subseteq G$ if G' is a subgraph of G, and $G' \subseteq_{ind} G$ means that G' is an induced subgraph. For disjoint subsets $U_1, U_2 \subseteq V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges between U_1 and U_2, and $q_g(U_1, U_2)$ is defined similarly. The set of all connected graphs G of order six and chromatic number $\chi(G) = s$ is denoted by $\mathcal{G}_s$.

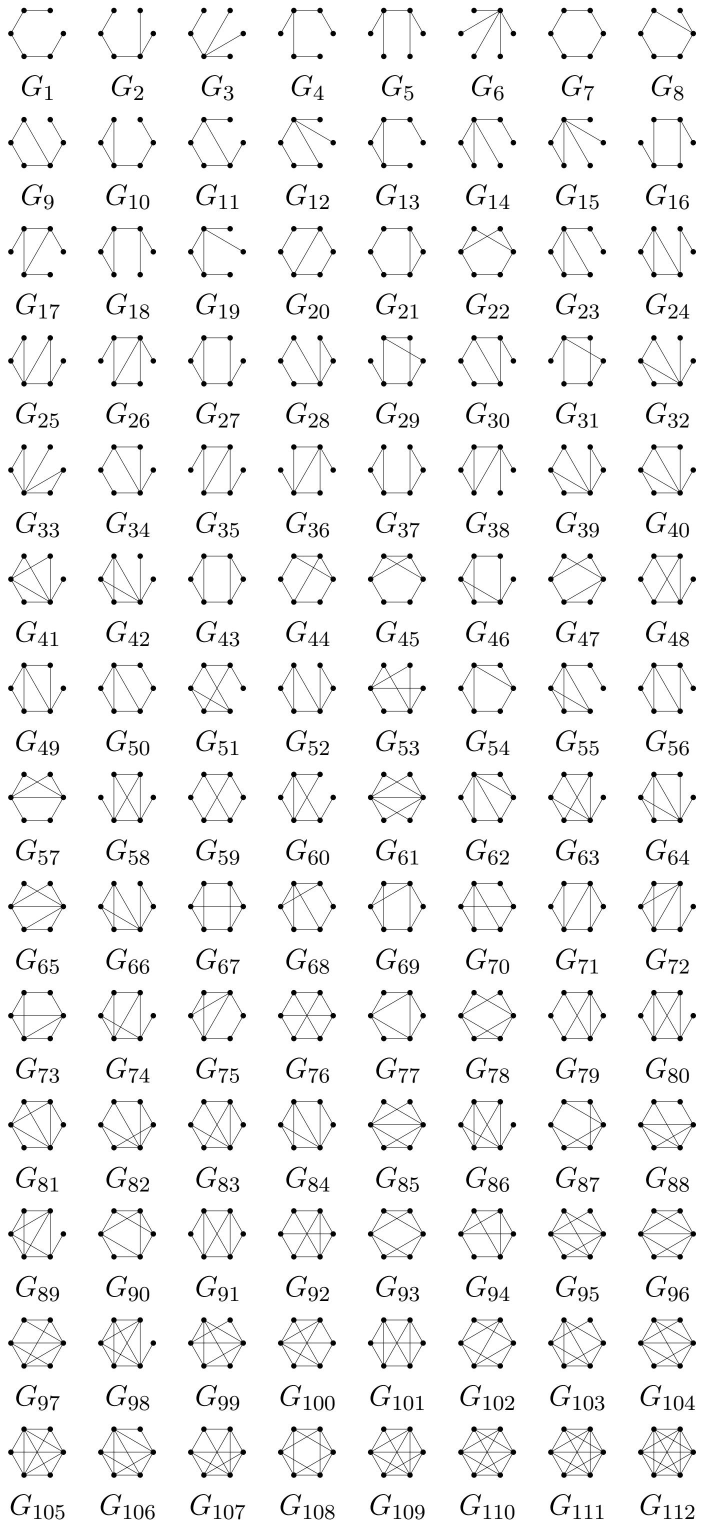

Table 1. The 112 connected graphs of order six.

2 The Ramsey Number $r(T_n, G)$ for $G \in \mathcal{G}_s$, $4 \le s \le 6$

Obviously, $K_6 = G_{112}$ is the only graph in $\mathcal{G}_6$, and $\mathcal{G}_5$ consists of the four connected graphs G of order six with clique number $cl(G) = 5$, i.e., $\mathcal{G}_5 = \{G_{98}, G_{106}, G_{109}, G_{111}\}$. If $G \in \mathcal{G}_4$, then either $cl(G) = 4$ or G is isomorphic to the wheel $W_5 = G_{82}$. This gives

$$\mathcal{G}_4 = \{G_{42}, G_{55}, G_{58}, G_{64}, G_{66}, G_{72}, G_{75}, G_{80}, G_{81}, G_{82}, G_{84}, G_{85}, G_{86}, G_{88}, G_{89}, G_{91}, G_{95}, G_{96}, G_{97}, G_{99}, G_{101}, G_{103}, G_{104}, G_{105}, G_{107}, G_{110}\}.$$

From (1), (2) and (3) we already know that $r(T_n, G)$ matches the lower bound in (10) for $G \in \mathcal{G}_s$ with $5 \le s \le 6$ and, in case of $T_n \ne S_n$, for $G \in \mathcal{G}_4 \setminus \{W_5\}$. Here we will show that the lower bound is also attained in the remaining cases with only one exception.

Theorem 2.1. *Let $n \ge 4$, $G \in \mathcal{G}_s$, $4 \le s \le 6$, and $(T_n, G) \ne (S_4, K_6 - K_3)$. Then*

$$r(T_n, G) = (n-1)(s-1) + 1.$$

Furthermore, $r(S_4, K_6 - K_3) = 11$.

Proof. To settle the remaining cases, i.e., $G \in \mathcal{G}_4$ where $T_n = S_n$, and $G = W_5$ where $T_n \ne S_n$, we first consider $G = G_{105} = K_6 - K_3$. By (5), $r(S_n, K_6 - K_3) = 3n - 2$ if $n \ge 5$. (The exceptional case $n = 4$ was overlooked in [11].) The coloring of K_{10} with $[V]_r = 2C_5$ implies that $r(S_4, K_6 - K_3) \ge 11$. To establish equality, take any coloring of K_{11} where $S_4 \not\subseteq [V]_r$ and consider some vertex $v \in V$. Since $d_g(v) \ge 8$ and $r(S_4, K_5 - K_3) = 8$ by (9), $K_6 - K_3 \subseteq [\{v\} \cup N_g(v)]_g$, and we are done.

Now let $G \in \mathcal{G}_4 \setminus \{K_6 - K_3\}$. Obviously, $G \subseteq G_{110} = K_6 - 2K_2$, and this implies $r(T_n, G) \le r(T_n, K_6 - 2K_2)$. Moreover, $r(T_n, G) \ge 3n - 2$ by (10). We already know that $r(T_n, K_6 - 2K_2) = 3n - 2$ if $T_n \ne S_n$. Thus, to complete the proof, it suffices to establish $r(S_n, K_6 - 2K_2) \le 3n - 2$. Suppose that we have an $(S_n, K_6 - 2K_2)$-coloring of K_{3n-2}. By (2), $r(T_n, K_5 - e) = 3n - 2$, and this yields $K_5 - e \subseteq [V]_g$ since $S_n \not\subseteq [V]_r$. Let U be the vertex set of a green $K_5 - e$ and $W = V \setminus U$.

Case 1: $[U]_g = K_5$. From $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \le n - 2$, we obtain $q_r(U, W) \le 5(n-2)$. Moreover, $K_6 - 2K_2 \not\subseteq [V]_g$ implies $q_r(w, U) \ge 2$ for every $w \in W$ yielding $q_r(U, W) \ge 2|W| = 6n - 14$. Hence, $6n - 14 \le 5n - 10$, a contradiction for $n \ge 5$. In case of $n = 4$ only $q_r(U, W) = 5n - 10$ is left. Consequently, $d_r(v) = 2$ for every $v \in V$ and $[W]_g = K_5$. This forces $[V]_r$ to be a bipartite graph and every component of $[V]_r$ to be an even cycle. Thus, $[V]_r = C_{10}$ or $[V]_r = C_6 \cup C_4$. In both cases, $K_6 - 2K_2 \subseteq [V]_g$, a contradiction.

Case 2: $[U]_g = K_5 - e$ and $K_5 \not\subseteq [V]_g$. Since $S_n \not\subseteq [V]_r$, $q_r(U, W) \le 3(n-2) + 2(n-3) = 5n - 12$. Moreover, $K_6 - 2K_2 \not\subseteq [V]_g$ and $K_5 \not\subseteq [V]_g$ imply $q_r(w, U) \ge 2$ for every $w \in W$ yielding $q_r(U, W) \ge 2|W| = 6n - 14$. Thus, $6n - 14 \le 5n - 12$, contradicting $n \ge 4$. ■

3 The Ramsey Number $r(S_n, G)$ for $G \in \mathcal{G}_3$

Here we consider the graphs $G \in \mathcal{G}_3$ except for $G = K_{2,2,2}$. The Ramsey number $r(S_n, K_{2,2,2})$ is separately studied in [27]. If $G \in \mathcal{G}_3$, then $G \subseteq K_{1,1,4} = G_{61}$, $G \subseteq K_{1,2,3} = G_{100}$ or $G \subseteq K_{2,2,2} = G_{108}$. We use this property to partition $\mathcal{G}_3 \setminus \{K_{2,2,2}\}$ into the following five subsets $\mathcal{G}_{3,i}$, $1 \leq i \leq 5$. Put

$$\begin{aligned}
\mathcal{G}_{3,1} &= \{G \in \mathcal{G}_3 \mid G \subseteq K_{1,1,4}\} = \{G_{15},\ G_{19},\ G_{32},\ G_{36},\ G_{41},\ G_{61}\},\\
\mathcal{G}_{3,2} &= \{G \in \mathcal{G}_3 \mid G \subseteq K_{2,2,2},\ G \neq K_{2,2,2},\ G \not\subseteq K_{1,2,3}, \text{and } G \not\subseteq K_{1,1,4}\}\\
&= \{G_{37},\ G_{43},\ G_{45},\ G_{52},\ G_{67},\ G_{68},\ G_{69},\ G_{71},\ G_{77},\ G_{87},\ G_{90},\ G_{93},\ G_{102}\},\\
\mathcal{G}_{3,3} &= \{G \in \mathcal{G}_3 \mid K_5 - 2K_2 \subseteq G \subseteq K_{1,2,3}\} = \{G_{63},\ G_{74},\ G_{83},\ G_{94},\ G_{100}\},\\
\mathcal{G}_{3,4} &= \{G_{39},\ G_{40},\ G_{49},\ G_{56},\ G_{57},\ G_{62},\ G_{65},\ G_{73}\},\\
\mathcal{G}_{3,5} &= \{G \in \mathcal{G}_3 \mid G \neq K_{2,2,2} \text{ and } G \notin \mathcal{G}_{3,1} \cup \mathcal{G}_{3,2} \cup \mathcal{G}_{3,3} \cup \mathcal{G}_{3,4}\}\\
&= \{G_8,\ G_{10},\ G_{13},\ G_{14},\ G_{17},\ G_{18},\ G_{21}, \ldots, G_{28},\ G_{30},\ G_{33},\ G_{34},\ G_{35},\\
&\qquad G_{38},\ G_{44},\ G_{46},\ G_{47},\ G_{48},\ G_{50},\ G_{51},\ G_{54},\ G_{60},\ G_{70},\ G_{78},\ G_{79},\ G_{92}\}.
\end{aligned}$$

The value of $r(S_n, G)$ depends on which of the subsets $\mathcal{G}_{3,i}$ the graph G belongs to. By (10), $r(T_n, G) \geq 2n$ if $G \in \mathcal{G}_{3,2}$ or if $G = K_{2,2,2}$, and $r(T_n, G) \geq 2n - 1$ for the remaining $G \in \mathcal{G}_3$. The following results show that $r(S_n, G) \leq 2n + 1$ for any $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$ if $n \geq 5$, whereas it is proved in [27] that $r(S_n, K_{2,2,2})$ can be significantly larger.

3.1 Results

By (4), $r(T_n, K_{1,1,4}) = 2n-1$ for any tree T_n with $n \geq 9$. This implies that $r(T_n, G) = 2n - 1$ for $n \geq 9$ and every $G \in \mathcal{G}_{3,1}$, since $2n - 1 \leq r(T_n, G) \leq r(T_n, K_{1,1,4})$. The following theorem closes the gap for $n \leq 8$ in case of $T_n = S_n$ with two exceptions. The evaluation of $r(S_5, G_{61})$ is due to Hua, Hongxue and Xiangyang [13].

Theorem 3.1. *Let $G \in \mathcal{G}_{3,1}$ and $n \geq 4$. If $G \neq G_{61}$ and $n \geq 5$ or if $G = G_{61}$ and $n \geq 9$, then $r(S_n, G) = 2n - 1$.*

Furthermore, $r(S_4, G_{19}) = 7$, $r(S_4, G) = 8$ if $G \notin \{G_{61},\ G_{19}\}$, $r(S_4, G_{61}) = 10$, $r(S_5, G_{61}) = 11$, $11 \leq r(S_6, G_{61}) \leq 13$, $13 \leq r(S_7, G_{61}) \leq 14$ and $r(S_8, G_{61}) = 16$.

The following three theorems show that $r(S_n, G)$ can differ from the bound given in (10) for $G \in \mathcal{G}_{3,i}$ with $2 \leq i \leq 4$ if special divisibility properties for n are fulfilled. The values of $r(S_n, G)$ are completely determined for $G \in \mathcal{G}_{3,2}$ and $G \in \mathcal{G}_{3,4}$; in case of $G \in \mathcal{G}_{3,3}$ some gaps are left for small n. The computation of $r(S_5, G_{100})$ is due to Hua, Hongxue and Xiangyang [13].

Theorem 3.2. *Let $G \in \mathcal{G}_{3,2}$ and $n \geq 4$.*

If $G \in \{G_{90},\ G_{102} = K_6 - (P_4 \cup K_2)\}$*, then*

$$r(S_n, G) = \begin{cases} 2n+1 & \text{for } n \equiv 0, 2, 4 \text{ or } 5 \pmod 6, \\ 2n & \text{otherwise.} \end{cases}$$

If $G \in \{G_{67},\ G_{71},\ G_{87} = K_6 - P_6\}$*, then*

$$r(S_n, G) = \begin{cases} 2n+1 & \text{for } n \equiv 2 \pmod 3, \\ 2n & \text{otherwise.} \end{cases}$$

If $G = G_{77}$*, then*

$$r(S_n, G) = \begin{cases} 2n+1 & \text{for } n \text{ even,} \\ 2n & \text{otherwise.} \end{cases}$$

If $G \in \{G_{37}, G_{43}, G_{45}, G_{52}, G_{68}, G_{69}, G_{93} = K_6 - (C_4 \cup K_2)\}$*, then* $r(S_n, G) = 2n$.

Theorem 3.3. *Let* $G \in \mathcal{G}_{3,3}$ *and* $n \geq 4$*. If* n *is even, then* $r(S_n, G) = 2n + 1$*.*

If n *is odd, where* $n \geq 13$ *for* $G = G_{100}$*,* $n \geq 9$ *for* $G = G_{94}$*, and* $n \geq 5$ *otherwise, then* $r(S_n, G) = 2n - 1$*.*

Furthermore, $r(S_5, G_{94}) = 10$*,* $13 \leq r(S_7, G_{94}) \leq 14$*,* $r(S_5, G_{100}) = 11$*, and* $2n - 1 \leq r(S_n, G_{100}) \leq 2n + 1$ *for* $n \in \{7, 9, 11\}$*.*

Theorem 3.4. *Let* $G \in \mathcal{G}_{3,4}$ *and* $n \geq 4$*. Then*

$$r(S_n, G) = \begin{cases} 2n & \text{if } n \text{ is even,} \\ 2n-1 & \text{if } n \text{ is odd.} \end{cases}$$

The next theorem shows that $r(S_n, G)$ attains the lower bound $2n - 1$ from (10) for any $G \in \mathcal{G}_{3,5}$, except for some small n.

Theorem 3.5. *Let* $G \in \mathcal{G}_{3,5}$*,* $\mathcal{S} = \{G_{33}, G_{60}, G_{78}, G_{79}, G_{92}\} \subseteq \mathcal{G}_{3,5}$ *and* $n \geq 4$*. If* $G \in \mathcal{G}_{3,5} \setminus \mathcal{S}$ *and* $n \geq 4$ *or if for* $G \in \mathcal{S}$ *the following conditions for* n *are fulfilled:*

(i) $n \geq 5$ *if* $G = G_{33}$*;*

(ii) $n = 5$ *or* $n \geq 7$ *if* $G \in \{G_{60}, G_{79}\}$*;*

(iii) $n = 5$ *or* $n \geq 9$ *if* $G = G_{78}$*; and*

(iv) $n \geq 13$ *if* $G = G_{92}$*; then*

$$r(S_n, G) = 2n - 1.$$

Futhermore, $r(S_4, G) = 8$ *if* $G \in \mathcal{S}$*,* $r(S_5, G_{92}) = 11$*,* $11 \leq r(S_6, G) \leq 13$ *if* $G \in \{G_{60}, G_{79}, G_{92}\}$*,* $2n-1 \leq r(S_n, G_{78}) \leq 2n$ *if* $6 \leq n \leq 8$*,* $2n-1 \leq r(S_n, G_{92}) \leq 2n+1$ *if* $7 \leq n \leq 12$*.*

Summarizing the results in the preceding theorems we see that $r(S_n, G)$ is determined for all $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$ with but a few exceptions for some G in case of some small n, namely $G = G_{60}$ or $G = G_{79}$ and $n = 6$, $G = G_{61}$ and $6 \leq n \leq 7$, $G = G_{78}$ and $6 \leq n \leq 8$, $G = G_{92}$ and $6 \leq n \leq 12$, $G = G_{94}$ and $n = 7$, $G = G_{100}$ and $n \in \{7, 9, 11\}$.

3.2 Some Useful Lemmas

The following lemmas are essential for proving the preceding theorems. The first lemma considers green subgraphs of order at most five in colorings of K_t, $2n-1 \le t \le 2n+1$, where $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \le n-2$.

Lemma 3.1. *Let $n \ge 4$, $2n-1 \le t \le 2n+1$, and let C be a coloring of K_t with $\Delta_r \le n-2$.*

(i) If $t = 2n+1$ or if n is odd and $2n-1 \le t \le 2n$, then $K_5 - 2K_2 \subseteq [V]_g$, i.e. $K_5 \subseteq [V]_g$, $K_5 - e \subseteq_{ind} [V]_g$ or $K_5 - 2K_2 \subseteq_{ind} [V]_g$.

(ii) If $t = 2n+1$ and $K_5 - e \not\subseteq [V]_g$, then $K_4 \not\subseteq [V]_g$.

(iii) If $t = 2n$, $K_5 - e \not\subseteq [V]_g$, and $K_4 \subseteq [V]_g$ with vertex set U, then $d_r(u) = n-2$ for every $u \in U$ and $q_r(w, U) = 2$ for every $w \in V \setminus U$.

(iv) If $t = 2n$ and $K_5 - 2K_2 \not\subseteq [V]_g$, then n has to be even and $K_4 \subseteq [V]_g$. Moreover, $K_5 - P_3 \subseteq_{ind} [V]_g$.

(v) If $t = 2n-1$ and $K_5 - 2K_2 \not\subseteq [V]_g$, then n has to be even and $K_5 - P_3 \subseteq_{ind} [V]_g$ or $K_5 - (P_3 \cup K_2) \subseteq_{ind} [V]_g$.

Proof. (i) Using that $r(S_n, K_5 - 2K_2) = 2n+1$ if n is even and $r(S_n, K_5 - 2K_2) = 2n-1$ if n is odd (see (10)), we obtain the desired result.

To prove (ii) and (iii), suppose that $t \ge 2n$, $K_5 - e \not\subseteq [V]_g$ and $K_4 \subseteq [V]_g$. Let U be the vertex set of a $K_4 \subseteq [V]_g$ and $W = V \setminus U$. Then $\Delta_r \le n-2$ yields $q_r(U, W) \le 4(n-2) = 4n-8$. Moreover, $q_r(w, U) \ge 2$ for every $w \in W$ since $K_5 - e \not\subseteq [V]_g$. Consequently, $q_r(U, W) \ge 2|W| = 2(t-4)$. It follows that $2(t-4) \le q_r(U, W) \le 4n-8$. Thus, only $t = 2n$ and $q_r(U, W) = 4n-8$ is left. This forces $d_r(u) = n-2$ for every $u \in U$ and $q_r(w, U) = 2$ for every $w \in W$.

(iv) Because of (i), n has to be even. By (2), $r(S_n, K_4 - e) = 2n-1$. Thus, a green $H = K_4 - e$ must occur since $S_n \not\subseteq [V]_r$. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of H and $W = V \setminus U$. If $[U]_g = K_4$ we are done. Otherwise we may assume that the edge u_1u_4 is red. From $\Delta_r \le n-2$ it follows that $q_r(U, W) \le 2(n-3) + 2(n-2) = 4n-10$. Consequently, $|W| = 2n-4$ forces a vertex $w \in W$ with $q_r(w, U) \le 1$. Since $K_5 - 2K_2 \not\subseteq [V]_g$, the edges wu_2 and wu_3 have to be green. Moreover, at least one of the edges wu_1 and wu_4 must be green. This yields a green K_4. Using (iii) we obtain $K_5 - P_3 \subseteq_{ind} [V]_g$.

(v) This follows from (i) and $r(S_n, K_5 - (P_3 \cup K_2)) = 2n-1$ (see (10)). ∎

In the following lemmas we consider colorings of K_t, $2n-1 \le t \le 2n+1$, where $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \le n-2$, and special green subgraphs of order five occur.

Lemma 3.2. *Let $n \ge 4$, $2n-1 \le t \le 2n+1$, and let C be a coloring of K_t with $\Delta_r \le n-2$ and $K_5 \subseteq [V]_g$.*

(i) *If* $t = 2n + 1$, *then* $G \subseteq [V]_g$ *for every* $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$.

(ii) *If* $t = 2n$ *and* $n = 4$ *or* $n \geq 6$, *then* $G \subseteq [V]_g$ *for every* $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$. *If* $n = 5$, *then* $G \subseteq [V]_g$ *for every* $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}, G_{100}\}$.

(iii) *If* $t = 2n - 1$ *and* $n = 4$ *or* $n \geq 9$, *then* $G \subseteq [V]_g$ *for every* $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$. *If* $5 \leq n \leq 8$, *then* $G \subseteq [V]_g$ *for every* $G \in \mathcal{G}_3$ *with* $G \subseteq G_{83}$, $G \subseteq G_{90}$ *or* $G \subseteq G_{94}$.

Proof. Let U be the vertex set of a $K_5 \subseteq [V]_g$ and $W = V \setminus U$. From $\Delta_r \leq n - 2$ we obtain

$$q_r(U, W) \leq 5(n - 2) = 5n - 10.$$

Consider first $t = 2n - 1 + a$, $0 \leq a \leq 2$, where $n \geq 4$ for $a = 2$, $n = 4$ or $n \geq 6$ for $a = 1$ and $n = 4$ or $n \geq 9$ for $a = 0$. We will prove that $q_r(w, U) \leq 2$ for some $w \in W$. If $n = 4$, this follows from $W \neq \emptyset$ and $\Delta_r \leq n - 2$. Assume now that $n > 4$ and $q_r(w, U) \geq 3$ for every $w \in W$. Then $q_r(U, W) \geq 3|W| = 3(t - 5) = 6n + 3a - 18$. Because of $q_r(U, W) \leq 5n - 10$ we obtain $6n + 3a - 18 \leq 5n - 10$. Hence, $n \leq 8 - 3a$, contradicting $n \geq 5$ for $a = 2$, $n \geq 6$ for $a = 1$ and $n \geq 9$ for $a = 0$. Thus, $K_6 - P_3 \subseteq [U \cup \{w\}]_g$ for some $w \in W$ with $q_r(w, U) \leq 2$. Since $G \subseteq K_6 - P_3$ for every $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$, we are done. The remaining cases are $t = 2n$ with $n = 5$ or $t = 2n - 1$ with $5 \leq n \leq 8$.

If $t = 2n$ and $n = 5$, then $|W| = 5$. In case of $q_r(w, U) \leq 2$ for some $w \in W$ again we are done. It remains that $q_r(w, U) \geq 3$ for every $w \in W$. Then $\Delta_r \leq n - 2 = 3$ forces $q_r(w, U) = 3$ for every $w \in W$, $[W]_g = K_5$ and $q_r(u, W) = 3$ for every $u \in U$. Let H be the bipartite graph $K_{5,5}$ with vertex classes U and W. The green subgraph H_g of H induced by the vertices of H contains only vertices of degree two, and this forces every component of H_g to be an even cycle. Hence, $H_g = C_4 \cup C_6$ or $H_g = C_{10}$. In both cases, $K_6 - K_{1,3}$, $K_6 - 2P_3$ and $G_{102} = K_6 - (P_4 \cup K_2)$ are contained in $[V]_g$. Consequently, any $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}, G_{100}\}$ occurs in $[V]_g$.

Finally let $t = 2n - 1$ and $5 \leq n \leq 8$. Then $q_r(w, U) \geq 4$ for every $w \in W$ is impossible as otherwise $q_r(U, W) \geq 4(2n - 6)$ contradicting $q_r(U, W) \leq 5n - 10$ for $n \geq 5$. Thus, $q_r(w, U) \leq 3$ for some $w \in W$ and $K_6 - K_{1,3} \subseteq [V]_g$. Since G_{83}, G_{90} and G_{94} are subgraphs of $K_6 - K_{1,3}$, we are done. ■

Lemma 3.3. *Let* $n \geq 4$, $2n - 1 \leq t \leq 2n + 1$, *and let* C *be a coloring of* K_t *where* $\Delta_r \leq n - 2$, $K_5 - e \subseteq [V]_g$ *and* $K_5 \not\subseteq [V]_g$.

(i) *If* $t = 2n+1$, *then* $G_{102} = K_6 - (P_4 \cup K_2) \subseteq [V]_g$ *and* $G_{100} = K_6 - (K_3 \cup K_2) \subseteq [V]_g$.

(ii) *If* $t = 2n$, *then either* $G_{102} \subseteq [V]_g$ *or* $n \equiv 2 (\mathrm{mod}\ 3)$ *and* $[V]_g = \overline{K_{n-1}} + \frac{n+1}{3} K_3$. *In any case,* $G_{94} = K_6 - ((K_{1,3} + e) \cup K_2) \subseteq [V]_g$, $G_{93} = K_6 - (C_4 \cup K_2) \subseteq [V]_g$, $G_{77} \subseteq [V]_g$ *and* $G_{68} \subseteq [V]_g$.

(iii) *If* $t = 2n - 1$, *then* $G_{100} \subseteq [V]_g$ *for* $n \geq 13$, $G_{94} \subseteq [V]_g$ *for* $n = 4$ *and for* $n \geq 6$, $G_{83} \subseteq [V]_g$ *and* $G_{78} = K_6 - ((K_4 - e) \cup K_2) \subseteq [V]_g$ *for* $n \geq 4$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of a $K_5 - e \subseteq [V]_g$ and let $W = V \setminus U$. We may assume that the edge u_1u_5 is red. From $\Delta_r \leq n - 2$ we obtain

$$q_r(U, W) \leq 2(n - 3) + 3(n - 2) = 5n - 12.$$

If $q_r(w, U) \leq 1$ for some $w \in W$, then $[U \cup \{w\}]_g$ contains every $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}\}$ and we are done. It remains that $q_r(w, U) \geq 2$ for every $w \in W$. Let $W_1 = \{w \in W \mid q_r(w, U) = 2\}$ and $W_2 = W \setminus W_1 = \{w \in W \mid q_r(w, U) \geq 3\}$. Then $q_r(U, W) \geq 2|W_1| + 3|W_2| = 3|W| - |W_1|$. Using $q_r(U, W) \leq 5n - 12$ we obtain

$$|W_1| \geq 3|W| - 5n + 12.$$

(i) If $t = 2n + 1$, then $|W| = 2n - 4$ and $|W_1| \geq 3|W| - 5n + 12 = n$. Since $\Delta_r \leq n - 2$, there must be a vertex $w \in W_1$ where u_1w is green. Hence, $G_{102} \subseteq [U \cup \{w\}]_g$. It remains to prove that $G_{100} \subseteq [V]_g$. If $N_r(w) \cap U = \{u_1, u_5\}$ or $N_r(w) \cap U \subseteq \{u_2, u_3, u_4\}$ for some $w \in W_1$, then $G_{100} \subseteq [U \cup \{w\}]_g$. Otherwise, $|N_r(w) \cap \{u_1, u_5\}| = 1$ for every $w \in W_1$, and $\Delta_r \leq n - 2$ forces $|W_1| \leq 2(n - 3)$. Since $|W_1| \geq n$, only $n \geq 6$ is left. Moreover, $|W_1| = 6$ in case of $n = 6$. If $n \geq 7$, then $|W_1| \geq 7$ and we may assume that four vertices of W_1 are joined red to u_1 and green to u_5. Among these four vertices there must be two vertices w_1 and w_2 with the same red neighbor in $\{u_2, u_3, u_4\}$, say u_2. Thus, $G_{100} \subseteq [\{u_2, u_3, u_4, u_5, w_1, w_2\}]_g$. If $n = 6$, then $|W| = 2n - 4 = 8$, and $|W_1| = 6$ implies $|W_2| = 2$. Because of $\Delta_r \leq n - 2 = 4$, in $[W]$ every vertex of W_1 is incident to at most two red edges and every vertex of W_2 to at most one red edge. Thus, every component of $[W]_r$ has to be a path or a cycle, where at least one path P_ℓ with $\ell \geq 2$ or at least two paths P_1 occur. Hence, the union of all paths in $[W]_r$ is a subgraph of a P_ℓ with $\ell \geq 2$, and $[W]_r \subseteq H$ where $H \in \{P_2 \cup C_3 \cup C_3,\ P_2 \cup C_6,\ P_3 \cup C_5,\ P_4 \cup C_4,\ P_5 \cup C_3,\ P_8\}$. In any case, $G_{100} \subseteq [W]_g$.

(ii) If $t = 2n$, then $|W| = 2n - 5$ and $|W_1| \geq 3|W| - 5n + 12 = n - 3$. Obviously, $G_{102} \subseteq [U \cup \{w\}]_g$ if $N_r(w) \cap \{u_2, u_3, u_4\} \neq \emptyset$ for some $w \in W_1$. It remains that $N_r(w) \cap U = \{u_1, u_5\}$ for every $w \in W_1$, and then $\Delta_r \leq n - 2$ implies $|W_1| \leq n - 3$. Consequently, $|W_1| = n - 3$ and $|W_2| = n - 2$. Moreover, $n \geq 5$ because of $W_2 \neq \emptyset$, $d_r(w) \geq 3$ for every $w \in W_2$ and $\Delta_r \leq n - 2$. Since $|W_1| = n - 3 \geq 2$ and $K_5 \not\subseteq [V]_g$, all edges in $[W_1]$ have to be red. Let $\widehat{W_1} = W_1 \cup \{u_1, u_5\}$ and $\widehat{W_2} = W_2 \cup \{u_2, u_3, u_4\}$. Clearly, $[\widehat{W_1}]$ is a red K_{n-1}, and all edges between $\widehat{W_1}$ and $\widehat{W_2}$ have to be green because of $\Delta_r \leq n - 2$. Consider now $[\widehat{W_2}]$. Since $|\widehat{W_2}| = n + 1$ and $\Delta_r \leq n - 2$, every vertex is incident to at least two green edges. If a green P_4 with vertex set W' occurs, then $G_{102} \subseteq [W' \cup \{u_1, u_5\}]_g$. It remains that every component of $[\widehat{W_2}]_g$ is a K_3. This is only possible if $|\widehat{W_2}| = n + 1 \equiv 0 (\mathrm{mod}\ 3)$, i.e. $n \equiv 2 (\mathrm{mod}\ 3)$, and leads to the desired coloring. Obviously, this coloring contains green subgraphs $K_6 - K_3$, $K_6 - (K_{1,3} \cup K_2)$ and G_{93}. Since $G_{77}, G_{94} \subseteq K_6 - K_3$, $G_{68} \subseteq K_6 - (K_{1,3} \cup K_2)$, and since G_{94}, G_{93}, G_{77} and G_{68} are also subgraphs of G_{102}, the additional statement is proved.

(iii) If $t = 2n - 1$, then $|W| = 2n - 6$ and $|W_1| \geq 3|W| - 5n + 12 = n - 6$. Hence, $|W_1| \geq 7$ for $n \geq 13$, and we can prove that $G_{100} \subseteq [V]_g$ as in (i) in case of

$|W_1| \geq 7$. If $|W_1| \geq 1$, then G_{94}, G_{83} and G_{78} occur in $[U \cup \{w\}]_g$ for any $w \in W_1$. It remains $W_1 = \emptyset$, i.e. $W = W_2$. This forces $n \leq 6$ since $|W_1| \geq n - 6$. Moreover, $n \geq 5$ because of $W_2 \neq \emptyset$, $d_r(w) \geq 3$ for every $w \in W_2$ and $\Delta_r \leq n - 2$. To settle the cases $n = 5$ and $n = 6$ we use $U' = \{u_2, u_3, u_4\}$.

If $n = 5$ we obtain $|W| = 4$. Moreover, $q_r(w, U) = 3$ for every $w \in W$ and $[W]_g = K_4$ are forced by $\Delta_r \leq n - 2 = 3$. Let $W = \{w_1, w_2, w_3, w_4\}$. To prove that $G_{83} \subseteq [V]_g$, note that $q_r(U', W) \leq 3|U'| = 9$. Thus, a vertex $w \in W$ exists where $q_r(w, U') \leq 2$, and this yields $G_{83} \subseteq [U \cup \{w\}]_g$. It remains to find a green G_{78}. If $q_r(w, U') = 3$ or $q_r(w, U') = 1$ for some $w \in W$, then $G_{78} \subseteq [U \cup \{w\}]_g$. Otherwise, $q_r(w, U') = 2$ and $q_r(w, \{u_1, u_5\}) = 1$ for every $w \in W$. Since $q_r(u, W) \leq \Delta_r \leq 3$ for every $u \in U'$, this guarantees a vertex $u \in U'$, say $u = u_2$, such that $q_r(u, W) = 2$. We may assume that u_2 is joined green to w_1 and w_2 and red to w_3 and w_4. Moreover, we may assume that the edges w_3u_1 and w_3u_3 are green. This yields $G_{78} \subseteq [\{u_1, u_2, u_3, w_1, w_2, w_3\}]_g$.

If $n = 6$ then we obtain $|W| = 6$. Again, $q_r(w, U) = 3$ for every $w \in W$, as otherwise $q_r(U, W) > 3|W| = 18$ contradicting $q_r(U, W) \leq 5n - 12$. Moreover, $\Delta_r \leq n - 2 = 4$ implies that all red edges in $[W]$ have to be independent, and we find G_{78} and G_{94} in $[W]_g$. Since $q_r(U', W) \leq 4|U'| = 12$, a vertex $w \in W$ exists such that $q_r(w, U') \leq 2$. This yields $G_{83} \subseteq [U \cup \{w\}]_g$. ■

Lemma 3.4. *Let $n \geq 4$, $2n - 1 \leq t \leq 2n + 1$, and let C be a coloring of K_t where $\Delta_r \leq n - 2$, $K_5 - 2K_2 \subseteq [V]_g$ and $K_5 - e \nsubseteq [V]_g$.*

(i) If $t \geq 2n$, then $G_{102} = K_6 - (P_4 \cup K_2) \subseteq [V]_g$.

(ii) If $t = 2n + 1$, then $G_{100} = K_6 - (K_3 \cup K_2) \subseteq [V]_g$.

(iii) If $t = 2n - 1$, then $G_{100} \subseteq [V]_g$ for $n \geq 13$ and $G_{94} = K_6 - ((K_{1,3} + e) \cup K_2) \subseteq [V]_g$ for $n \geq 9$.

(iv) If $t = 2n - 1$, then $G_{83} \subseteq [V]_g$ for $n \geq 5$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of a $K_5 - 2K_2 \subseteq [V]_g$ and let $W = V \setminus U$. We may assume that the edges u_1u_5 and u_2u_4 are red. Let $U' = \{u_1, u_2, u_4, u_5\}$. From $\Delta_r \leq n - 2$ we obtain

$$q_r(U, W) \leq 4(n - 3) + (n - 2) = 5n - 14 \text{ and } q_r(U', W) \leq 4(n - 3) = 4n - 12.$$

Let $W_1 = N_g(u_3) \cap W$ and $W_2 = W \setminus W_1 = N_r(u_3) \cap W$. If $q_r(w, U) \leq 1$ for some $w \in W_1$, then $[U \cup \{w\}]_g$ contains every $G \in \mathcal{G}_3 \setminus \{K_{2,2,2}, K_{1,1,4}\}$ and we are done. It remains $q_r(w, U) \geq 2$ for every $w \in W_1$.

(i) It suffices to consider $t = 2n$. If $q_r(w, U') \leq 1$ for some $w \in W_2$, then $G_{102} \subseteq [U \cup \{w\}]_g$. Otherwise, $q_r(U', W) \geq 2|W_1| + 2|W_2| = 2|W| = 2(2n - 5) = 4n - 10$ contradicting $q_r(U', W) \leq 4n - 12$.

To prove (ii) and (iii) we look at W_1 and W_2 in more detail. Let $W_{i,j} = \{w \in W_i \mid q_r(w, U) = j\}$. Using $q_r(w, U) \geq 2$ for every $w \in W_1$, we obtain $q_r(U, W) \geq |W_{2,1}| +$

$2(|W_{1,2}|+|W_{2,2}|)+3(|W|-|W_{1,2}|-|W_{2,1}|-|W_{2,2}|)) = 3|W|-|W_{1,2}|-2|W_{2,1}|-|W_{2,2}|$. From $q_r(U,W) \leq 5n-14$ it follows that

$$|W_{1,2}| + 2|W_{2,1}| + |W_{2,2}| \geq 3|W| - 5n + 14.$$

(ii) If $t = 2n+1$, then $|W| = 2n-4$ and $|W_{1,2}| + 2|W_{2,1}| + |W_{2,2}| \geq n+2$. Since $|W_{2,1}| + |W_{2,2}| \leq |W_2| \leq \Delta_r \leq n-2$, we obtain $|W_{1,2}| + |W_{2,1}| \geq 4$. First consider the case $|W_{1,2}| \geq 1$. Let $w \in W_{1,2}$. If $\{u_1, u_5\} \subseteq N_r(w)$ or $\{u_2, u_4\} \subseteq N_r(w)$, then $G_{100} \subseteq [U \cup \{w\}]_g$. Otherwise w is joined green to vertices u and u' where $u \in \{u_1, u_5\}$ and $u' \in \{u_2, u_4\}$. But then $[\{w, u_3, u, u'\}]_g = K_4$ contradicting Lemma 3.1(*ii*). It remains $|W_{2,1}| \geq 4$, and we obtain $G_{100} \subseteq [\{w_1, w_2, u_2, u_3, u_4, u_5\}]_g$ for any $w_1, w_2 \in W_{2,1}$.

(iii) If $t = 2n-1$, then $|W| = 2n-6$ and $|W_{1,2}| + 2|W_{2,1}| + |W_{2,2}| \geq n-4$. Note that $G_{94} \subseteq G_{100}$. First consider the case $|W_{1,2}| \geq 5$. If $\{u_1, u_5\} \subseteq N_r(w)$ or $\{u_2, u_4\} \subseteq N_r(w)$ for some $w \in W_{1,2}$, then $G_{100} \subseteq [U \cup \{w\}]_g$. Otherwise, every $w \in W_{1,2}$ has one green neighbor in $\{u_1, u_5\}$ and one in $\{u_2, u_4\}$. Thus, for $|W_{1,2}| \geq 5$ there are vertices $w_1, w_2 \in W_{1,2}$ with the same green neighbors $u \in \{u_1, u_5\}$ and $u' \in \{u_2, u_4\}$. But then $K_5 - e \subseteq [\{w_1, w_2, u_3, u, u'\}]_g$, a contradiction. It remains $|W_{1,2}| \leq 4$. Consequently, $2|W_{2,1}| + |W_{2,2}| \geq n-8$. If $n \geq 9$, then $W_{2,1} \cup W_{2,2} \neq \emptyset$ and $G_{94} \subseteq [U \cup \{w\}]_g$ for any $w \in W_{2,1} \cup W_{2,2}$. If $n \geq 13$, then $2|W_{2,1}| + |W_{2,2}| \geq 5$. In case of $|W_{2,2}| \geq 5$ there must be two vertices $w_1, w_2 \in W_{2,2}$ with the same red neighbor $u \in U'$, say u_1, and $G_{100} \subseteq [\{w_1, w_2, u_2, u_3, u_4, u_5\}]_g$. It remains $|W_{2,1}| \geq 1$ where $W_{2,2} \neq \emptyset$ if $|W_{2,1}| = 1$. Let $w_1 \in W_{2,1}$ and $w_2 \in W_{2,1} \cup W_{2,2}$ where $w_1 \neq w_2$. We may assume that $u_2, u_4, u_5 \in N_g(w_2)$. Then $G_{100} \subseteq [\{w_1, w_2, u_2, u_3, u_4, u_5\}]_g$.

(iv) Since $|W_2| \leq \Delta_r \leq n-2$ we obtain $|W_1| = |W|-|W_2| \geq 2n-6-(n-2) = n-4$. Thus, $|W_1| \geq 1$ for $n \geq 5$. If $q_r(w, U') \leq 3$ for some $w \in W_1$, then $G_{83} \subseteq [U \cup \{w\}]_g$. Otherwise, all edges between W_1 and U' are red, forcing $n \geq 6$, as $d_r(w) \geq 4$ for every $w \in W_1$ and $\Delta_r \leq n-2$. Moreover, $d_r(u) \geq |W_1|+1$ for every $u \in U'$, yielding $|W_1| \leq n-3$. Thus, only $n-4 \leq |W_1| \leq n-3$ is possible. First we consider $|W_1| = n-3$. It implies $|W_2| = n-3 \geq 3$ and $q_r(w, U') = 0$ for every $w \in W_2$. Hence, $G_{83} \subseteq [\{w_1, w_2, u_1, u_2, u_3, u_4\}]_g$ for any $w_1, w_2 \in W_2$. The remaining case is $|W_1| = n-4$ and $|W_2| = n-2 \geq 4$. Due to $\Delta_r \leq n-2$ every $u \in U'$ has at most one red neighbor in W_2, and we obtain $q_r(U', W_2) \leq 4$. If $q_r(w, U') = 0$ for some $w \in W_2$, then $q_r(U', W_2) \leq 4$ guarantees a vertex $w' \neq w$ in W_2 with $q_r(w', U') \leq 1$. We may assume that $\{u_1, u_2, u_4\} \subseteq N_g(w')$ and obtain $G_{83} \subseteq [\{w, w', u_1, u_2, u_3, u_4\}]_g$. It remains $q_r(w, U') \geq 1$ for every $w \in W_2$. Because of $q_r(U', W_2) \leq 4$ only $|W_2| = 4$ and $q_r(w, U') = 1$ for every $w \in W_2$ is left. Moreover, $q_r(u, W_2) = 1$ for every $u \in U'$. Hence, $G_{83} \subseteq [\{w, w', u_1, u_2, u_3, u_4\}]_g$ for $w, w' \in W_2$ where $w \in N_r(u_2)$ and $w' \in N_r(u_5)$. ∎

Lemma 3.5. *Let $n \geq 4$ be even, $2n-1 \leq t \leq 2n$, and let C be a coloring of K_t where $\Delta_r \leq n-2$, $K_5 - P_3 \subseteq [V]_g$ and $K_5 - 2K_2 \not\subseteq [V]_g$.*

(i) If $t = 2n$, then $G_{62} \subseteq [V]_g$, $G_{65} \subseteq [V]_g$ and $G_{87} = K_6 - P_6 \subseteq [V]_g$ for $n \geq 4$.

(ii) *If* $t = 2n - 1$*, then* $G_{70} \subseteq [V]_g$*,* $G_{73} \subseteq [V]_g$*, and* $G_{79} \subseteq [V]_g$ *for* $n \geq 4$*.*

(iii) *If* $t = 2n - 1$*, then* $G_{78} = K_6 - ((K_4 - e) \cup K_2) \subseteq [V]_g$ *for* $n \geq 8$ *and* $G_{92} = K_6 - (K_3 \cup P_3) \subseteq [V]_g$ *for* $n \geq 10$*.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of a $K_5 - P_3 \subseteq [V]_g$. We may assume that the edges u_2u_3 and u_3u_4 are red. Let $W = V \setminus U$, $U' = \{u_1, u_2, u_4, u_5\}$ and $U'' = \{u_2, u_3, u_4\}$. Note that $[U']$ is a green K_4. From $\Delta_r \leq n - 2$ we obtain

$$\begin{aligned} q_r(U, W) &\leq 2(n-2) + 2(n-3) + n - 4 = 5n - 14, \\ q_r(U'', W) &\leq 2(n-3) + n - 4 = 3n - 10. \end{aligned}$$

(i) Consider $W_1 = N_g(u_1) \cap W$ and $W_2 = N_g(u_3) \cap W$. Note that $|W| = 2n - 5$. From $\Delta_r \leq n-2$ it follows that $|W_1| \geq |W| - (n-2) = n-3$ and $|W_2| \geq |W| - (n-4) = n-1 \geq 3$. Since $q_r(U'', W) \leq 3n-10$ and $|W_1| \geq n-3$, there is a vertex $w \in W_1$ with $q_r(w, U'') \leq 2$, yielding G_{62} and G_{65} in $[U \cup \{w\}]_g$. To prove that $G_{87} \subseteq [V]_g$ consider vertices $w_1, w_2 \in W_2$. Note that $K_5 - e \not\subseteq [V]_g$. Hence, $q_r(\{w_1, w_2\}, \{u_1, u_5\}) \geq 1$, and we may assume that w_1u_1 is red. Moreover, $q_r(w_1, U') = 2$ by Lemma 3.1(iii). Thus, $G_{87} \subseteq [U \cup \{w_1\}]_g$.

(ii) Now let $W_1 = N_g(u_3) \cap W$ and $W_2 = W \setminus W_1 = N_r(u_3) \cap W$. From $\Delta_r \leq n-2$ we obtain $|W_2| \leq n-4$. If $q_r(w, U'') \leq 1$ for some $w \in W_1$, then G_{70}, G_{73} and G_{79} occur in $[U \cup \{w\}]_g$. Otherwise, $q_r(U'', W) \geq 2|W_1| + |W_2| = 2|W| - |W_2| \geq 2|W| - (n-4) = 2(2n-6) - (n-4) = 3n-8$, contradicting $q_r(U'', W) \leq 3n - 10$.

(iii) Note that $K_5 - e \not\subseteq [V]_g$ forces $q_r(w, U') \geq 2$ for every $w \in W$. Now let $W_1 = \{w \in W \mid q_r(w, U) = 2\}$ and $W_2 = W \setminus W_1$. Clearly, every $w \in W_1$ has to be joined green to u_3. Put $W_{1,1} = \{w \in W_1 \mid wu_1 \text{ and } wu_5 \text{ are red}\}$, $W_{1,2} = \{w \in W_1 \mid wu_2 \text{ and } wu_4 \text{ are red}\}$ and $W_{1,3} = W_1 \setminus (W_{1,1} \cup W_{1,2})$. From $q_r(U, W) \leq 5n - 14$, $q_r(U, W) \geq 2|W_1| + 3|W_2| = 3|W| - |W_1|$ and $|W| = 2n - 6$ it follows that

$$|W_1| = |W_{1,1}| + |W_{1,2}| + |W_{1,3}| \geq n - 4.$$

First we will prove that $G_{78} \subseteq [V]_g$ for $n \geq 8$. Note that $|W_1| \geq n - 4 \geq 4$ in case of $n \geq 8$. If $|W_{1,1}| \geq 2$ and $w_1, w_2 \in W_{1,1}$, then $G_{78} \subseteq [U' \cup \{w_1, w_2\}]_g$. If $|W_{1,2}| \geq 1$ and $w \in W_{1,2}$, then $G_{78} \subseteq [U \cup \{w\}]_g$. Otherwise, $|W_{1,3}| \geq 3$. Then u_2 or u_4, say u_2, must have two red neighbors $w_1, w_2 \in W_{1,3}$, and we obtain $G_{78} \subseteq [\{w_1, w_2, u_1, u_3, u_4, u_5\}]_g$.

It remains to prove that $G_{92} \subseteq [V]_g$ for $n \geq 10$. Note that $|W_1| \geq n - 4 \geq 6$ in case of $n \geq 10$. If $|W_{1,2}| \geq 2$ and $w_1, w_2 \in W_{1,2}$, then $K_5 - e \subseteq [\{w_1, w_2, u_1, u_3, u_5]_g$, a contradiction. If $|W_{1,3}| \geq 5$, then there are two vertices $w_1, w_2 \in W_{1,3}$ joined red to the same vertices in U', say to u_1 and u_2. But then $K_5 - 2K_2 \subseteq [\{w_1, w_2, u_3, u_4, u_5\}]_g$, a contradiction. The case $|W_{1,1}| \geq 1$ remains, yielding $G_{92} \subseteq [U \cup \{w\}]_g$ for any $w \in W_{1,1}$. ■

Lemma 3.6. *Let* $n \geq 4$ *be even and let* C *be a coloring of* K_{2n-1} *where* $\Delta_r \leq n-2$*,* $K_5 - (P_3 \cup K_2) \subseteq [V]_g$*,* $K_5 - P_3 \not\subseteq [V]_g$ *and* $K_5 - 2K_2 \not\subseteq [V]_g$*.*

(i) *If* $n \geq 4$*, then* $G_{46} \subseteq [V]_g$*,* $G_{54} \subseteq [V]_g$ *and* $G_{70} \subseteq [V]_g$*.*

(ii) If $n \geq 8$, then $G_{78} = K_6 - ((K_4 - e) \cup K_2) \subseteq [V]_g$ and $G_{92} = K_6 - (K_3 \cup P_3) \subseteq [V]_g$.

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of a $K_5 - (P_3 \cup K_2) \subseteq [V]_g$ and $W = V \setminus U$. We may assume that the edges u_1u_5, u_2u_3 and u_3u_4 are red. From $\Delta_r \leq n - 2$ we obtain

$$q_r(U, W) \leq 4(n-3) + n - 4 = 5n - 16.$$

Note that $K_5 - P_3 \not\subseteq [V]_g$ and $K_5 - 2K_2 \not\subseteq [V]_g$ force $q_r(w, U) \geq 2$ for every $w \in W$. Let $W_1 = \{w \in W \mid q_r(w, U) = 2\}$ and $W_2 = W \setminus W_1$. Every $w \in W_1$ has to be joined green to u_3 as otherwise $K_5 - 2K_2 \subseteq [\{w, u_1, u_2, u_4, u_5\}]_g$ or $K_5 - P_3 \subseteq [\{w, u_1, u_2, u_4, u_5\}]_g$. Put $W_{1,1} = \{w \in W_1 \mid wu_1 \text{ and } wu_5 \text{ are red}\}$, $W_{1,2} = \{w \in W_1 \mid wu_2 \text{ and } wu_4 \text{ are red}\}$, and $W_{1,3} = W_1 \setminus (W_{1,1} \cup W_{1,2})$. From $q_r(U, W) \leq 5n - 16$ and $q_r(U, W) \geq 2|W_1| + 3|W_2| = 3|W| - |W_1| = 3(2n-6) - |W_1|$ we derive

$$|W_1| = |W_{1,1}| + |W_{1,2}| + |W_{1,3}| \geq n - 2.$$

Note that $|W_{1,1}| \leq n - 3$ because of $\Delta_r \leq n - 2$. Hence $|W_1| \geq n - 2$ implies $|W_{1,2}| + |W_{1,3}| \geq 1$. Moreover, $|W_{1,2}| \leq 1$, as otherwise any two vertices $w_1, w_2 \in W_{1,2}$ together with u_1, u_3 and u_5 yield a green $K_5 - 2K_2$. If $|W_{1,3}| \geq 5$, then two vertices $w_1, w_2 \in W_{1,3}$ have to be joined red to the same vertices in $\{u_1, u_2, u_4, u_5\}$, say to u_1 and u_2. But then $K_5 - 2K_2 \subseteq [\{w_1, w_2, u_3, u_4, u_5\}]_g$, a contradiction. Consequently, $|W_{1,3}| \leq 4$ and $|W_{1,2}| + |W_{1,3}| \leq 5$.

(i) If $|W_{1,3}| \geq 1$, then any $w \in W_{1,3}$ and the vertices in U induce a green $K_6 - P_6$. Thus, G_{46}, G_{54} and G_{70} occur in $[V]_g$. It remains that $|W_{1,3}| = 0$. Then $|W_{1,2}| + |W_{1,3}| \geq 1$ and $|W_{1,2}| \leq 1$ force $|W_{1,2}| = 1$. Consequently, $|W_{1,1}| \geq n - 3 \geq 1$ because of $|W_1| \geq n - 2$. Consider now vertices $w_1 \in W_{1,1}$ and $w_2 \in W_{1,2}$. Then $G_{70} \subseteq [U \cup \{w_1\}]_g$, whereas G_{46} and G_{54} occur in $[U \cup \{w_2\}]_g$.

(ii) If $n \geq 8$, then $|W_1| \geq n - 2 \geq 6$. Note that $1 \leq |W_{1,2}| + |W_{1,3}| \leq 5$. Hence, $|W_{1,1}| \geq 1$. Let $w_1 \in W_{1,1}$ and $w_2 \in W_{1,2} \cup W_{1,3}$. Then $G_{92} \subseteq [U \cup \{w_1\}]_g$ and $G_{78} \subseteq [U \cup \{w_2\}]_g$ if $w_2 \in W_{1,2}$. If $w_2 \in W_{1,3}$ we may assume that the edges w_2u_1 and w_2u_2 are red. This yields $G_{78} \subseteq [\{w_1, w_2, u_1, u_3, u_4, u_5\}]_g$. ■

3.3 Proofs of the Theorems

Proof of Theorem 3.1. First we establish suitable lower bounds for $r(S_n, G)$. In any case, $r(S_n, G) \geq 2n - 1$ by (10). The coloring of K_9 with $[V]_r = 3K_3$ shows that $r(S_4, G_{61}) \geq 10$. The coloring of K_7 with $[V]_r = C_3 \cup C_4$ implies $(S_4, G) \geq 8$ for $G \notin \{G_{61}, G_{19}\}$. From [13] we use that $r(S_5, G_{61}) \geq 11$, and $r(S_8, G_{61}) \geq 16$ was shown in [8]. To prove equality, i.e., to establish suitable upper bounds for $r(S_n, G)$, we refine the method used in [34].

Consider any coloring of K_t where $n \geq 4$, $t = 2n - 1 + a$, $a \geq 0$ and $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \leq n - 2$. Hence, $d_g(v) \geq n + a$ for every $v \in V$. Let $u_1 \in V$ with $d_g(u_1) = \Delta_g$ and $u_2 \in N_g(u_1)$. Since $|N_g(u_1)| \geq n$ and $\Delta_r \leq n - 2$, a vertex $u_3 \in N_g(u_1)$ exists such that u_2u_3 is green. Let $U = \{u_1, u_2, u_3\}$ and $W = V \setminus U$. Put $W_i = N_g(u_i) \cap W$. We obtain

$$|W| \geq \sum_{i=1}^{3} |W_i| - \sum_{1\leq i<j\leq 3} |W_i \cap W_j| \geq \Delta_g - 2 + 2(n + a - 2) - \sum_{1\leq i<j\leq 3} |W_i \cap W_j|.$$

Consequently, since $|W| = 2n - 4 + a$ and $\Delta_g \geq n + a$,

$$\sum_{1\leq i<j\leq 3} |W_i \cap W_j| \geq \Delta_g + a - 2 \geq n + 2a - 2.$$

First let $n + 2a \geq 9$. This gives $\sum_{1\leq i<j\leq 3} |W_i \cap W_j| \geq 7$ implying $|W_i \cap W_j| \geq 3$ for some i, j where $1 \leq i < j \leq 3$. Thus, $G_{61} \subseteq [U \cup (W_i \cap W_j)]_g$, and we obtain $r(S_n, G_{61}) \leq 2n - 1$ if $n \geq 9$, $r(S_n, G_{61}) \leq 2n$ if $7 \leq n \leq 8$, $r(S_n, G_{61}) \leq 2n + 1$ if $5 \leq n \leq 6$ and $r(S_4, G_{61}) \leq 10$.

Now let $n = 4$, $a = 1$ or $n \geq 5$, $a = 0$. Note that in case of $n = 5$, $a = 0$, i.e. $K_t = K_9$, we have $\Delta_g \geq 6$, as otherwise $\Delta_r \leq n - 2 = 3$ would force a 5-regular green subgraph of order 9 which is impossible. From $\sum_{1\leq i<j\leq 3} |W_i \cap W_j| \geq \Delta_g + a - 2 \geq n + 2a - 2$ we obtain $\sum_{1\leq i<j\leq 3} |W_i \cap W_j| \geq 4$. Hence, $|W_i \cap W_j| \geq 2$ for some i, j with $1 \leq i < j \leq 3$. Consequently, $G_{41} \subseteq [U \cup \{w_1, w_2, w_3\}]_g$ where $w_1, w_2 \in W_i \cap W_j$ and $w_3 \in W_i \setminus \{w_1, w_2\}$. Note that $G \subseteq G_{41}$ for every $G \neq G_{61}$. Thus, for $G \neq G_{61}$, $r(S_n, G) \leq 2n - 1$ if $n \geq 5$ and $r(S_4, G) \leq 8$. It remains to prove that $r(S_4, G_{19}) \leq 7$. If a coloring of K_7 does not contain a red S_4, then $[V]_r \subseteq H$ where $H \in \{C_7,\ K_1 \cup C_6,\ K_1 \cup C_3 \cup C_3,\ K_2 \cup C_5,\ C_3 \cup C_4\}$. In any case, $G_{19} \subseteq [V]_g$ and we are done. ■

Proof of Theorem 3.2. As already mentioned, $r(S_n, G) \geq 2n$ for every $G \in \mathcal{G}_{3,2}$. To prove that $r(S_n, G) \geq 2n + 1$ for n even and $G \in \{G_{102}, G_{90}, G_{77}\}$, consider the coloring of K_{2n} where $[V]_g = \frac{n}{2}K_2 + \frac{n}{2}K_2$. For $n \equiv 2 (\mathrm{mod}\ 3)$ and $G \in \{G_{102}, G_{90}, G_{87}, G_{71}, G_{67}\}$ the coloring of K_{2n} with $[V]_g = \overline{K_{n-1}} + \frac{n+1}{3}K_3$ implies $r(S_n, G) \geq 2n + 1$.

Next we will show that $r(S_n, G) \leq 2n + 1$ for all $G \in \mathcal{G}_{3,2}$. Note that $G \subseteq G_{102}$ if $G \in \mathcal{G}_{3,2}$. Consider any coloring of K_{2n+1} where $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \leq n - 2$. By Lemma 3.1(i), $K_5 - 2K_2 \subseteq [V]_g$. Using Lemmas 3.2($i$), 3.3($i$), and 3.4($i$) we obtain that $G_{102} \subseteq [V]_g$, and we are done. It remains to establish $r(S_n, G) \leq 2n$ in the following special cases.

Case 1: $G \in \{G_{77}, G_{90}, G_{102}\}$, n odd, and, additionally, $n \not\equiv 2 (\mathrm{mod}\ 3)$ if $G \in \{G_{102}, G_{90}\}$. Consider any coloring of K_{2n} where $S_n \not\subseteq [V]_r$. By Lemma 3.1(i), $K_5 - 2K_2 \subseteq [V]_g$. Hence, Lemmas 3.2($ii$), 3.3($ii$), and 3.4($i$) guarantee that $G \subseteq [V]_g$.

Case 2: $G \in \{G_{67}, G_{71}, G_{87}\}$ and $n \not\equiv 2 (\mathrm{mod}\ 3)$. Note that G_{71} and G_{67} are subgraphs of G_{87}. Consider any coloring of K_{2n} where $S_n \not\subseteq [V]_r$. If $K_5 - 2K_2 \subseteq [V]_g$, then again Lemmas 3.2($ii$), 3.3($ii$), and 3.4($i$) guarantee that $G \subseteq [V]_g$. If $K_5 - 2K_2 \not\subseteq [V]_g$, then $K_5 - P_3 \subseteq [V]_g$ by Lemma 3.1(iv), and Lemma 3.5(i) yields $G \subseteq [V]_g$.

Case 3: $G \in \{G_{37}, G_{43}, G_{45}, G_{52}, G_{68}, G_{69}, G_{93}\}$. Note that $G \subseteq G_{93}$ for $G \in \{G_{37}, G_{43}, G_{45}, G_{52}, G_{69}\}$. Consider any coloring of K_{2n} where $S_n \not\subseteq [V]_r$. If $K_5 - 2K_2 \subseteq [V]_g$, then Lemmas 3.2($ii$), 3.3($ii$), and 3.4($i$) imply $G_{93} \subseteq [V]_g$ and $G_{68} \subseteq [V]_g$. Thus, by Lemma 3.1(i) and (iv), only the case n even and $K_4 \subseteq [V]_g$ is left. Let U be the vertex set of a green K_4 and $W = V \setminus U$. From Lemma 3.1(iii) we obtain $d_r(u) = n - 2$ for every $u \in U$ and $q_g(w, U) = q_r(w, U) = 2$ for every $w \in W$. Now we use induction on n. If $n = 4$, then it follows from $\Delta_r \leq n - 2 = 2$ that $[W]_g = K_4$ and $d_r(v) = 2$ for every vertex $v \in V$. Hence, $[V]_r$ is bipartite and every component of $[V]_r$ is an even cycle. This implies $[V]_r = C_4 \cup C_4$ or $[V]_r = C_8$. In both cases, $G_{93} \subseteq [V]_g$ and $G_{68} \subseteq [V]_g$. Now let $n \geq 6$. As induction hypothesis we use that any coloring of $K_{2(n-2)}$ without a red subgraph S_{n-2} contains green subgraphs G_{93} and G_{68}. Note that $|W| = 2(n - 2)$. A red S_{n-2} in $[W]$ is impossible since otherwise $q_r(w, U) = 2$ for every $w \in W$ would force $S_n \subseteq [V]_r$. Thus, $G_{93} \subseteq [W]_g$ and $G_{68} \subseteq [W]_g$, and we are done. ■

Proof of Theorem 3.3. Note that $K_5 - 2K_2 \subseteq G \subseteq G_{100}$ for every $G \in \mathcal{G}_{3,3}$. Consider any coloring of K_t where $2n - 1 \leq t \leq 2n + 1$, $n \geq 4$ and $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \leq n - 2$. If $t = 2n + 1$, then $K_5 - 2K_2 \subseteq [V]_g$ by Lemma 3.1(i). Hence, Lemmas 3.2(i), 3.3(i) and 3.4(ii) yield $G_{100} \subseteq [V]_g$. Consequently, $r(S_n, G) \leq 2n+1$ for every $G \in \mathcal{G}_{3,3}$. If n is even, then equality holds since $r(S_n, G) \geq r(S_n, K_5 - 2K_2) = 2n + 1$ (see (10)).

Now let n be odd. Again, $K_5 - 2K_2 \subseteq [V]_g$ by Lemma 3.1(i). If $t = 2n - 1$, then we obtain $G_{100} \subseteq [V]_g$ for $n \geq 13$, $G_{94} \subseteq [V]_g$ for $n \geq 9$ and $G_{83} \subseteq [V]_g$ for $n \geq 5$ using Lemmas 3.2(iii), 3.3(iii), 3.4(iii) and (iv). Note that $G_{63} \subseteq G_{83}$ and $G_{74} \subseteq G_{83}$. Thus, $r(S_n, G_{100}) \leq 2n - 1$ for $n \geq 13$, $r(S_n, G_{94}) \leq 2n - 1$ for $n \geq 9$ and $r(S_n, G) \leq 2n - 1$ for $G \in \{G_{63}, G_{74}, G_{83}\}$ if $n \geq 5$. Equality holds since $r(S_n, G) \geq 2n - 1$ for every $G \in \mathcal{G}_3$. For $t = 2n$, $n \in \{5, 7\}$, we obtain $G_{94} \subseteq [V]_g$ using Lemmas 3.2(ii), 3.3(ii) and 3.4(i). This implies $r(S_n, G_{94}) \leq 2n$ if $n \in \{5, 7\}$. Moreover, the (S_5, G_{94})-coloring of K_9 in Figure 1 proves that equality holds if $n = 5$. To complete the proof we have to consider $G = G_{100}$ where $n \in \{5, 7, 9, 11\}$. The computation of $r(S_5, G_{100})$ can be found in [13], and the bounds for $r(S_n, G)$ if $n \in \{7, 9, 11\}$ are obvious. ■

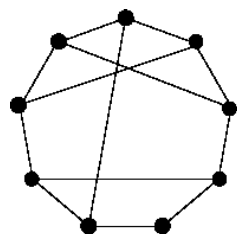

Figure 1: The red subgraph of a (S_5, G_{94})-coloring of K_9.

Proof of Theorem 3.4. Note that $G \subseteq G_{62}$, $G \subseteq G_{65}$ or $G \subseteq G_{73}$ for every $G \in \mathcal{G}_{3,4}$. Moreover, $G \subseteq G_{83}$ for every $G \in \mathcal{G}_{3,4}$ and $G_{73} \subseteq G_{87}$. First let n be odd.

Since $r(S_n, G) \geq 2n-1$ for any $G \in \mathcal{G}_3$ we only have to prove that $r(S_n, G) \leq 2n-1$. Consider any coloring of K_{2n-1} where $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \leq n-2$. By Lemma 3.1(i), $K_5 - 2K_2 \subseteq [V]_g$. Using Lemmas 3.2($iii$), 3.3($iii$) and 3.4($iv$) we obtain $G \subseteq [V]_g$ for any $G \in \mathcal{G}_{3,4}$. Now let n be even. The coloring of K_{2n-1} where $[V]_g = \overline{\frac{n}{2}K_2 + K_{n-1}}$ does not contain a red S_n. Moreover, every green subgraph of order six is contained in $K_6 - K_4$, $K_6 - (K_3 \cup P_3)$, $K_6 - (C_4 \cup K_2)$ or $K_6 - (K_5 - 2K_2)$. This implies $G \not\subseteq [V]_g$ for every $G \in \mathcal{G}_{3,4}$. Thus, $r(S_n, G) \geq 2n$. To prove that $r(S_n, G) \leq 2n$ consider any coloring of K_{2n} where $S_n \not\subseteq [V]_r$. If $K_5 - 2K_2 \subseteq [V]_g$, then we take a suitable subgraph of order $2n-1$ and are done as in the case n odd. Otherwise, Lemma 3.1(iv) forces that $K_5 - P_3 \subseteq [V]_g$. Now Lemma 3.5($i$) yields subgraphs G_{62}, G_{65} and G_{73} in $[V]_g$ and the proof is complete. ■

Proof of Theorem 3.5. First we will prove that $r(S_n, G) = 2n-1$ for $G \in \mathcal{G}_{3,5} \setminus \mathcal{S}$ if $n \geq 4$ and for $G \in \mathcal{S}$ under the conditions given in the theorem. Since $r(S_n, G) \geq 2n-1$ by (10) it remains to establish $r(S_n, G) \leq 2n-1$. Consider any coloring of K_{2n-1} where $S_n \not\subseteq [V]_r$, i.e. $\Delta_r \leq n-2$. We distinguish four cases depending on G and n.

Case 1: $G \in \mathcal{G}_{3,5} \setminus \mathcal{S}$ and $n \geq 5$ or $G \in \mathcal{S} \setminus \{G_{33}\}$ where $n = 5$ or $n \geq 7$ if $G \in \{G_{60}, G_{79}\}$, $n \geq 9$ if $G = G_{78}$ and $n \geq 13$ if $G = G_{92}$. First let $K_5 - 2K_2 \subseteq [V]_g$. Note that $G \subseteq G_{83}$ for every $G \in \mathcal{G}_{3,5} \setminus \{G_{78}, G_{92}\}$, $G_{78} \subseteq G_{94}$ and $G_{92} \subseteq G_{100}$. Consequently, the desired result follows from Lemmas 3.2(iii), 3.3(iii), 3.4(iii) and 3.4(iv). Now let $K_5 - 2K_2 \not\subseteq [V]_g$. By Lemma 3.1($v$), n has to be even and $K_5 - P_3 \subseteq_{ind} [V]_g$ or $K_5 - (P_3 \cup K_2) \subseteq_{ind} [V]_g$. Note that $G \subseteq G_{70}$ for every $G \in \mathcal{G}_{3,5} \setminus (\mathcal{S} \cup \{G_{25}, G_{35}, G_{38}, G_{46}, G_{54}\})$ and $G \subseteq G_{73}$ for every $G \in \{G_{25}, G_{35}, G_{38}, G_{46}, G_{54}\}$. Moreover, $G_{35}, G_{38} \subseteq G_{46}$, $G_{25} \subseteq G_{54}$ and $G_{60} \subseteq G_{79} \subseteq G_{92}$. Hence, the desired result follows from Lemmas 3.5(ii), 3.5(iii) and 3.6.

Case 2: $G = G_{33}$, $n \geq 5$. If $d_g(v) \geq n+1$ for some $v \in V$, then $\Delta_r \leq n-2$ guarantees two independent green edges in $[N_g(v)]$. Hence, $G_{33} \subseteq [N_g(v) \cup \{v\}]_g$. It remains $d_g(v) = n$ and $d_r(v) = n-2$ for any $v \in V$. Assume that $G_{33} \not\subseteq [V]_g$. Then any two green edges in $[N_g(v)]$ have to be adjacent, and $\Delta_r \leq n-2$ forces $[N_g(v)]_g = K_{1,n-1}$ and $[N_g(v)]_r = K_{n-1} \cup K_1$. Let U be the vertex set of the red $K_{n-1} \subseteq [N_g(v)]$ and $W = V \setminus U$. All edges between U and W have to be green because of $\Delta_r \leq n-2$. But then $d_g(v) = n$ for every $v \in V$ guarantees two independent green edges in $[W]$ c.ontradicting $G_{33} \not\subseteq [V]_g$.

Case 3: $G = G_{78}$, $n = 5$. Then $\Delta_r \leq n-2 = 3$. Since $[V]_r$ cannot be 3-regular, there is a vertex $v \in V$ with $d_g(v) \geq 6$. Moreover, a vertex $w \in V$ exists such that $|N_g(v) \cap N_g(w)| \geq 4$. Let $U = \{u_1, u_2, u_3, u_4\} \subseteq N_g(v) \cap N_g(w)$. If $[U]$ contains a green edge, then $G_{78} \subseteq [U \cup \{v, w\}]_g$. Otherwise, $[U]_r = K_4$, and $\Delta_r \leq 3$ forces only green edges between U and $W = V \setminus U$. Furthermore, $[W]$ must contain a green edge w_1w_2. Consequently, a green G_{78} occurs in the subgraph induced by u_1, u_2, w_1, w_2 and two other vertices $w_3, w_4 \in W$.

Case 4: $G \in \mathcal{G}_{3,5} \setminus \mathcal{S}$, $n = 4$. Then $G \subseteq G_{70}$, $G \subseteq G_{54}$ or $G \subseteq G_{46}$. From $\Delta_r \leq n-2 = 2$ we obtain that $[V]_r \subseteq H$ where $H \in \{K_1 \cup K_3 \cup K_3,\ K_1 \cup C_6,\ K_2 \cup$

$C_5,\ K_3 \cup C_4,\ C_7\}$. In any case, G_{70}, G_{54} and G_{46} are subgraphs of $[V]_g$ and we are done.

Now let us prove the additional results given in the theorem. We first consider $r(S_4, G)$ for $G \in \mathcal{S}$. The coloring of K_7 where $[V]_r = C_7$ establishes $r(S_4, G) \geq 8$. For any coloring of K_8 with $S_4 \not\subseteq [V]_r$ we obtain that $[V]_r \subseteq H$ with $H \in \{K_1 \cup K_3 \cup C_4,\ K_1 \cup C_7,\ K_2 \cup K_3 \cup K_3,\ K_2 \cup C_6,\ K_3 \cup C_5,\ C_4 \cup C_4,\ C_8\}$. In any case we find green subgraphs G_{92}, G_{78} and G_{33}. Since G_{60}, $G_{79} \subseteq G_{92}$ we are done. To prove $r(S_5, G_{92}) = 11$ we use that $K_{3,3} \subseteq G_{92} \subseteq G_{100}$. It is known that $r(S_5, K_{3,3}) = 11$ (see [24]) and, by Theorem 3.3, $r(S_5, G_{100}) = 11$. This implies the desired result. To complete the proof note that $G \subseteq G_{100}$ for every $G \in \mathcal{G}_{3,5}$ and $G_{78} \subseteq G_{93}$. Thus, $r(S_n, G) \leq 2n+1$ for every $G \in \mathcal{G}_{3,5}$ by Theorem 3.3 and $r(S_n, G_{78}) \leq 2n$ by Theorem 3.2. Since $r(S_n, G) \geq 2n - 1$ for any $G \in \mathcal{G}_3$, we are done. ■

4 The Ramsey Number $r(S_n, G)$ for $G \in \mathcal{G}_2$

The set $\mathcal{G}_2$ consists of all graphs from Table 1 which have not yet been considered, i.e. all connected spanning subgraphs of $K_{1,5} = G_6$, $K_{2,4} = G_{53}$ or $K_{3,3} = G_{76}$. This gives

$$\mathcal{G}_2 = \{G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_9, G_{11}, G_{12}, G_{16}, G_{20}, G_{29}, G_{31}, G_{53}, G_{59}, G_{76}\}.$$

In the following theorem $r(S_n, G)$ is evaluated for all $G \in \mathcal{G}_2$ and $4 \leq n \leq 5$.

Theorem 4.1.

$$r(S_4, G) = \begin{cases} 6 & \text{if } G \in \{G_1,\ G_4,\ G_5,\ G_7,\ G_9,\ G_{11}\}, \\ 7 & \text{if } G \in \{G_2,\ G_3,\ G_{12},\ G_{16},\ G_{20},\ G_{29},\ G_{31},\ G_{59}\}, \\ 8 & \text{if } G \in \{G_6,\ G_{53},\ G_{76}\}. \end{cases}$$

$$r(S_5, G) = \begin{cases} 7 & \text{if } G \in \{G_1,\ G_2,\ G_3,\ G_4,\ G_5,\ G_9,\ G_{12}\}, \\ 8 & \text{if } G \in \{G_7,\ G_{11},\ G_{16},\ G_{20}\}, \\ 9 & \text{if } G \in \{G_6,\ G_{29},\ G_{31},\ G_{53},\ G_{59}\}, \\ 11 & \text{if } G = G_{76}. \end{cases}$$

Proof. We first determine $r(S_4, G)$. Let $G \in \{G_1,\ G_4,\ G_5,\ G_7,\ G_9,\ G_{11}\}$. Clearly, $r(S_4, G) \geq 6$. To establish equality, consider any coloring of K_6 where $S_4 \not\subseteq [V]_r$. Consequently, $[V]_r \subseteq H$ with $H \in \{C_6,\ C_5 \cup K_1,\ C_4 \cup K_2,\ 2K_3\}$. In any case, $G \subseteq [V]_g$. Now let $G \in \{G_2,\ G_3,\ G_{12},\ G_{16},\ G_{20},\ G_{29},\ G_{31},\ G_{59}\}$. Since $G \subseteq G_{70}$, $r(S_4, G) \leq 7$ follows from Theorem 3.5. To prove that $r(S_4, G) \geq 7$ we use three different colorings of K_6. If $[V]_r = 2K_3$, then we obtain an (S_4, G)-coloring for $G \in \{G_2,\ G_3,\ G_{12},\ G_{16},\ G_{31}\}$, $[V]_r = C_4 \cup K_2$ yields an (S_4, G_{20})-coloring, and $[V]_r = C_6$ gives an (S_4, G)-coloring for $G \in \{G_{29},\ G_{59}\}$. Finally let $G \in \{G_6,\ G_{53},\ G_{76}\}$. The coloring of K_7 where $[V]_r = C_7$ proves $r(S_4, G) \geq 8$. Because $G_6 \subseteq G_{62}$, $G_{53} \subseteq G_{93}$ and $G_{76} \subseteq G_{92}$, we obtain $r(S_4, G) \leq 8$ using Theorems 3.4, 3.2 and 3.5.

Consider now $r(S_5, G)$. First let $G \in \{G_1, G_2, G_3, G_4, G_5, G_9, G_{12}\}$. The coloring of K_6 where $[V]_r = K_{3,3}$ implies $r(S_5, G) \geq 7$. Since $G_1, G_4 \subseteq G_9$ and $G_2, G_3 \subseteq G_{12}$ it remains to prove that $r(S_5, G) \leq 7$ for $G \in \{G_5, G_9, G_{12}\}$. Consider any coloring of K_7 with $S_5 \not\subseteq [V]_r$, i.e. $d_r(v) \leq 3$ for every $v \in V$. As $r(S_5, C_4) = 7$ (see [7]), a green C_4 must occur. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green C_4 where the edges u_1u_2, u_2u_3, u_3u_4 and u_1u_4 are green. Moreover, let $W = \{w_1, w_2, w_3\} = V \setminus U$. Because of $S_5 \not\subseteq [V]_r$, $q_g(w, U) \geq 1$ for every $w \in W$, and $q_g(w, U) = 1$ implies only green edges incident to w in $[W]$. Consider first that two edges in $[W]$, say w_1w_2 and w_1w_3, are red. Then $S_5 \not\subseteq [V]_r$ implies $q_g(w_1, U) \geq 3$ and $q_g(w_i, U) \geq 2$ for $i = 2$ and $i = 3$. We may assume that the edges from w_1 to u_1, u_2 and u_3 are green. Because one of the edges from w_2 to u_1, u_2 and u_3 has to be green, $G_5, G_{12} \subseteq [V]_g$. Obviously, $G_9 \subseteq [V]_g$ if w_2w_3 is green. If w_2w_3 is red, then $q_g(w_2, U) \geq 3$, and this also yields $G_9 \subseteq [V]_g$. The remaining case is that two edges in $[W]$, say w_1w_2 and w_1w_3, are green. Since $q_g(w_1, U) \geq 1$, $G_5, G_9 \subseteq [V]_g$, and it remains to prove that $G_{12} \subseteq [V]_g$. Clearly, $G_{12} \subseteq [V]_g$ if $q_g(u, W) \geq 2$ for some $u \in U$. Otherwise, $q_g(U, W) \leq 4$, and this yields $q_g(w_i, U) = q_g(w_j, U) = 1$ for two vertices $w_i, w_j \in W$. Thus, also w_2w_3 has to be green. Furthermore we may assume that the edges w_1u_1, w_2u_2 and w_3u_3 are green. Then $d_r(u_4) \leq 3$ forces one of the edges from u_4 to $\{u_2, w_1, w_2, w_3\}$ to be green and again we obtain $G_{12} \subseteq [V]_g$.

Now let $G \in \{G_7, G_{11}, G_{16}, G_{20}\}$. The coloring of K_7 where $[V]_g$ consists of two green copies of K_4 with exactly one common vertex implies $r(S_5, G) \geq 8$. Since $G_7, G_{11} \subseteq G_{20}$ it remains to establish $r(S_5, G) \leq 8$ for $G \in \{G_{16}, G_{20}\}$. Consider any coloring of K_8 where $S_5 \not\subseteq [V]_r$. To prove that $G_{16} \subseteq [V]_g$ we use $r(S_5, G_{12}) = 7$. Consequently, $G_{12} \subseteq [V]_g$. Let $U = \{u_1, u_2, \ldots, u_6\}$ be the vertex set of a green G_{12} where the edges from u_1 to u_2, u_3, u_4, u_5 and the edges u_6u_2, u_6u_3 are green. Since $S_5 \not\subseteq [V]_r$, one of the edges from u_6 to $\{u_4, u_5\} \cup (V \setminus U)$ has to be green and this yields $G_{16} \subseteq [V]_g$. To prove that $G_{20} \subseteq [V]_g$ we use $r(C_4, G_{20}) = 7$ (see [20]). Suppose that $G_{20} \not\subseteq [V]_g$. Then a red C_4 must occur. Let U be the vertex set of a red C_4 and $W = V \setminus U$. As $S_5 \not\subseteq [V]_r$, $q_g(u, W) \geq 3$ for every $u \in U$. Hence we find three vertices in U and three vertices in W yielding a green $G_{20} = K_{3,3} - 2K_2$, a contradiction.

Consider now $G \in \{G_6, G_{29}, G_{31}, G_{53}, G_{59}\}$. The coloring of K_8 where $[V]_r = 2K_4$ shows that $r(S_5, G_6) \geq 9$. For $G \neq G_6$ we obtain $r(S_5, G) \geq 9$ from $K_{2,3} \subseteq G$ and $r(S_5, K_{2,3}) = 9$ (see [17]). To prove $r(S_5, G) \leq 9$, note that $G_6, G_{29}, G_{59} \subseteq G_{83}$ and $G_{31}, G_{53} \subseteq G_{78}$. Thus, the desired result follows from $r(S_5, G_{78}) = r(S_5, G_{83}) = 9$, proved in Theorem 3.5 and Theorem 3.3. For the remaining case $G = G_{76} = K_{3,3}$ the value of $r(S_5, G)$ has been determined in [24]. ■

For the six trees $G \in \mathcal{G}_2$, the values of $r(S_n, G)$ are almost completely known from general results obtained for $r(S_n, T_m)$. Harary [16] proved that

$$r(S_n, S_m) = n + m - 3 + \epsilon \tag{11}$$

where $\epsilon = 1$ if n or m is even and $\epsilon = 0$ otherwise. Burr [2] obtained the following result:

$$r(S_n, T_m) = n + m - 2 \text{ if } n, m \geq 3 \text{ and } n - 2 \equiv 0 \pmod{m-1}. \tag{12}$$

Guo and Volkmann [14] showed that

$$r(S_n, T_m) \leq n + m - 3 \text{ if } m, n \geq 3,\ n - 2 \not\equiv 0 \pmod{m-1} \text{ and } T_m \neq S_n, \tag{13}$$

and that equality holds if $n = m \geq 4$ or if in case of $n > m$ one of the following conditions is fulfilled: $n - 2 = k(m-1) + 1$ with $k \in \mathbb{N}$ or $n - 2 = k(m-1) + r$ with $k \in \mathbb{N}$, $2 \leq r \leq m - 2$ and $\Delta(T_m) = m - 2$ or $k + r + 2 - m \geq 0$. Parsons [30] determined $r(S_n, P_m)$ for the path P_m on m vertices by explicit formulas and a recurrence, in particular he obtained the following result:

$$r(S_{m+k}, P_m) = 2m - 1 \text{ if } 1 \leq k < (m+4)/3. \tag{14}$$

Here we will determine the missing values of $r(S_n, G)$ for the trees $G \in \mathcal{G}_2$ and summarize the results in the following theorem.

Theorem 4.2. *Let $n \geq 6$ and $G \in \{G_1, G_2, G_3, G_4, G_5, G_6\}$. Then*

$$r(S_n, G) = \begin{cases} n + 4 & \text{if } G = G_6 \text{ or if } n \equiv 2 \pmod 5 \text{ and } G \neq G_6, \\ n + 2 & \text{if } n = 9 \text{ and } G \in \{G_1,\ G_4,\ G_5\}, \\ n + 3 & \text{otherwise.} \end{cases}$$

Proof. The case $G = G_6 = S_6$ is settled by (11), and for $G \neq G_6$, $n \equiv 2 \pmod 5$ we are done by (12). Using (13) where equality holds, we obtain $r(S_n, G)$ for $G = G_3$, and for $G \in \{G_1, G_2, G_4, G_5\}$ only $n = 9$ is left. From (14) we derive $r(S_9, G_1) = 11$. By (13), $r(S_9, G_2) \leq 12$, and the coloring of K_{11} where $[V]_g = K_5 \cup K_{3,3}$ yields equality. It remains to prove $r(S_9, G) = 11$ for $G \in \{G_4,\ G_5\}$. The coloring of K_{10} with $[V]_g = 2K_3 \cup K_4$ implies $r(S_9, G) \geq 11$. To establish equality, consider any coloring of K_{11} where $S_9 \not\subseteq [V]_r$. Since $r(S_9, G_1) = 11$, a green P_6 must occur. Let $U = \{u_1, u_2, \ldots, u_6\}$ be the vertex set of a green P_6 where the edges $u_i u_{i+1}$ are green for $i = 1, \ldots, 5$. Moreover, let $W = V \setminus U$. If one of the edges from u_2 to u_4, u_5 or u_6 is green, then $G_4 \subseteq [V]_g$. Otherwise, $S_9 \not\subseteq [V]_r$ implies that $u_2 w$ is green for some $w \in W$. Similarly, at least one edge from w to $(W \setminus \{w\}) \cup \{u_3, u_4, u_5, u_6\}$ has to be green, and again we find a green G_4. It remains to prove that $G_5 \subseteq [V]_g$. A vertex $v \in V(K_{11})$ with $d_r(v) \neq 7$ must exist. Consequently, $S_9 \not\subseteq [V]_r$ forces $d_r(v) \leq 6$, i.e. $d_g(v) \geq 4$. Let $U = \{u_1, u_2, u_3, u_4\} \subseteq N_g(v)$, $U' = U \cup \{v\}$, and $W = V \setminus U' = \{w_1, \ldots, w_6\}$. Suppose $G_5 \not\subseteq [V]_g$. From $r(S_4, G_5) = 6$ we obtain $S_4 \subseteq [W]_r$. We may assume that the edges from w_1 to w_2, w_3 and w_4 are red. Because of $S_9 \not\subseteq [V]_r$, $q_g(w_1, U') \geq 1$. If $q_g(w_1, U) \geq 1$, say $w_1 u_1$ is green, then $S_9 \not\subseteq [V]_r$ forces $q_g(u_1, (W \setminus \{w_1\}) \cup (U \setminus \{u_1\})) \geq 1$. This gives $G_5 \subseteq [V]_g$, a contradiction. It remains that $w_1 v$ is green and all edges from w_1 to U are red. But then $S_9 \not\subseteq [V]_r$

forces only green edges from w_1 to w_5 and w_6. Again $G_5 \subseteq [V]_g$, and we are done. ■

Next we consider the six non-tree graphs $G \in \mathcal{G}_2$ where $G \neq K_{2,4}$ and $C_6 \nsubseteq G$. Since $C_4 \subseteq G$, $r(S_n, G) \geq r(S_n, C_4)$ for $G \in \{G_9,\ G_{11},\ G_{12},\ G_{16}\}$, and $K_{2,3} \subseteq G$ implies $r(S_n, G) \geq r(S_n, K_{2,3})$ for $G \in \{G_{29},\ G_{31}\}$. We will show that in both cases equality holds if n is sufficiently large. The following lemma is essential for proving this result.

Lemma 4.1. *If $r(S_n, C_4) \geq n+4$ and $G \in \{G_9,\ G_{11},\ G_{12},\ G_{16}\}$, then $r(S_n, G) = r(S_n, C_4)$. If $r(S_n, K_{2,3}) \geq n+4$ and $G \in \{G_{29},\ G_{31}\}$, then $r(S_n, G) = r(S_n, K_{2,3})$.*

Proof. It suffices to establish the missing upper bounds for $r(S_n, G)$. Assume first that $r(S_n, C_4) \geq n+4$ and consider any coloring of K_t where $t = r(S_n, C_4)$ and $S_n \nsubseteq [V]_r$. Then $C_4 \subseteq [V]_g$ and $d_g(v) \geq 5$ for every $v \in V$. Let U be the vertex set of a green C_4. Since $|N_g(u) \setminus U| \geq 2$ for any $u \in U$, $G_i \subseteq [V]_g$ for $i \in \{11, 12, 16\}$. To find a green G_9, take a vertex $v \in N_g(u) \setminus U$ for some $u \in U$. As $|N_g(v) \setminus U| \geq 1$, the desired result follows. Assume now that $r(S_n, K_{2,3}) \geq n+4$ and consider any coloring of K_t where $t = r(S_n, K_{2,3})$ and $S_n \nsubseteq [V]_r$. Then $K_{2,3} \subseteq [V]_g$ and $d_g(v) \geq 5$ for every $v \in V$. Let U be the vertex set of a green $K_{2,3}$. Because $|N_g(u) \setminus U| \geq 1$ for every $u \in U$, $G_{29} \subseteq [V]_g$ and $G_{31} \subseteq [V]_g$, and we are done. ■

By (8) and $r(S_n, C_4) \leq r(S_n, K_{2,3})$, the conditions on $r(S_n, C_4)$ and $r(S_n, K_{2,3})$ in Lemma 4.1 are satisfied if n is sufficiently large, and we obtain the following result.

Theorem 4.3. *If n is sufficiently large, then $r(S_n, G) = r(S_n, C_4)$ for $G \in \{G_9, G_{11}, G_{12}, G_{16}\}$ and $r(S_n, G) = r(S_n, K_{2,3})$ for $G \in \{G_{29},\ G_{31}\}$.*

It remains an open problem to determine the exact values of $r(S_n, G)$ if $G \in \{G_9,\ G_{11}, G_{12},\ G_{16},\ G_{29},\ G_{31}\}$ and all $n \geq 6$. For $G \in \{G_9,\ G_{11},\ G_{12},\ G_{16}\}$ it follows from Lemma 4.1, (6), (7) and (8), that the exact value of $r(S_n, G)$ is known for infinitely many n and

$$n - 1 + \left\lfloor \sqrt{n-1} - 6(n-1)^{11/40} \right\rfloor < r(S_n, G) \leq n + \left\lceil \sqrt{n-1} \right\rceil$$

for n sufficiently large. In [3] it is shown that $r(S_n, K_{2,3}) < n + 2\sqrt{n}$ for all sufficiently large n. Consequently, for $G \in \{G_{29},\ G_{31}\}$ and n sufficiently large,

$$n - 1 + \left\lfloor \sqrt{n-1} - 6(n-1)^{11/40} \right\rfloor < r(S_n, G) < n + 2\sqrt{n}.$$

The remaining non-tree graphs in $\mathcal{G}_2$ are $G_{53} = K_{2,4}$ and the four subgraphs of $K_{3,3}$ containing a subgraph isomophic to C_6, namely $G_7 = C_6$, $G_{20} = K_{3,3} - 2K_2$, $G_{59} = K_{3,3} - K_2$ and $G_{76} = K_{3,3}$. The values of $r(S_n, C_6)$ for $6 \leq n \leq 12$ can be found in [36]: $r(S_n, C_6) = n+4$ if $6 \leq n \leq 7$ or $10 \leq n \leq 12$ and $r(S_n, C_6) = n+3$ if $8 \leq n \leq 9$. Moreover, $r(S_6, K_{2,4}) = 11$, $r(S_6, K_{3,3}) = 12$ and $r(S_7, K_{2,4}) = r(S_7, K_{3,3}) = 13$ (see [24]). From [3] we know that, for n sufficiently large, $r(S_n, K_{2,4}) < n + 3\sqrt{n}$ and $r(S_n, G) < n + 3\, n^{2/3}$ for all $G \in \{G_7, G_{20}, G_{59}, G_{76}\}$, but it remains an unsolved problem to determine further exact values.

References

[1] L. Boza, Corrections to "The Ramsey numbers for a quadrilateral vs. all graphs on six vertices", *J. Combin. Math. Combin. Comput.* **89** (2014), 155–156.

[2] S. A. Burr, Generalized Ramsey theory for graphs—a survey, in: *Graphs and Combinatorics*, Lec. Notes in Math. **406** (R. A. Bari and F. Harary, eds.), Springer, Berlin, 1974, 52–75.

[3] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Some complete bipartite graph-tree Ramsey numbers, *Ann. Discrete Math.* **41** (1989), 79–89.

[4] G. Chartrand, R. J. Gould and A. D. Polimeni, On Ramsey numbers of forests versus nearly complete graphs, *J. Graph Theory* **4** (1980), 233–239.

[5] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

[6] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.

[7] M. Clancy, Some small Ramsey numbers, *J. Graph Theory* **1** (1977), 89–91.

[8] P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, The book-tree Ramsey numbers, *Scientia, Ser. A: Mathematical Sciences* **1** (1988), 111–117.

[9] R. J. Faudree, C. C. Rousseau, and R. H. Schelp, All triangle–graph Ramsey numbers for connected graphs of order six, *J. Graph Theory* **4** (1980), 293–300.

[10] R. J. Faudree, C. C. Rousseau and R. H. Schelp, Small order graph-tree Ramsey numbers, *Discrete Math.* **72** (1988), 119–127.

[11] R. J. Faudree, R. H. Schelp and C. C. Rousseau, Generalizations of a Ramsey result of Chvátal, in: *Proc. Fourth Int. Conf. on the Theory and Applications of Graphs*, Kalamazoo, 1980, 351–361.

[12] R. J. Gould and M. S. Jacobson, On the Ramsey number of trees versus graphs with large clique number, *J. Graph Theory* **7** (1983), 71–78.

[13] G. Hua, S. Hongxue and L. Xiangyang, Ramsey numbers $r(K_{1,4}, G)$ for all three-partite graphs G of order six, *J. Southeast Univ. (English Edition)* **20** (2004), 378–380.

[14] Y. B. Guo and L. Volkmann, Tree-Ramsey numbers, *Australas. J. Combin.* **11** (1995), 169–175.

[15] F. Harary, *Graph Theory*, Addison-Wesley, Reading (Massachusetts), 1969.

[16] F. Harary, Recent results on generalized Ramsey theory for graphs, in: *Graph Theory and Applications*, Lec. Notes in Math. **303** (Y. Alavi et al., eds.), Springer, Berlin, 1972, 125–138.

[17] G. R. T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory* **13** (1989), 245–248.

[18] M. Hoeth and I. Mengersen, Ramsey numbers for graphs of order four versus connected graphs of order six, *Util. Math.* **57** (2000), 3–19.

[19] C. J. Jayawardene and C. C. Rousseau, Ramsey numbers $r(C_6, G)$ for all graphs G of order less than six, *Congr. Numer.* **136** (1999), 147–159.

[20] C. J. Jayawardene and C. C. Rousseau, The Ramsey numbers for a quadrilateral vs. all graphs on six vertices, *J. Combin. Math. Combin. Comput.* **35** (2000), 71–87, Erratum, *J. Combin. Math. Combin. Comput.* **51** (2004), 221.

[21] C. J. Jayawardene and C. C. Rousseau, The Ramsey number for a cycle of length five vs. a complete graph of order six, *J. Graph Theory* **35** (2000), 99–108.

[22] C. J. Jayawardene and C. C. Rousseau, Ramsey numbers $r(C_5, G)$ for all graphs G of order six, *Ars Combin.* **57** (2000), 163–173.

[23] M. Krone and I. Mengersen, The Ramsey numbers $r(K_5-2K_2, 2K_3)$, $r(K_5-e, 2K_3)$, and $r(K_5, 2K_3)$, *J. Combin. Math. Combin. Comput.* **81** (2012), 257–260.

[24] R. Lortz and I. Mengersen, Further Ramsey numbers for small complete bipartite graphs, *Ars Combin.* **79** (2006), 195–203.

[25] R. Lortz and I. Mengersen, Ramsey numbers for small graphs versus small disconnected graphs, *Australas. J. Combin.* **51** (2011), 89–108.

[26] R. Lortz and I. Mengersen, On the Ramsey numbers of certain graphs of order five versus all connected graphs of order six, *J. Combin. Math. Combin. Comput.* **90** (2014), 197–222.

[27] R. Lortz and I. Mengersen, On the Ramsey numbers $r(S_n, K_6 - 3K_2)$, *J. Combin. Math. Combin. Comput.* (to appear).

[28] R. Lortz and I. Mengersen, On the Ramsey numbers of non-star trees versus connected graphs of order six, (in preparation).

[29] J. McNamara, $r(K_4 - e, K_6) = 21$, (unpublished).

[30] T. D. Parsons, Path-star Ramsey numbers, *J. Combin. Theory (B)* **17** (1974), 51–58.

[31] T. D. Parsons, Ramsey graphs and block designs I, *Trans. Amer. Math. Soc.* **209** (1975), 33–44.

[32] S. P. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* (2014) DS 1.14.

[33] C. C. Rousseau and C. J. Jayawardene, The Ramsey number for a quadrilateral vs. a complete graph on six vertices, *Congr. Numer.* **123** (1997), 97–108.

[34] C. C. Rousseau and J. Sheehan, A class of Ramsey problems involving trees, *J. London Math. Soc.*(2) **18** (1978), 392–396.

[35] Y. Wu, Y. Sun, R. Zhang and S. P. Radziszowski, Ramsey numbers of C_4 versus wheels and stars, *Graphs Combin.* **31** (2015), 2437–2446.

[36] Y. Zhang, H. Broersma and Y. Chen, Narrowing down the gap on cycle-star Ramsey numbers, *J. Comb.* **7** (2016), 481–493.

(Received 22 Feb 2017; revised 20 Apr 2018, 14 Sep 2018)