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Independent domination bicritical graphs

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Abstract

Let G be a graph. An *independent dominating set* of G is a subset $D \subseteq V(G)$ such that no two vertices in D are adjacent, and every vertex of G either belongs to D or is adjacent to a vertex in D. The size of a smallest independent dominating set of G is the *independent domination number*, $i(G)$. The graph G is *i-critical* if $i(G-x) < i(G)$ for all vertices x, and is *i-bicritical* if $i(G - \{x, y\}) < i(G)$ for all 2-subsets of vertices $\{x, y\}$. It is shown that i-bicritical graphs differ structurally from γ-bicritical graphs, which are those in the corresponding collection defined with respect to the domination number. Several methods of constructing i-bicritical graphs from other graphs are described. Conditions that must be satisfied by the constituent graphs in order for the resulting graph to be i-bicritical are given. Some of these graphs are also i-critical.

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A subset $D \subseteq V(G)$ is called an *independent dominating set* of the graph G if it is both a dominating set and an independent set. The minimum cardinality among all independent dominating sets of G is the *independent domination number*, $i(G)$. An independent dominating set of minimum cardinality is called an *i-set*.

For a vertex $v \in V(G)$, the number $i(G - v)$ may be greater than, less than, or equal to $i(G)$. A graph G is *independent domination critical*, or *i-critical* if $i(G - v) < i(G)$ for every $v \in V(G)$. More generally, for an integer $t \geq 1$, a graph G is (i, t)-critical if $i(G - S) < i(G)$ for any $S \subseteq V(G)$ with $|S| = t$. Independent domination critical graphs are $(i, 1)$-critical graphs. For surveys about independent domination and independent domination critical graphs, see [9, 11].

In this paper we study $(i, 2)$-critical graphs, which we refer to as *i-bicritical graphs*. These were first considered by Xu, Xu, and Zhang [15], who described some of their basic properties and gave a construction that produces a new i-bicritical graph from a graph which is both i-critical and i-bicritical. Examples of i-bicritical graphs given in [15] include $K_{n,n}$, $K_{n,n+1}$ and the Cartesian product $K_n \square K_n$, where $n \geq 3$ in each case. The graph G shown in Figure 1 can also be seen to be i-bicritical.

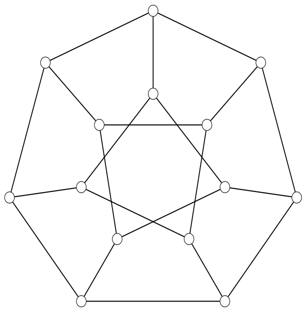

Figure 1: An i-bicritical graph.

Bicriticality for domination was first studied in [2]. We shall reference these results, in context, throughout this paper. The (γ, t)-critical graphs, defined analogously to the (i, t)-critical graphs, were introduced by Mojdeh, Firoozi, and Hasni [13]. The $(\gamma, 1)$-critical graphs are the γ-critical graphs. The $(\gamma, 2)$-critical graphs are commonly referred to as γ-*bicritical* graphs. Constructions of bicritical graphs with edge connectivity 2 can be found in [3]. It is easy to observe that if G is i-bicritical and $\gamma(G) = i(G)$, then G is γ-bicritical. For each $n \geq 3$, the Cartesian product $K_n \square K_n$ is an example of such a graph. The (γ, k)-critical graphs have been further studied in [12] and [7].

This paper is organized as follows. Notation, terminology and basic properties of i-bicritical graphs are reviewed in the next section. It is shown that i-bicritical graphs have different structural properties than γ-bicritical graphs. In particular, they may have cut vertices or cut-edges. In Section 3 we characterize the i-bicritical graphs with independent domination number 2, and show that for each $k \geq 4$ and every graph G there exists an i-bicritical graph H with $i(H) = k$ such that G is an induced subgraph of H. When $i(G) \geq 4$ the graph H can be chosen so that $i(G) = i(H)$. When $i(G) = 3$ it is an open question whether there exists such an H with $i(H) = 3$. In the remaining sections we consider constructions of i-bicritical graphs using the operations of disjoint union, join, coalescence, identification on a subgraph, and wreath product.

2 Notation, terminology and basic properties

Definitions and notation for graphs and domination are followed from [9, 10], and [14].

For a set of vertices $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by the vertices in S. For a set $S \subseteq V(G)$, $G - S$ is the graph $\langle V(G) - S \rangle$ and for a vertex $v \in V(G)$, $G - v$ is $\langle V(G) - \{v\} \rangle$. For a vertex $x \in V(G)$, the *open neighbourhood*, $N_G(x)$, is the set $\{y \mid xy \in E(G)\}$, and the *closed neighbourhood*, $N_G[x]$, is the set $N_G[x] = N_G(x) \cup \{x\}$. Analogously, for a set $S \subseteq V(G)$, the *open neighbourhood of* S, $N_G(S)$, is the set $\{x \mid xy \in E(G)$ for some $y \in S\}$, and the *closed neighbourhood of* S, $N_G[S]$, is the set $N_G[S] = N_G(S) \cup S$. When the graph G is obvious from context, we simply write $N(x)$, $N[x]$, $N(S)$, and $N[S]$.

Let G_1 and G_2 be graphs. The *union of* G_1 *and* G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Note that the graphs G_1 and G_2 may have vertices or edges in common. For $k \geq 3$, the union of the graphs $G_1, G_2, \ldots, G_k$ can be recursively defined by

$$\bigcup_{i=1}^{k} G_i = \left(\bigcup_{i=1}^{k-1} G_i \right) \bigcup G_k.$$

The operation of *disjoint union* of graphs corresponds to the union of disjoint graphs.

Let D be a subset of vertices of a graph G. We say that D *dominates* a vertex v if either $v \in D$ or v is adjacent to a vertex in D. For a set of vertices $S \subseteq V$ we say that D *dominates* S if it dominates every vertex of S. The set D is a *dominating set* if it dominates V. The *domination number* of G, $\gamma(G)$, is the smallest cardinality of a dominating set of G.

Let G be a graph. We identify the following three disjoint subsets whose union is V:

(i) $V_i^+ = \{v : i(G - v) > i(G)\}$;

(ii) $V_i^0 = \{v : i(G - v) = i(G)\}$;

(iii) $V_i^- = \{v : i(G - v) < i(G)\}$.

The set V_i^0 is the set of *i-stable* vertices, and the set V_i^- is the set of *i-critical* vertices. A graph is i-critical if and only if $V = V_i^-$. If G is i-critical and $i(G) = k$ we say that G is *k-i-critical.* The corresponding concepts for the domination number, γ, are defined similarly. Notice that the cycles C_4 and C_7 are both i-critical and γ-critical.

The following properties of i-bicritical graphs are proved in [15].

Theorem 2.1. [15] *Let G be an i-bicritical graph with at least two vertices. Then,*

(a) $i(G) - 2 \leq i(G - \{x, y\}) \leq i(G) - 1$;

(b) for any vertex v, $i(G) - 1 \leq i(G - v) \leq i(G)$;

(c) if $i(G - v) = i(G)$, then $G - v$ is i-critical;

(d) if $x, y \in V$ are such that $i(G - \{x, y\}) = i(G) - 2$, then $d(x, y) \geq 2$;

(e) G has no vertex of degree 2;

(f) G is not a tree.

Statements (a), (b) and (c) above follow from the general observation that, for any graph G and $S \subseteq V(G)$, $i(G - S) \geq i(G) - |S|$. To see that both extremes can occur in the inequality in (a), consider $K_{n,n}$ and $K_{n,n+1}$. The graph $K_{n,n+1}$ also demonstrates that both extremes can occur in the inequalities in (b). Statement (e) can be seen as the i-bicritical equivalent of the result that a γ-bicritical graph can not have a vertex of degree 1 [1]. In fact, a connected γ-bicritical graph must have $\delta \geq 3$, $\gamma \geq 3$ and edge connectivity at least 2. To see that i-bicritical graphs can have cut vertices and cut edges, consider the graph constructed from $K_{3,4}$ by adding a new vertex and joining it to one of the vertices in the independent set of size 4.

Graphs with the property that $i(G - \{x, y\}) = i(G) - 2$ for any two independent vertices x and y are called *strongly i-bicritical* graphs. These are studied in detail in [5]. They have more structure than i-bicritical graphs. For example, if G is strongly i-bicritical then G is 2-connected and has minimum degree $\delta \geq 3$.

We now establish several other properties of i-bicritical graphs.

Proposition 2.2. *If G is i-bicritical, then there does not exist $v \in V(G)$ such that $\langle N(v) \rangle$ has $K_{2,m}, m \geq 0$, as a spanning subgraph.*

Proof. Suppose G is i-bicritical and let $v \in V(G)$ such that $\langle N(v) \rangle$ has $K_{2,m}$ as a spanning subgraph. Let $\{v_1, v_2\}$ be the vertices in the independent set of size 2 in this copy of $K_{2,m}$ and let D be an i-set of $G - \{v_1, v_2\}$. If there is a vertex x with $x \in (N[v] - \{v_1, v_2\}) \cap D$, then D is also an independent dominating set of G, a contradiction. If $(N[v] - \{v_1, v_2\}) \cap D = \emptyset$, then D does not dominate v, a contradiction. The result follows. □

Proposition 2.3. *If G is connected and i-bicritical, then at most one vertex of G has a neighbour of degree 1. Furthermore, if $v \in V(G)$ has a neighbour of degree 1, then v is the only vertex of G which is not an i-critical vertex.*

Proof. Suppose there exist $u, v \in V(G)$ such that $u \neq v$ and both of these vertices have a neighbour of degree 1. Let u' be a degree 1 neighbour of u, and v' be a degree 1 neighbour of v.

Since G is i-bicritical, $G - \{u, v\}$ has an independent dominating set, D, of size at most $i(G) - 1$. But u', v' are isolated vertices of $G - \{u, v\}$, hence $u', v' \in D$. Thus

D is an independent dominating set of G, a contradiction. Therefore at most one vertex of G has a neighbour of degree 1.

Suppose $v \in V(G)$ is the only vertex of G with a neighbour of degree 1, say v'. As above, since v' is in any independent dominating set of $G - v$, any such set dominates G. Therefore, $i(G - v) = i(G)$, so that v is not an i-critical vertex of G.

Now let $x \in V(G) - \{v\}$. We claim that x is a critical vertex of G. Since G is i-bicritical, $i(G - \{v, x\}) < i(G)$. Since v' is in any independent dominating set of $G - \{v, x\}$, any such set dominates $G - x$. This completes the proof. □

For $n \geq 3$, the graph constructed from $K_{n,n+1}$ by adding a new vertex and joining it to one of the vertices in the independent set of size $n+1$ is a connected i-bicritical graph with a vertex of degree 1 (also, see Figure 2). We do not know if it is possible for an i-bicritical graph with at least 3 vertices to have more than one vertex of degree 1.

Proposition 2.4. *If G is connected and i-bicritical, then $diam(G) \leq 2i(G) - 1$.*

Proof. We use the fact that $2i-2$ is a sharp upper bound on the diameter of connected i-critical graphs [4]. Suppose G is a connected, i-bicritical graph. If G is i-critical, then the above bound holds. Otherwise, by Theorem 2.1, there exists a vertex v such that $G - v$ is i-critical. If $G - v$ is connected, then $diam(G) \leq diam(G - v) + 1 \leq 2i(G - v) - 2 + 1 = 2i(G) - 2 + 1 \leq 2i(G) - 1$. Suppose $G - v$ is disconnected. Then each component is i-critical. Suppose $G_1, G_2, \ldots, G_k$ are the components of $G - v$ and assume G_1, G_2 are the components with the largest and second largest diameter. Then $diam(G) \leq diam(G_1) + diam(G_2) + 2 \leq 2i(G_1) - 2 + 2i(G_2) - 2 + 2 \leq 2i(G) - 2$. □

The simplicity of the above proof suggests that the bound in the proposition is weak. On the other hand, the diameter of a connected γ-critical graph is at most $2\gamma - 2$ [6], and the diameter of a connected γ-bicritical graph is at most $2\gamma - 3$ [8]. Both bounds are sharp. The diameter of a strongly i-bicritical graph G is at most $3i(G)/2$. The bound is not known to be sharp [5].

The following construction was introduced by Brigham et al. [2] as a way of producing γ-bicritical graphs that are not γ-critical, and was considered by Xu, Xu, and Zhang in the context of i-bicritical graphs [15]. For a graph G and a vertex $v \in V(G)$, the *expansion of G via v* is the graph $G_{[v]}$ with vertex set $V(G_{[v]}) = V(G) \cup \{v'\}$ (where $v' \notin V(G)$) and edge set $E(G_{[v]}) = E(G) \cup \{uv' : u \in N_G[v]\}$. We note that, for any graph G, $i(G_{[v]}) = i(G)$ and $G_{[v]}$ is not i-critical since $G_{[v]} - v' \cong G$.

Proposition 2.5. [15] *If G is i-bicritical and i-critical, then $G_{[v]}$ is i-bicritical.*

The previous proposition is formally identical to a statement about γ-bicritical graphs from [2]. The graph in Figure 2 provides an example which shows that the hypothesis that G is i-critical can neither be deleted, nor be replaced by the hypothesis that $i(G - v) = i(G) - 1$. Referring to the figure, note that G is i-bicritical, $i(G) = 4$ and $i(G - d) = i(G)$; hence G is not i-critical. The graph $G_{[v]}$ is not i-bicritical because $i(G_{[v]}) = i(G_{[v]} - \{v', d\}) = 4$.

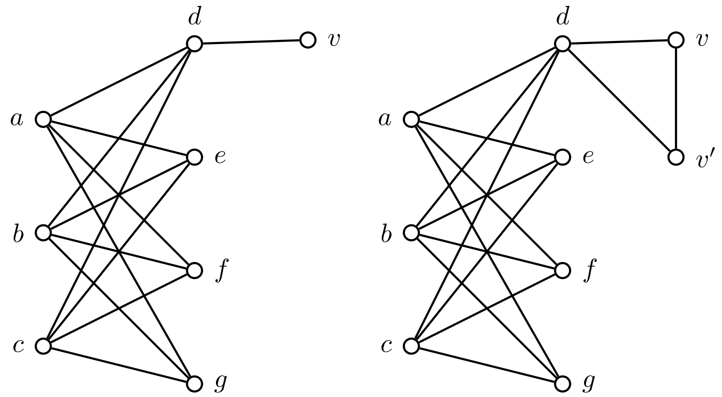

Figure 2: graphs G and $G_{[v]}$ from left to right

Graphs that are both i-critical and i-bicritical, for example $K_{n,n}$ or $K_n \Box K_n$, where $n \geq 3$, have no i-stable vertices, that is, $|V_i^0| = 0$. For $n \geq 3$, the complete bipartite graph $K_{n,n+1}$ is an i-bicritical graph with $|V_i^0| = n + 1 \geq 4$.

The expansion construction is useful in creating i-bicritical graphs with $|V_i^0| = 2$. If G is both i-critical and i-bicritical, then for any vertex $v \in V(G)$ the only stable vertices of $G_{[v]}$ are v and v'. To see this, let $x \in V(G_{[v]}) - \{v, v'\}$, and let D be an i-set of $G - x$. Since D dominates v in $G - x$, D dominates v' in $G_{[v]} - x$. Thus D is an independent dominating set of $G_{[v]} - x$ and $i(G_{[v]} - x) \leq |D| < i(G) = i(G_{[v]})$.

3 Characterizations

In this section we characterize the 2-i-bicritical graphs, and show that for $k \geq 4$, there is no characterization of the k-i-bicritical graphs in terms of a finite collection of forbidden subgraphs. Characterizing the 3-i-bicritical graphs is an open problem.

The only 2-i-critical graphs are $K_{2n} - F$, where F is a 1-factor [1]. We show that there are only two 2-i-bicritical graphs.

Theorem 3.1. *The only* 2-*i*-*bicritical graphs are* $\overline{K}_2$ *and the disjoint union* $K_1 \cup K_2$.

Proof. Let G be a 2-i-bicritical graph. Since $i(G) = 2$, there exists an independent dominating set $\{x, y\} \subseteq V(G)$. Consider $G - \{x, y\}$. If $i(G - \{x, y\}) = 0$ then $G \cong K_1 \cup K_1$. If $i(G - \{x, y\}) = 1$, then there exists a vertex $w \in V(G - \{x, y\})$ that dominates $G - \{x, y\}$. In addition, w is not adjacent to at least one of x and y in G, say y. Then $xw \in E(G)$ since $\{x, y\}$ is an independent dominating set. Consider $G - \{w, y\}$. Since $i(G - \{w, y\}) = 1$ there exists a vertex $z \in V(G - \{w, y\})$ that dominates $G - \{w, y\}$. Since w dominates $G - \{x, y\}$, $z \in N(w)$. Then $zy \notin E(G)$ for otherwise $i(G) = 1$.

Suppose $z \neq x$. Consider $G - \{w, z\}$. Since $i(G - \{w, z\}) = 1$, there exists a vertex $v \in V(G - \{w, z\})$ such that v dominates $G - \{w, z\}$. Notice that $v \neq y$ since $yx \notin E(G)$ and likewise $v \neq x$. Also, $vw \in E(G)$ since w dominates $G - \{x, y\}$ and

$vz \in E(G)$ since z dominates $G - \{w, y\}$. Then v dominates G and $i(G) = 1$, a contradiction.

Suppose $z = x$ and $N(w) - \{x\} \neq \emptyset$. Consider $G - \{w, x\}$. Since $i(G - \{w, x\}) = 1$ there exists a vertex $v \in V(G - \{w, x\})$ that dominates $G - \{w, x\}$. Then $vx \in E(G)$ since $x = z$ dominates $G - \{w, y\}$ and $vw \in E(G)$ since w dominates $G - \{x, y\}$. Thus v dominates G and $i(G) = 1$, a contradiction. Therefore $N(w) - \{x\} = \emptyset$ and $G \cong K_1 \cup K_2$. □

Ao used the following construction to prove that for any graph G there is a 3-i-critical graph $H_1 = H_1(G)$ such that G is an induced subgraph of H_1 [1]. Let G be a graph. Construct $H_1 = H_1(G)$ from the disjoint union $G' = G \cup \overline{K}_2$ as follows: For each $v \in V(G')$, add independent vertices $\{v_1, v_2\}$ and all edges between $V(G' - v)$ and $\{v_1, v_2\}$. Additionally, for all pairs $x, y \in V(G')$ add all edges between $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Then $i(H_1) = 3$, the graph H_1 is 3-i-critical, and G is an induced subgraph of H_1.

It can be seen from considering $G = \overline{K}_3$ that H_1 may not be i-bicritical. We now use a similar construction to obtain a similar result for i-bicritical graphs.

Let G be a graph. For $j \geq 1$, let $H_j = H_j(G)$ be the graph constructed from the disjoint union $G' = G \cup \overline{K}_{j+2}$ as follows: For each vertex $v \in V(G')$ add independent vertices $I_v = \{v_1, v_2, \ldots, v_{j+1}\}$ and add all edges between $V(G' - v)$ and I_v. Additionally, for all pairs $x, y \in V(G')$ add all edges between I_x and I_y. Observe that $i(H_j) = j + 2$, and that G is an induced subgraph of H_j.

Theorem 3.2. *For $j \geq 2$, the graph H_j is $(j+2)$-i-critical and $(j+2)$-i-bicritical.*

Proof. Consider $z \in V(H_j)$. If $z \in V(G')$, then I_z is an independent dominating set of $H_j - z$. If $z \in I_v$ for some $v \in V(G')$, then $\{v\} \cup (I_v - \{z\})$ is an independent dominating set of $H_j - z$. Thus $i(H_j - z) \leq j + 1 < i(H_j)$ so H_j is i-critical.

Now consider $\{x, y\} \subseteq V(H_j)$. If $\{x, y\} \subseteq V(G')$, then I_x is an independent dominating set of $H_j - \{x, y\}$. If $x \in V(G')$ and $y \in I_z$ for some $z \in V(G')$, then $I_x - \{y\}$ is an independent dominating set of $H_j - \{x, y\}$. If $x \in I_u$ for some $u \in V(G')$ and $y \in I_v$ for some $v \in V(G')$, then $\{u\} \cup (I_u - \{x\})$ is an independent dominating set of $H_j - \{x, y\}$. Finally, if $\{x, y\} \subseteq I_v$ for some $v \in V(G')$, then $\{v\} \cup (I_v - \{x, y\})$ is an independent dominating set of $H_j - \{x, y\}$. It now follows that H_j is i-bicritical. □

Corollary 3.3. *For any graph G and for all $k \geq 4$, there exists a k-i-bicritical graph H such that G is an induced subgraph of H.*

When $i(G) \geq 4$, the graph H can be chosen so that $i(H) = i(G)$. Consequently, for $k \geq 4$ there is no characterization of the k-i-bicritical graphs in terms of a finite collection of forbidden subgraphs. It is unknown whether the same statement holds when $k = 3$.

Since the characterization problem is difficult it is useful to know ways to produce i-bicritical graphs. In the next several sections, operations such as disjoint union, join, and coalesence are used to present a collection of methods to construct

i-bicritical graphs. Many of the constructions presented rely on the use of already known i-bicritical graphs to create new i-bicritical graphs.

We conclude this section by noting that a slight strengthening of the statement about i-critical graphs is also a consequence of Theorem 3.2.

Corollary 3.4. *For any graph G and for all $k \geq 3$, there exists a k-i-critical graph H such that G is an induced subgraph of H.*

4 Construction of i-Bicritical Graphs via Disjoint Union

Let $G_1, G_2, \ldots, G_k$ be disjoint graphs. Note that $i(\bigcup_{t=1}^{k} G_t) = \sum_{t=1}^{k} i(G_t)$. Also note that K_1 is both i-critical and i-bicritical.

Theorem 4.1. *Let $G_1, G_2, \ldots, G_k$ be disjoint graphs. For $k \geq 2$, the graph $\bigcup_{t=1}^{k} G_t$ is i-bicritical if and only if each of $G_1, G_2, \ldots, G_k$ is i-bicritical and at most one of these graphs is not i-critical.*

Proof. For convenience, let $G = \bigcup_{t=1}^{k} G_t$.

Suppose G is i-bicritical. Any component of G which is isomorphic to K_1 is i-bicritical. Let $1 \leq j \leq k$ and suppose $|V(G_j)| \geq 2$. Let $\{u, v\} \subseteq V(G_j)$. By hypothesis, $i(G - \{u, v\}) \leq i(G) - 1$. But $i(G - \{u, v\}) = \left(\sum_{t=1, t \neq j}^{k} i(G_t) \right) + i(G_j - \{u, v\})$, so that

$$1 \leq i(G) - i(G - \{u, v\}) = i(G_j) - i(G_j - \{u, v\}).$$

Therefore G_j is i-bicritical. Therefore each of $G_1, G_2, \ldots, G_k$ is i-bicritical.

Suppose $u \in V(G_j)$ and $v \in V(G_\ell)$ for some $1 \leq j < \ell \leq k$. Then

$$i(G) - 1 \geq i(G - \{u, v\}) = \left(\sum_{t=1, t \neq j, \ell}^{k} i(G_t) \right) + i(G_j - u) + i(G_\ell - v),$$

so that

$$1 \leq i(G) - i(G - \{u, v\}) = i(G_j) - i(G_j - u) + i(G_\ell) - i(G_\ell - v).$$

Since u and v are arbitrary vertices of G_j and G_ℓ, respectively, at most one of these graphs is not i-critical. Therefore at most one of $G_1, G_2, \ldots, G_k$ is not i-critical.

For the converse, suppose each of $G_1, G_2, \ldots, G_k$ is i-bicritical and at most one of them is not i-critical. Without loss of generality, say G_k may not be i-critical.

Consider $G - \{u, v\}$ for some $\{u, v\} \subseteq V(G)$. If $u, v \in V(G_j)$ for some $1 \leq j \leq k$, then

$$\begin{aligned} i(G - \{u,v\}) &= \left(\sum_{t=1, t\neq j}^{k} i(G_t)\right) + i(G_j - \{u,v\}) \\ &\leq \left(\sum_{t=1, t\neq j}^{k} i(G_t)\right) + i(G_j) - 1 \\ &= i(G) - 1 \end{aligned}$$

If $u \in V(G_j)$ and $v \in V(G_\ell)$ for some $1 \leq j < \ell \leq k$, then

$$\begin{aligned} i(G - \{u,v\}) &= \left(\sum_{t=1, t\neq j,\ell}^{k} i(G_t)\right) + i(G_j - u) + i(G_l - v) \\ &\leq i(G) - 1 \end{aligned}$$

since at most one of G_j and G_l is not i-critical. It now follows that G is i-bicritical. □

5 Construction of i-Bicritical Graphs via Join

Let G and H be disjoint graphs. Recall that the *join of* G *and* H, denoted $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The graph $\bigvee_{t=1}^{k} G_t$ is defined recursively by $\bigvee_{t=1}^{k} G_t = \left(\bigvee_{t=1}^{k-1} G_t\right) \vee G_k$. Note that $i(\bigvee_{t=1}^{k} G_t) = \min\{i(G_t), 1 \leq t \leq k\}$.

Note that, if $G \ncong K_1$, then $K_1 \vee G$ is not i-bicritical. Thus, in studying the join of graphs, we only consider graphs with at least two vertices.

Theorem 5.1. *Let $G_1, G_2, \ldots, G_k$ be disjoint graphs with $|V(G_t)| \geq 2$ for each $t \in \{1, 2, \ldots, k\}$. Then $\bigvee_{t=1}^{k} G_t$ is i-bicritical if and only if each of $G_1, G_2, \ldots, G_k$ is i-bicritical and either*

(a) $i(G_1) = i(G_2) = \cdots = i(G_k)$, and at most one of $G_1, G_2, \ldots, G_k$ is not i-critical, or

(b) $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$, the graph G_1 has no edges, and each of $G_2, G_3, \ldots, G_k$ is i-critical.

Proof. Let $G = \bigvee_{t=1}^{k} G_t$.

Suppose G is i-bicritical. Suppose, without loss of generality, that G_1 is not i-bicritical. Let $\{x, y\} \subseteq V(G_1)$ such that $i(G_1 - \{x, y\}) \geq i(G_1)$. Let D be an i-set of $G - \{x, y\}$. By definition of join, $D \subseteq V(G_j)$ for some subscript j. If $D \subseteq V(G_1)$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) \geq i(G)$, a contradiction. If $D \subseteq V(G_j)$ for $j > 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$, a contradiction. Therefore, each of $G_1, G_2, \ldots, G_k$ is i-bicritical.

We claim that at most one of $G_1, G_2, \ldots, G_k$ is not i-critical. Let $x \in V(G_j)$ and $y \in V(G_\ell)$, where $j \neq \ell$. Let D be an i-set of $G - \{x, y\}$. As above, $D \subseteq V(G_p)$ for some subscript p. We show that, further, $p \in \{j, \ell\}$. Suppose not. Then $i(G) \leq i(G_p) = i(G - \{x, y\}) \leq i(G) - 1$, a contradiction. Thus, $p \in \{j, \ell\}$. If $D \subseteq V(G_j)$, then $i(G) - 1 \geq i(G - \{x, y\}) = i(G_j - x)$. Therefore, G_j is i-critical. Similarly, if $D \subseteq V(G_\ell)$ then G_ℓ is i-critical. It follows that at least one graph among each pair of graphs chosen from $G_1, G_2, \ldots, G_k$ is i-critical. This proves the claim.

We now claim that independent domination numbers of $G_1, G_2, \ldots, G_k$ differ by at most one. Suppose, without loss of generality, that $i(G_1) \geq i(G_2)+2$. Let $\{x, y\} \subseteq V(G_1)$ and let D be an i-set of $G - \{x, y\}$. As above, $D \subseteq V(G_p)$ for some subscript p. If $D \subseteq V(G_1)$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) - 2 \geq i(G_2) \geq i(G)$, a contradiction. If $D \subseteq V(G_j)$ for $j > 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$, a contradiction. This proves the claim.

We claim that either $i(G_1) = i(G_2) = \cdots = i(G_k)$ or $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$. The statement follows immediately from the argument above if $k = 2$. Suppose $k \geq 3$ and, without loss of generality, $i(G_1) + 1 = i(G_2) = i(G_3)$. Let $x \in V(G_2)$ and $y \in V(G_3)$, and let D be an i-set of $G - \{x, y\}$. If $D \subseteq V(G_2 - x)$, then $i(G_1) - 1 \geq i(G) - 1 \geq i(G - \{x, y\}) = i(G_2 - x) \geq i(G_2) - 1$, so that $i(G_1) \geq i(G_2)$, a contradiction. The case where $D \subseteq V(G_3 - y)$ similarly leads to a contradiction. If $D \subseteq V(G_j)$ for $j \notin \{2, 3\}$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$, a contradiction. Since independent domination numbers of $G_1, G_2, \ldots, G_k$ differ by at most one, the claim is now proved.

Finally, we claim that if $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$, then G_1 has no edges and each of $G_2, G_3, \ldots, G_k$ is i-critical. Suppose that $i(G_1) - 1 = i(G_2) = \cdots = i(G_k)$ and G_1 has at least one edge. Let $xy \in E(G_1)$ and let D be an i-set of $G - \{x, y\}$. Note that $i(G_1 - \{x, y\}) \geq i(G_1) - 1$ since $xy \in E(G_1)$. If $D \subseteq V(G_1)$, then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \geq i(G_1) - 1 = i(G_2) = i(G)$, a contradiction. If $D \subseteq V(G_j)$ for $j > 1$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$, a contradiction. Hence G_1 has no edges.

Continuing the proof of the claim, suppose, without loss of generality, that G_2 is not i-critical. Let $x \in V(G_1)$ and $y \in V(G_2)$ such that $i(G_2 - y) \geq i(G_2)$. Let D be an i-set of $G - \{x, y\}$. If $D \subseteq V(G_1)$, then $i(G - \{x, y\}) = i(G_1 - x) = i(G_1) - 1 = i(G_2) \geq i(G)$, a contradiction. If $D \subseteq V(G_2)$, then $i(G - \{x, y\}) = i(G_2 - y) \geq i(G_2) \geq i(G)$, a contradiction. If $D \cap V(G_j) \neq \emptyset$ for $j > 2$, then $i(G - \{x, y\}) = i(G_j) \geq i(G)$, a contradiction. Therefore each of $G_2, G_2, \ldots, G_k$ is i-critical. The claim is now proved.

Now suppose each of $G_1, G_2, \ldots G_k$ is i-bicritical and either (a) or (b) holds. Let $\{x, y\} \subseteq V(G)$.

Suppose first that $x, y \in V(G_j)$ for some j. If (a) holds, then suppose, without loss of generality, that $j = 1$. Then $i(G - \{x, y\}) = i(G_1 - \{x, y\}) \leq i(G_1) - 1 < i(G)$. Now suppose (b) holds. If $j = 1$, then since G_1 has no edges and $i(G) = i(G_1) - 1$, we have $i(G - \{x, y\}) = i(G_1 - \{x, y\}) = i(G_1) - 2 = i(G) - 1 < i(G)$. If $j > 1$, then since $i(G) = i(G_j)$, we have $i(G - \{x, y\}) \leq i(G_j) - 1 = i(G) - 1 < i(G)$.

Now suppose that $x \in V(G_j)$ and $y \in V(G_\ell)$, where $1 \leq j < \ell \leq k$. If (a) holds, then since at most one of $G_1, G_2, \ldots, G_k$ is not i-critical we have $i(G - \{x, y\}) \leq \min\{i(G_j - x), i(G_\ell - y)\} < i(G)$. Suppose (b) holds. Then, since $\ell > 1$, $i(G) = i(G_\ell)$, and G_ℓ is i-critical, we have $i(G - \{x, y\}) = i(G_\ell - y) = i(G_\ell) - 1 < i(G)$.

It now follows that G is i-bicritical. □

6 Construction of i-Bicritical Graphs via Coalescence

Let G and H be disjoint graphs. Let $x \in V(G)$ and $y \in V(H)$. The *coalescence G and H with respect to x and y* is the graph $G \cdot_{xy} H$ with vertex set $V(G \cdot_{xy} H) = (V(G) - \{x\}) \cup (V(H) - \{y\}) \cup \{v_{xy}\}$, where $v_{xy} \notin V(G) \cup V(H)$, and edge set $E(G \cdot_{xy} H) = E(G - x) \cup E(H - y) \cup \{v_{xy}w : w \in N_G(x) \cup N_H(y)\}$. If the context is clear, or if the vertices x and y are not important, $G \cdot H$ is used instead of $G \cdot_{xy} H$.

We first consider the independent domination number of $G \cdot_{xy} H$ and show that $i(G) + i(H) - 1 \leq i(G \cdot_{xy} H) \leq i(G) + i(H)$. When $G \cdot_{xy} H$ is i-bicritical, either possibility can arise. This is in contrast to the situation when $G \cdot_{xy} H$ is i-critical. In that case the only possibility is that $i(G \cdot_{xy} H) = i(G) + i(H) - 1$. We are able to give necessary and sufficient conditions for $G \cdot_{xy} H$ to be i-bicritical with independent domination number $i(G) + i(H) - 1$, and necessary conditions for i-bicriticality when $i(G \cdot_{xy} H) = i(G) + i(H)$.

Proposition 6.1. *For any disjoint graphs G and H with $x \in V(G)$ and $y \in V(H)$, we have $i(G \cdot_{xy} H) \geq i(G) + i(H) - 1$.*

Proof. Let S be an i-set of $G \cdot_{xy} H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Note that v_{xy}, the vertex arising from the identification of x and y, is in neither of these sets as it is not an element of $V(G) \cup V(H)$.

If $v_{xy} \in S$, then $S_G \cup \{x\}$ is an independent dominating set of G and $S_H \cup \{y\}$ is an independent dominating set of H. Thus, $i(G \cdot_{xy} H) = |S| = |S_G| + |S_H| + 1 \geq i(G) - 1 + i(H) - 1 + 1$.

If $v_{xy} \notin S$, then a vertex of either $G - x$ or $H - y$ dominates v_{xy}. Suppose a vertex of $G - x$ dominates v_{xy}. Then S_G is an i-set of G and S_H is an i-set of $H - y$. Since $i(H - y) \geq i(H) - 1$, we have $i(G \cdot_{xy} H) = |S| = |S_G| + |S_H| \geq i(G) + i(H) - 1$.

Thus, in either case the inequality holds. □

Proposition 6.2. *Let G and H be disjoint graphs with $x \in V(G)$ and $y \in V(H)$. If x is an i-critical vertex of G or y is an i-critical vertex of H, then $i(G \cdot_{xy} H) = i(G) + i(H) - 1$.*

Proof. It suffices to prove the statement only in the case where x is an i-critical vertex of G.

Suppose first that y is in an i-set of H. Then $i(H)$ vertices of H, including y, can be used to dominate $(H - y) \cup \{v_{xy}\}$. Since x is an i-critical vertex of G, $i(G) - 1$ vertices of G can be used to dominate $G - x$. Further, since x is i-critical in G, none of these vertices of G are adjacent to x. Thus, $i(G \cdot_{xy} H) \leq i(G) + i(H) - 1$, and equality holds by Proposition 6.1.

Now suppose y is not in any i-set of H. Let S_H be an i-set of H. The vertex v_{xy} is dominated by S_H and, again, $i(G) - 1$ vertices of G can be used to dominate $G - x$. Thus, $i(G \cdot_{xy} H) \leq i(G) + i(H) - 1$, and equality holds by Proposition 6.1. □

Propositions 6.1 and 6.2 imply that if one of the vertices of identification x or y is i-critical in its corresponding graph, then $i(G \cdot_{xy} H) = i(G) + i(H) - 1$. It is not necessary for either to be i-critical in its corresponding graph, however, as we now show.

Proposition 6.3. *Let G and H be disjoint graphs. If x is in an i-set of G and y is in an i-set of H, then $i(G \cdot_{xy} H) = i(G) + i(H) - 1$.*

Proof. Let S_G be an i-set of G such that $x \in S_G$ and let S_H be an i-set of H such that $y \in S_H$. Then $S = (S_G \cup S_H \cup \{v_{xy}\}) \setminus \{x, y\}$ is an independent dominating set of $G \cdot_{xy} H$. Thus, $i(G \cdot_{xy} H) = i(G) + i(H) - 1$. □

We now consider the two remaining possibilities: neither x is an i-set of G nor y is in an i-set of H, and one of these vertices is in an i-set of its graph and the other is not.

Proposition 6.4. *Let G and H be disjoint graphs. If x is not in any i-set of G and y is not in any i-set of H, then $i(G \cdot_{xy} H) = i(G) + i(H)$.*

Proof. Since the union of an i-set of G and an i-set of H is an independent dominating set of $G \cdot_{xy} H$, we have $i(G \cdot_{xy} H) \leq i(G) + i(H)$.

Since x is not in any i-set of G, it is not i-critical in G. Likewise, y is not i-critical in H. Let S be an i-set of $i(G \cdot_{xy} H)$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

Suppose $v_{xy} \in S$. Then $S'_G = S_G \cup \{x\}$ and $S'_H = S_H \cup \{y\}$ are independent dominating sets of G and H, respectively. Since $x \in S'_G$ and $y \in S'_H$, neither of these are i-sets. Therefore $|S| > i(G) + i(H) - 1$, and thus $|S| = i(G) + i(H)$.

Now suppose that $v_{xy} \notin S$. If S_G dominates v_{xy} then S_G is an independent dominating set of G and $|S_G| \geq i(G)$. Hence S_H is an independent dominating set of $H - y$ so $|S_H| \geq i(H - y) = i(H)$. Thus $|S| \geq i(G) + i(H)$. If S_G does not dominate v_{xy}, then S_H does and the same statement follows similarly. □

If x is in an i-set of G and y is not in any i-set of H, it is possible to have either $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ or $i(G \cdot_{xy} H) = i(G) + i(H)$. For example, if $G = K_{3,3}$ and $H = K_{3,4}$, then for any vertex x of $K_{3,3}$ and for any vertex y of degree 3 in $K_{3,4}$, we have $i(G \cdot_{xy} H) = 5 = i(G) + i(H) - 1$. If $G = K_{3,3}[v]$ (the expansion via v of $K_{3,3}$, where v is any vertex in $K_{3,3}$) and $H = K_{3,4}$ where x is v', the vertex added to $K_{3,3}$ in the expansion, and y is a vertex of degree 3 in $K_{3,4}$, then $G \cdot_{xy} H$ has $i(G \cdot_{xy} H) = 6 = i(G) + i(H)$. These two cases are pictured below in Figure 3.
The following result, when combined with the propositions above, helps explain when these two cases arise.

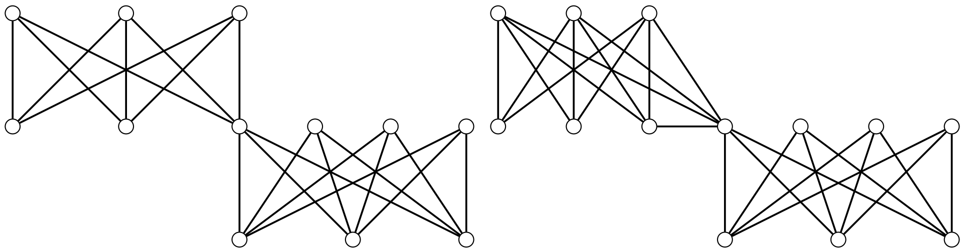

Figure 3: The graphs $K_{3,3} \cdot K_{3,4}$ and $K_{3,3_{[v]}} \cdot K_{3,4}$.

Proposition 6.5. *Let G and H be disjoint graphs. Suppose x is in an i-set of G, y is not in any i-set of H. Then*

(a) if $i(G \cdot_{xy} H) = i(G) + i(H) - 1$, then x is i-critical in G; and

(b) if $i(G \cdot_{xy} H) = i(G) + i(H)$, then x is not i-critical in G.

Proof. Suppose $i(G \cdot_{xy} H) = i(G) + i(H)$. Then, by Proposition 6.2, x is not i-critical in G.

Now suppose $i(G \cdot_{xy} H) = i(G) + i(H) - 1$. Let S be an i-set of $G \cdot_{xy} H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

We claim $v_{xy} \notin S$. Suppose the contrary. Then $S_G \cup \{x\}$ and $S_H \cup \{y\}$ are independent dominating sets of G and H, respectively. Thus $|S_G \cup \{x\}| \geq i(G)$ and $|S_H \cup \{y\}| > i(H)$, as y is not in an i-set of H. Hence $|S| \geq i(G) + i(H) + 1 - 1 = i(G) + i(H)$, a contradiction. This proves the claim.

Next, we claim S_G does not dominate v_{xy}. Suppose the contrary. Then S_G and S_H are independent dominating sets G and $H - y$, respectively. Therefore, $|S_G| \geq i(G)$ and $|S_H| \geq i(H)$, which implies $|S| \geq i(G) + i(H)$, a contradiction. This proves the claim.

It now follows that S_G and S_H are independent dominating sets of $G - x$ and H, respectively. Thus $|S_G| \geq i(G) - 1$ and $|S_H| \geq i(H)$. Since $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ we have that $|S_G| = i(G) - 1$ and $|S_H| = i(H)$. Therefore, x is i-critical in G. □

Having considered the possibilities for the independent domination number of the coalescence of G and H, we now consider the situations in which $G \cdot H$ is i-bicritical.

Theorem 6.6. *Let G and H be disjoint graphs. Let $x \in V(G)$ and $y \in V(H)$ each have degree at least one. The graph $G \cdot_{xy} H$ is i-bicritical with $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ if and only if*

(a) G and H are i-bicritical;

(b) x is i-critical in G, and y is i-critical in H; and

(c) G or H is i-critical.

Proof. Suppose that $G \cdot_{xy} H$ is i-bicritical with $i(G \cdot_{xy} H) = i(G) + i(H) - 1$.

We first show that (b) holds. By symmetry it suffices to show that y is an i-critical vertex of H. Let $u \in N_G(x)$. Then, since u and x are adjacent, $i(G \cdot_{xy} H - \{u, v_{xy}\}) = (i(G) + i(H) - 1) - 1$. Hence, $i(G) + i(H) - 2 = i(G - \{u, x\}) + i(H - y) \geq i(G) - 1 + i(H) - 1$. Therefore, $i(G - \{u, x\}) = i(G) - 1$ and $i(H - y) = i(H) - 1$, so y is i-critical in H.

Next, we show that (a) holds. By symmetry it suffices to show that G is i-bicritical. Suppose G is not i-bicritical and let $\{w, z\} \subseteq V(G)$ be such that $i(G - \{w, z\}) \geq i(G)$. Let S be an i-set of $G \cdot_{xy} H - \{w, z\}$ and let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Note that v_{xy} belongs to neither S_G nor S_H, as it is not an element of $V(G) \cup V(H)$.

Suppose $x = w$. Then, $i(G \cdot_{xy} H - \{v_{xy}, z\}) = i(G - \{w, z\}) + i(H - y) \geq i(G) + i(H) - 1 = i(G \cdot_{xy} H)$, a contradiction to the i-bicriticality of $G \cdot_{xy} H$.

Now suppose $x \notin \{w, z\}$. If $v_{xy} \in S$ then $S_G \cup \{x\}$ and $S_H \cup \{y\}$ are independent dominating sets of $G - \{w, z\}$ and H, respectively. Therefore, $|S| \geq i(G) + i(H) - 1$, a contradiction. Hence we may assume $v_{xy} \notin S$. If S_G dominates x, then S_G and S_H are independent dominating sets of $G - \{w, z\}$ and $H - y$, respectively, and $|S| \geq i(G) + i(H) - 1$, a contradiction. If S_G does not dominate x, then S_G and S_H are independent dominating sets of $G - \{w, z, x\}$ and H, respectively. Thus $|S_H| \geq i(H)$ and $|S_G| \geq i(G - \{w, z, x\}) = i((G - \{w, z\}) - x) \geq i(G - \{w, z\}) - 1 \geq i(G) - 1$. Therefore, $|S| \geq i(G) + i(H) - 1$, a contradiction. This completes the proof that (a) holds.

Finally, we show that (c) holds. Suppose neither G nor H is i-critical. By (b), there exists $w \in V(G - x)$ such that $i(G - w) \geq i(G)$ and $z \in V(H - y)$ such that $i(H - z) \geq i(H)$. Let S be an i-set of $G \cdot_{xy} H - \{w, z\}$ and let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

If $v_{xy} \in S$, then $S_G \cup \{x\}$ is an independent dominating set of $G - w$ and $S_H \cup \{y\}$ is an independent dominating set of $H - z$. Therefore $|S| \geq i(G) + i(H) - 1$, a contradiction.

On the other hand, suppose $v_{xy} \notin S$. Since S dominates v_{xy}, either S_G dominates x or S_H dominates y. By symmetry, assume the former. Thus, S_G is an independent dominating set of $G - w$ and S_H is an independent dominating set of $H - \{y, z\}$. In this case, $|S_G| \geq i(G)$, and $|S_H| \geq i(H - \{y, z\}) \geq i(H) - 1$ by (a). Therefore $|S| \geq i(G) + i(H) - 1$, a contradiction. This completes the proof that (c) holds, and the proof that if $G \cdot_{xy} H$ is i-bicritical with $i(G \cdot_{xy} H) = i(G) + i(H) - 1$ than (a), (b) and (c) hold.

Now suppose that (a), (b) and (c) hold. Let $\{w, z\} \subseteq V(G \cdot_{xy} H)$ and consider $G \cdot_{xy} H - \{w, z\}$. We want to show that $i(G \cdot_{xy} H - \{w, z\}) \leq i(G) + i(H) - 2$.

Suppose first that $v_{xy} = w$, say. By symmetry we may assume $z \in V(G)$. Let S_G be an i-set of $G - \{x, z\}$ and S_H be an i-set of $H - y$. Then $i(G \cdot_{xy} H - \{w, z\}) \leq |S_G| + |S_H| \leq i(G) - 1 + i(H) - 1$, as needed.

Hence, in what follows, we may assume $v_{xy} \notin \{w, z\}$.

Suppose that $\{w, z\} \subseteq V(G)$. Let S_G be an i-set of $G - \{w, z\}$. Then S_G dominates x. Let S_H be an i-set of $H - y$. Since y is i-critical in H, we have that $N_H(y) \cap S_H = \emptyset$. Thus, $S = S_G \cup S_H$ is an independent dominating set of

$G \cdot_{xy} H - \{w,z\}$ and $i(G \cdot_{xy} H - \{w,z\}) = i((G - \{w,z\}) \cdot_{xy} H) \leq |S| \leq i(G) - 1 + i(H) - 1 = i(G) + i(H) - 2$, as needed.

If $\{w,z\} \subseteq V(H)$ we similarly obtain an independent dominating set of the required size.

Finally, suppose $w \in V(G) - \{x\}$ and $z \in V(H) - \{y\}$. By (c), we may assume without loss of generality that G is i-critical. Let S_H be an i-set of $H - \{z,y\}$. Then $|S_H| \leq i(H) - 1$. If S_H dominates y, then let S_G be an i-set of $G - \{w,x\}$. Then $|S_G| \leq i(G) - 1$. By definition of S_G and S_H we have that $S_G \cup S_H$ is an independent dominating set of $G \cdot_{xy} H - \{w,z\}$, and $i(G \cdot_{xy} H - \{w,z\}) \leq |S_G \cup S_H| \leq i(G) - 1 + i(H) - 1 = i(G) + i(H) - 2$, as needed. If S_H does not dominate y, let S_G be an i-set of $G - w$. Then $|S_G| \leq i(G) - 1$ by (c). By definition of S_G and S_H we have that $S_G \cup S_H$ is an independent set of $G \cdot_{xy} H - \{w,z\}$, and $i(G \cdot_{xy} H - \{w,z\}) \leq |S_G \cup S_H| \leq i(G) - 1 + i(H) - 1 = i(G) + i(H) - 2$, as needed.

It follows from the above that $G \cdot_{xy} H$ is i-bicritical. □

The previous theorem is not true if x can be an isolated vertex of G. For example, let $G = \overline{K}_2$ and $H = K_{2,3}$ be disjoint graphs. For any vertices $x \in V(G)$ and $y \in V(H)$, the graph $G \cdot_{xy} H \cong K_1 \cup K_{2,3}$ is i-bicritical with independent domination number $3 = i(G) + i(H) - 1$. But statement (b) does not hold when y belongs to the independent set of size 3 in H. No such vertex is i-critical in H.

We now give an example to show that, in Theorem 6.6, if G is i-critical and H is not i-critical, then it is necessary for vertex y to be i-critical in H. Let G and H be the disjoint graphs shown in Figure 4 (overleaf). Note that G is both i-bicritical and i-critical, and H is i-bicritical. However, the vertex y is not i-critical in H. The coalescence $G \cdot_{xy} H$ has $i(G \cdot_{xy} H) = 6$. On the other hand, $G \cdot_{xy} H - \{v,k\} \cong K_{3,3} \cup K_{2,2} \cup K_1$ (disjoint union), thus $i(G \cdot_{xy} H - \{v,k\}) = 6$. Therefore, $G \cdot_{xy} H$ is not i-bicritical.

Using a proof similar to the one in Theorem 6.6, we can show the following.

Theorem 6.7. *Let G and H be disjoint graphs with $x \in V(G)$ and $y \in V(H)$. If $G \cdot_{xy} H$ is γ-bicritical with $\gamma(G \cdot_{xy} H) = \gamma(G) + \gamma(H) - 1$ then x is γ-critical in G, y is γ-critical in H, both G and H are γ-bicritical, and at most one of G and H is not γ-critical.*

We are also able to give necessary and sufficient conditions for $G \cdot_{xy} H$ to be both i-critical and i-bicritical. In light of Theorem 6.6, the following theorem is useful if the coalescence construction is applied iteratively to a collection of graphs.

Theorem 6.8. *Let G and H be disjoint graphs with $x \in V(G)$ and $y \in V(H)$. The graph $G \cdot_{xy} H$ is i-critical and i-bicritical if and only if*

(a) $i(G \cdot_{xy} H) = i(G) + i(H) - 1$;

(b) G and H are i-critical; and

(c) G and H are i-bicritical.

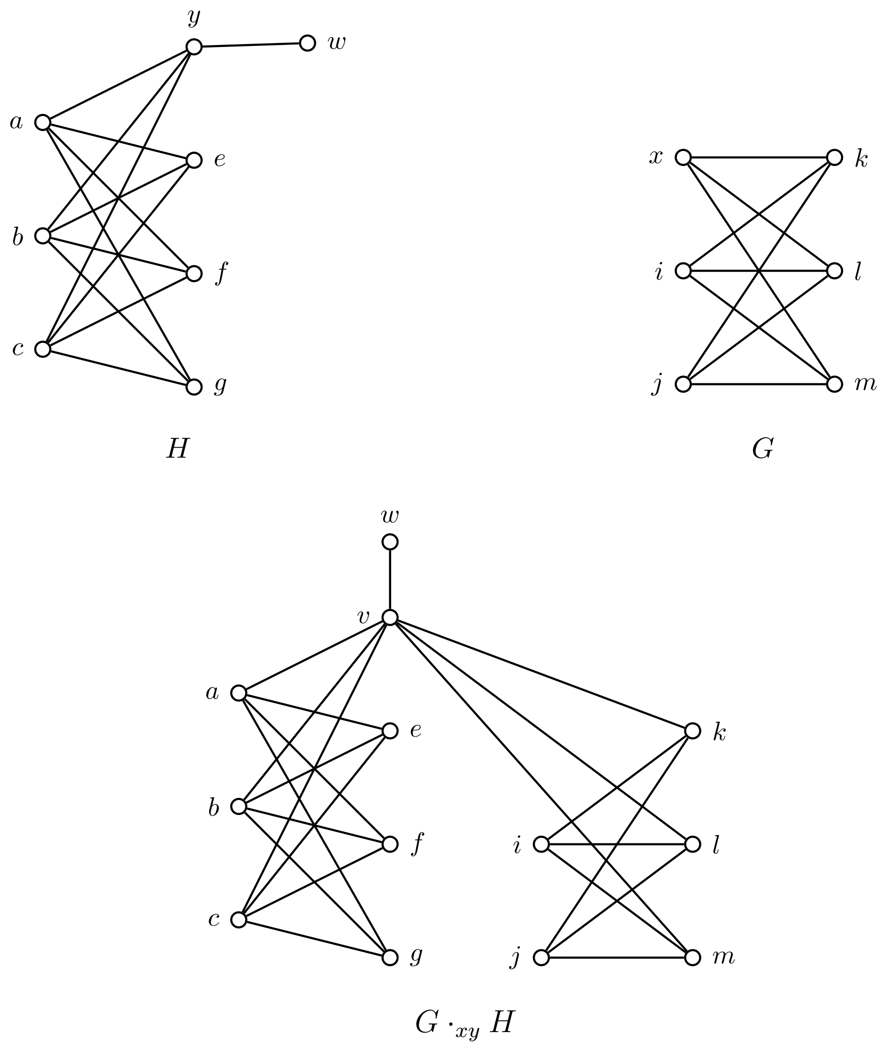

Figure 4: Showing the condition that y is critical in H is needed in Theorem 6.6.

Proof. Suppose $G \cdot_{xy} H$ is i-critical and i-bicritical. Then $i(G) + i(H) - 1 \leq i(G \cdot_{xy} H) \leq i(G)+i(H)$. Suppose equality holds in the upper bound. Then, by Proposition 6.2, x is not i-critical in G and y is not i-critical in H. Any independent dominating set of $G \cdot_{xy} H - v_{xy}$ must be the union of an independent dominating set S_x of $G - x$ and an independent dominating set S_y of $H - y$. Since x is not i-critical in G, $|S_x| \geq i(G)$. Similarly, $|S_y| \geq i(H)$. Thus $i(G \cdot_{xy} H - v_{xy}) \geq i(G) + i(H)$, a contradiction to i-criticality. Hence, (a) holds.

It now follows from Theorem 6.6 that condition (c) holds.

It remains to show that (b) holds. By symmetry it suffices to show that G is i-critical. The vertex x is i-critical in G by Theorem 6.6. Suppose G is not i-critical and let $w \in V(G - x)$ be such that $i(G - w) \geq i(G)$. Since w can not be an i-critical vertex of the i-bicritical graph $G \cdot_{xy} H$, it follows that $G \cdot_{xy} H - w$ is i-critical, it has an independent dominating set S of size $i(G) + i(H) - 2$. Let

$S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Note that v_{xy} does not belong to either of these sets as it is not an element of $V(G) \cup V(H)$. If v_{xy} is in S, then $S_G \cup \{x\}$ and $S_H \cup \{y\}$ are independent dominating sets of $G - w$ and H, respectively. Thus, $i(G \cdot_{xy} H - w) = |S_G| + |S_H| + 1 \geq i(G) - 1 + i(H) - 1 + 1 = i(G) + i(H) - 1$, a contradiction. Suppose, then, that $v_{xy} \notin S$. If S_G dominates x, then S_G is an independent dominating set of $G - w$ and S_H is an independent dominating set of $H - y$, so that $i(G \cdot_{xy} H - w) = |S_G| + |S_H| \geq i(G) + i(H) - 1$, a contradiction. If S_G does not dominate x, then it is an independent dominating set of $G - \{w, x\}$. S_H is an independent dominating set of H. In this case, $i(G \cdot_{xy} H - w) = |S_G| + |S_H| \geq i(G) - 1 + i(H)$, a contradiction. This completes the proof that (b) holds, and that (a), (b), and (c) hold.

Now suppose (a), (b) and (c) hold. Then $G \cdot_{xy} H$ is i-bicritical by Theorem 6.6. It remains to show that it is also i-critical. Let $w \in V(G \cdot_{xy} H)$. If $w = v_{xy}$, then the union of an independent dominating set of $G - x$ and an independent dominating set of $H - y$ is an independent dominating set of $G \cdot_{xy} H - w$ of size $i(G) - 1 + i(H) - 1$, as needed. Otherwise, without loss of generality suppose $w \in V(G - x)$. Let S_G be an i-set of $G - w$. Since G is i-critical, $|S_G| = i(G) - 1$, and S_G dominates x. Let S_H be an i-set of $H - y$. Then, since H is i-critical, $|S_H| = i(H) - 1$, and $S_H \cap N_H(y) = \emptyset$. Therefore $S_G \cup S_H$ is an independent dominating set of $G \cdot_{xy} H - w$ of size $i(G) - 1 + i(H) - 1$, as needed. Therefore G is i-critical, and the proof is complete. □

It remains to consider the situation where $i(G \cdot_{xy} H) = i(G) + i(H)$. By Propositions 6.4 and 6.5, there are two cases: (i) x is not in any i-set of G and y is not in any i-set of H; and (ii) x is in an i-set of G but is not i-critical in G, and y is not in any i-set of H. We are able to give necessary and sufficient conditions for i-bicriticality of $G \cdot_{xy} H$ in the first case, but not in the second case.

Theorem 6.9. *Let G and H be disjoint graphs. Let $x \in V(G)$ and $y \in V(H)$ be such that x is not in any i-set of G, and y is not in any i-set of H. Then $G \cdot_{xy} H$ is i-bicritical if and only if*

(a) G and H are i-bicritical;

(b) $G - x$ is i-bicritical or there exists an independent set $D_H \subseteq V(H)$ such that $y \in D_H$ and $|D_H| = i(H) + 1$; and

(c) $H - y$ is i-bicritical or there exists an independent set $D_G \subseteq V(G)$ such that $x \in D_G$ and $|D_G| = i(G) + 1$.

Proof. By our assumptions on x and y we have $i(G \cdot_{xy} H) = i(G) + i(H)$. Further, x is not an i-critical vertex of G and y is not an i-critical vertex of H.

Suppose $G \cdot_{xy} H$ is i-bicritical. Let $u, v \in V(G - x) \cup \{v_{xy}\}$. The graph $G \cdot_{xy} H - \{u, v\}$ has an i-set, S, of size at most $i(G) + i(H) - 1$. Let $S_G = S \cap V(G)$, and $S_H = S \cap V(H)$. There are two cases to consider, depending on whether $v_{xy} \in S$. We show that, in each case, $i((G - x) - \{u, v\}) \leq |S_G| \leq i(G) - 1$, so that G is i-bicritical. The i-bicriticality of H is established similarly.

Suppose $v_{xy} \notin S$ (this case must arise when $v_{xy} \in \{u, v\}$, and may arise at other times). Then S_H is an independent dominating set of H, so that $|S_H| \geq i(H)$. Similarly, S_G is an independent dominating set of $G - \{u, v\}$, and $|S_G| \leq i(G) + i(H) - 1 - |S_H| \leq i(G) - 1$.

Now suppose $v_{xy} \in S$. Then $S_H \cup \{y\}$ is an independent dominating set of H. Since y is not in any i-set of H, we have $|S_H| \geq i(H) + 1$. Similarly, $S_G \cup \{x\}$ is an independent dominating set of $G - \{u, v\}$, and $|S_G| \leq i(G) + i(H) - 1 - |S_H| - 1 \leq i(G) - 1$. This completes the proof that statement (a) holds.

We now show that (b) holds. Suppose that $G - x$ is not i-bicritical. Then there exist $u, v \in V(G - x)$ such that $i((G - x) - \{u, v\}) \geq i(G - x) = i(G)$ (since (a) holds, we have $i(G - x) \leq i(G)$ by Theorem 2.1, and equality holds since x is not in any i-set of G). Since $G \cdot_{xy} H$ is i-bicritical, it has an i-set S of size at most $i(G) + i(H) - 1$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

We claim that $v_{xy} \in S$. Suppose not. Then S_H is an independent dominating set of $H - y$, and $|S_H| \geq i(H)$. It then follows that S_G is an independent dominating set of $G - x$ with $|S_G| < i(G - x)$, a contradiction. This proves the claim.

Since x is not in any i-set of G, we must have that $S_G \cup \{x\}$ has size $i(G) - 1$ and dominates neither u nor v. The set $S_H \cup \{y\}$ is an independent dominating set of H. By our assumption on y and work above, we have

$$i(H) + 1 \leq |S_H \cup \{y\}| = |S| - |S_G| + 1 \leq i(G) + i(H) - 1 - (i(G) - 2) = i(H) + 1.$$

Hence (b) holds. Statement (c) is shown to hold by a similar argument.

Now suppose (a), (b) and (c) hold. Let $u, v \in V(G \cdot_{xy} H)$.

Suppose $u, v \in V(G - x)$. If $G - x$ is i-bicritical, then the union of an i-set of $(G - x) - \{u, v\}$ and an i-set of H (which exists, and necessarily dominates y but does not contain it since y is not in any i-set of H), is an independent dominating set of $G \cdot_{xy} H - \{u, v\}$ of size at most $i(G) - 1 + i(H)$. Suppose, then, that $G - x$ is not i-bicritical. If there is an i-set of $G - \{u, v\}$, of size at most $i(G) - 1$ which does not contain x then, as above there is an independent dominating set of $G \cdot_{xy} H - \{u, v\}$ of size $i(G) - 1 + i(H)$. Otherwise, every i-set of $S_G \subseteq G - \{u, v\}$ of size at most $i(G) - 1$ contains x. By (b) there exists an independent set $D_H \subseteq V(H)$ such that $y \in D_H$ and $|D_H| = i(H) + 1$. Then $S_G \cup D_H$ is an independent dominating set of $G \cdot_{xy} H - \{u, v\}$ of size at most $(i(G) - 1) + (i(H) + 1) - 1$, as needed. Similar considerations apply when $u, v \in V(H - y)$.

Suppose $u = v_{xy}$ and $v \in V(G - x)$. Since G is i-bicritical there exists an i-set $S_G \subseteq V(G) - \{x, v\}$ of size at most $i(G) - 1$. Let S_H be any i-set of H. By hypothesis, $y \notin S_H$. Then $S_G \cup S_H$ is an independent dominating set of $G \cdot_{xy} H - \{v_{xy}, v\} = G \cdot_{xy} H - \{u, v\}$ of size at most $i(G) - 1 + i(H)$, as needed. Similar considerations apply when $u = v_{xy}$ and $v \in V(H - y)$.

Finally, suppose $u \in V(G)$ and $v \in V(H)$. Since G is i-bicritical, $G - \{x, u\}$ has an i-set, S_G, of size at most $i(G) - 1$. Similarly, $H - \{y, v\}$ has an i-set of size, S_H, at most $i(H) - 1$. Consider the independent set $S_G \cup S_H$. If it dominates v_{xy}, then it is an independent dominating set of $G \cdot_{xy} H - \{u, v\}$ of size at most $i(G) - 1 + i(H) - 1$. Otherwise, $S_G \cup S_H \cup \{v_{xy}\}$ is an independent dominating set of $G \cdot_{xy} H - \{u, v\}$ of size at most $(i(G) - 1) + (i(H) - 1) + 1$. This completes the proof. □

Corollary 6.10. *Let G and H be disjoint graphs. Let $x \in V(G)$ and $y \in V(H)$ be such that x is not in any i-set of G, and y is not in any i-set of H. If the graphs $G, H, G - x$ and $H - y$ are all i-bicritical, then $G \cdot_{xy} H$ is i-bicritical*

Let $m, n \geq 3$ be integers. We can obtain families of i-bicritical graphs by letting $G = K_{m,m+1}$ and $H = K_{n,n+1}$, and x, y be vertices in the larger independent set of G, H, respectively. By Corollary 6.10, the graph $G \cdot_{xy} H$ is i-bicritical. Furthermore, this graph has $m+n+1$ vertices which do not belong to an i-set, so the corollary can be applied again. If the other graph in the coalescence is, for example, $K_{t,t+1}, t \geq 3$, then similar considerations hold and the construction can be applied iteratively.

7 Construction of i-Bicritical Graphs via Identification on a Subgraph

Let H be a graph. Let G_1 and G_2 be graphs for which H is the subgraph of each one induced by $V(G_1) \cap V(G_2)$. The graph $G_1(H)\widehat{\odot}G_2(H)$ is obtained from $G_1 \cup G_2$ by adding the set of edges $\{x_1x_2 : x_1 \in V(G_1) - V(H) \text{ and } x_2 \in V(G_2) - V(H)\}$. This construction can be informally described as coinciding G_1 and G_2 on their common subgraph H, and then adding all possible edges between vertices of $G_1 - H$ and vertices of $G_2 - H$.

It follows from the definition that $i(G_1(H)\widehat{\odot}G_2(H)) = \min\{i(G_1), i(G_2)\}$, and that any independent dominating set of this graph is a subset of $V(G_1)$ or of $V(G_2)$.

Let G be a graph. In what follows, we call a pair of different vertices $x, y \in V(G)$ a *bicritical pair* of G if $i(G - \{x, y\}) < i(G)$.

We first consider the case where $i(G_1) = i(G_2)$ and characterize the situations where $G_1(H)\widehat{\odot}G_2(H)$ is i-bicritical. Somewhat remarkably, it is not required that G_1 or G_2 be bicritical.

Theorem 7.1. *Let H be a graph. Let G_1 and G_2 be graphs for which H is the subgraph of each one induced by $V(G_1) \cap V(G_2)$, and are such that $i(G_1) = i(G_2)$. Then, $G = G_1(H)\widehat{\odot}G_2(H)$ is i-bicritical if and only if, for all pairs of vertices x, y,*

(a) for each $j \in \{1, 2\}$, if $x, y \in V(G_j - H)$ then x, y is an i-bicritical pair of G_j;

(b) if $x, y \in V(H)$, then x, y is an i-bicritical pair of G_1 or of G_2;

(c) for each $j \in \{1, 2\}$, if $x \in V(G_j - H)$ and $y \in V(H)$, then either x, y is an i-bicritical pair of G_j, or y is an i-critical vertex of G_{2-j+1};

and every vertex of $G_1 - H$ is an i-critical vertex of G_1 or every vertex of $G_2 - H$ is an i-critical vertex of G_2.

Proof. We have $i(G) = i(G_1) = i(G_2)$.

Suppose G is i-bicritical. Let $x, y \in V(G)$ and consider $G - \{x, y\}$. Since, by definition of G, any independent dominating set of G is a subset of $V(G_1) - \{x, y\}$ or $V(G_2) - \{x, y\}$, it is clear that conditions (a) through (c) must hold. Suppose, without loss of generality, that the vertex $x \in V(G_1 - H)$ is not an i-critical vertex of G_1. Since, for any vertex $y \in V(G_2 - H)$ we must have $i(G - \{x, y\}) < i(G)$, it follows that y must be an i-critical vertex of G_2. Therefore, every vertex of $G_1 - H$ is an i-critical vertex of G_1 or every vertex of $G_2 - H$ is an i-critical vertex of G_2.

Now suppose the given conditions all hold. Let $x, y \in V(G)$ and consider $G - \{x, y\}$. If $x, y \in V(G_1 - H)$ then, by (a), $i(G_1 - \{x, y\}) < i(G_1) = i(G)$. Therefore $i(G - \{x, y\}) < i(G)$. Similarly, if $x, y \in V(G_2 - H)$, then $i(G - \{x, y\}) < i(G)$. If $x, y \in V(H)$ then, by (b), either $G_1 - \{x, y\}$ or $G_2 - \{x, y\}$ has an independent dominating set of size less than $i(G)$. Since any such set dominates $G - \{x, y\}$, we have $i(G - \{x, y\}) < i(G)$. Suppose $x \in V(G_1 - H)$ and $y \in V(H)$. If x, y is an i-bicritical pair of G_1, then $i(G - \{x, y\}) < i(G)$ as before. If y is an i-critical vertex of G_2, then $G_2 - y$ has an independent dominating set of size less than $i(G_2) = i(G)$, and $i(G - \{x, y\}) < i(G)$ as before. A similar argument applies if $x \in V(G_2 - H_2)$ and $y \in V(H)$. Finally, suppose $x \in V(G_1 - H)$ and $y \in V(G_2 - H)$. Then either x is an i-critical vertex of G_1 or y is an i-critical vertex of G_2, and $i(G - \{x, y\}) < i(G)$ as before. □

Corollary 7.2. *Let H be a graph. Let G_1 and G_2 be graphs for which H is the subgraph of each one induced by $V(G_1) \cap V(G_2)$, and are such that $i(G_1) = i(G_2)$. If G_1 is i-critical and i-bicritical, and G_2 is i-bicritical, then $G_1(H)\widehat{\odot}G_2(H)$ is i-bicritical.*

Corollary 7.3. *Let H be a graph. Let G_1 and G_2 be graphs for which H is the subgraph of each one induced by $V(G_1) \cap V(G_2)$, and are such that $i(G_1) = i(G_2)$. Then $G_1(H)\widehat{\odot}G_2(H)$ is i-critical and i-bicritical if and only if*

(a) for each $j \in \{1, 2\}$, any pair of vertices $x, y \in V(G_j - H)$ is an i-bicritical pair of G_j;

(b) any pair of vertices $x, y \in V(H)$ is an i-bicritical pair of G_1 or G_2;

(c) for each $j \in \{1, 2\}$, all vertices in $V(G_j - H)$ are i-critical vertices of G_j; and

(d) every vertex in $V(H)$ is an i-critical vertex of G_1 or G_2.

Proof. Suppose $G = G_1(H)\widehat{\odot}G_2(H)$ is i-critical and i-bicritical. Then (a) and (b) hold by Theorem 7.1.

Let $x \in V(G_1 - H)$. Since G is i-critical, and every independent dominating set of G is a subset of $V(G_1)$ or of $V(G_2)$, we must have $i(G_1 - x) < i(G_1)$, so that x is an i-critical vertex of G_1. Therefore, all vertices of $G_1 - H$ are i-critical vertices of G_1. Similarly, all vertices of $G_2 - H$ are i-critical vertices of G_2.

Let $x \in V(H)$. Then $G - x$ has an independent dominating set, D, of size less than $i(G)$. Thus either $G_1 - x$ has an independent dominating set of size less than $i(G_1)$ or $G_2 - x$ has an independent dominating set of size less than $i(G_2)$. Therefore, every vertex of H is an i-critical vertex of G_1 or of G_2.

For the converse, suppose G_1 and G_2 are different graphs such that $V(G_1) \cap V(G_2) = V(H)$, $i(G_1) = i(G_2)$, and conditions (a) through (d) hold. Then $G = G_1(H)\widehat{\odot}G_2(H)$ is i-bicritical by Theorem 7.1.

Let $x \in V(G)$. If $x \in V(G_1 - H)$, then since x is an i-critical vertex of G_1, the graph G_1 has an independent dominating set of size less than $i(G_1)$. The same set is an independent dominating set of $G - x$, hence x is an i-critical vertex of G. Similarly, if $x \in V(G_2 - H)$, then x is an i-critical vertex of G. And similarly again, if $x \in V(H)$, then x is an i-critical vertex of G. Therefore, G is i-critical. □

More generally, let $G_1, G_2, \ldots, G_k$ be graphs for which H is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \le j < \ell \le k$. The graph $G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is the graph obtained from $\bigcup_{t=1}^{k} G_t$ by adding the set of edges $\{x_j x_\ell : x_j \in V(G_j) - V(H)$ and $x_\ell \in V(G_\ell) - V(H), j \ne \ell\}$. The same graph is obtained iteratively as $(((G_1(H)\widehat{\odot}G_2(H))\widehat{\odot}G_3(H))\widehat{\odot}\cdots\widehat{\odot}G_k(H))$. This construction can be informally described as coinciding the graphs $G_1, G_2, \ldots, G_k$ on their common subgraph H, and then adding all possible edges between vertices in $G_j - H$ and $G_\ell - H$, where $j \ne \ell$.

As in the case when $k = 2$, it follows from the definition that

$$i(G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)) = \min\{i(G_1), i(G_2), \ldots, i(G_k)\},$$

and that any independent dominating set of this graph is a subset of $V(G_j)$ for some $j, 1 \le j \le k$. Essentially the same arguments as above prove the following.

Theorem 7.4. *Let H be a graph. Let $G_1, G_2, \ldots, G_k$ be graphs for which H is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \le j < \ell \le k$, and are such that $i(G_1) = i(G_2) = \cdots = i(G_k)$. Then, $G = G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is i-bicritical if and only if, for all pairs of vertices x, y,*

(a) if $x, y \in V(G_j - H)$, then x, y is an i-bicritical pair of G_j;

(b) if $x, y \in V(H)$, then there exists $j, 1 \le j \le k$ such that x, y is an i-bicritical pair of G_j;

(c) if $x \in V(G_j - H)$ and $y \in V(H)$, then either x, y is an i-bicritical pair of G_j, or there exists $\ell, 1 \le \ell \le k, \ell \ne j$ such that y is an i-critical vertex of G_ℓ;

and there is at most one subscript j such that not all vertices of $G_j - H_j$ are i-critical vertices of G_j.

Corollary 7.5. *Let H be a graph. Let $G_1, G_2, \ldots, G_k$ be graphs for which H is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \le j < \ell \le k$, and are such that $i(G_1) = i(G_2) = \cdots = i(G_k)$. If G_1 is i-bicritical, and $G_2, G_3, \ldots, G_k$ are both i-critical and i-bicritical, then $G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is i-bicritical.*

Corollary 7.6. *Let H be a graph. Let $G_1, G_2, \ldots, G_k$ be graphs for which H is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \le j < \ell \le k$, and are such that $i(G_1) = i(G_2) = \cdots = i(G_k)$. Then, $G = G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is i-critical and i-bicritical if and only if*

(a) *for each* $j \in \{1, 2, \ldots, k\}$*, any pair of vertices* $x, y \in V(G_j - H)$ *is an* i*-bicritical pair of* G_j*;*

(b) *for two vertices* $x, y \in V(H)$ *there exists* ℓ *such that* x, y *is an* i*-bicritical pair of* G_ℓ*;*

(c) *for each* $j \in \{1, 2, \ldots, k\}$*, all vertices of* $G_j - H_j$ *are* i*-critical vertices of* G_j*; and*

(d) *for every vertex* $x \in V(H)$ *there exists* ℓ *such that* x *is an* i*-critical vertex of* G_ℓ*.*

We now consider i-bicriticality of $G_1(H)\widehat{\odot}G_2(H)$ when $i(G_1) \neq i(G_2)$. Note that, if $i(G_1) < i(G_2)$, then $G_1(H)\widehat{\odot}G_2(H)$ can not be i-critical because, for any $x \in V(G_2 - H)$ we have $i(G_1(H)\widehat{\odot}G_2(H) - x) \geq i(G_2 - x) \geq i(G_2) - 1 \geq i(G_1)$.

Another definition is needed. A pair x, y of different vertices of a graph G is called a *strongly* i*-bicritical pair* if $i(G - \{x, y\}) = i(G) - 2$. Observe that a strongly i-bicritical pair of vertices are non-adjacent.

Theorem 7.7. *Let* H *be a graph. Let* G_1 *and* G_2 *be graphs for which* H *is the subgraph of each one induced by* $V(G_1) \cap V(G_2)$*, and are such that* $i(G_1) < i(G_2)$*. Then,* $G = G_1(H)\widehat{\odot}G_2(H)$ *is* i*-bicritical if and only if*

(a) *either* $i(G_2) = i(G_1) + 1$*, or* $|V(G_2 - H)| = 1$*;*

(b) $E(G_2 - H) = \emptyset$*, and any pair of vertices* $x, y \in V(G_2 - H)$ *is a strongly* i*-bicritical pair of* G_2*;*

(c) *if* $x, y \in V(H)$*, then either* x, y *is an* i*-bicritical pair of* G_1*, or* $i(G_2) = i(G_1)+1$ *and* x, y *are a strongly* i*-bicritical pair of* G_2*;*

(d) G_1 *is bicritical; and*

(e) *every vertex in* $V(G_1 - H)$ *is* i*-critical.*

Proof. Suppose first that G is i-bicritical. Note that $i(G) = i(G_1)$.

Suppose $|V(G_2 - H)| > 1$ and let x, y be vertices in $V(G_2 - H)$. In order for $G - \{x, y\}$ to have an independent dominating set of size less than $i(G) = i(G_1)$, we must have $i(G_2 - \{x, y\}) < i(G_1)$. Since $i(G_1) \leq i(G_2) - 1$ and $i(G_2) - 2 \leq i(G_2 - \{x, y\})$, it follows that $i(G_2 - \{x, y\}) = i(G_2) - 2$ and $i(G_2) = i(G_1) + 1$. Hence (a) holds. If x and y are adjacent then $i(G_2 - \{x, y\}) \geq i(G_2) - 1$; hence (b) also holds.

Let $x, y \in V(H)$. An independent dominating set of $G - \{x, y\}$ of size less than $i(G) = i(G_1)$ is either a subset of $V(G_1)$ or a subset of $V(G_2)$. In the former case x, y is an i-bicritical pair of G_1. In the latter case, as above $i(G_2) = i(G_1) + 1$ and x, y is a strongly i-bicritical pair of G_2. Hence (c) holds.

Let $x, y \in V(G_1)$. Since an independent dominating set of $G - \{x, y\}$ of size less than $i(G) = i(G_1)$ must be a subset of $V(G_1)$, it follows that x, y is an i-bicritical pair of G_1. Hence (d) holds.

Finally, let $x \in V(G_1 - H)$, and $y \in V(G_2 - H)$. Since $i(G_2 - y) \geq i(G_2) - 1 \geq i(G_1) = i(G)$, an independent dominating set of $G - \{x, y\}$ of size less than $i(G) = i(G_1)$ must be a subset of $V(G_1 - x)$. Hence x is an i-critical vertex of G_1, and (e) holds.

Now suppose that conditions (a) through (e) hold. Let $x, y \in V(G)$ and consider $G - \{x, y\}$. If $x, y \in V(H)$, then $i(G - \{x, y\}) < i(G) = i(G_1)$ by (c) and (a). If $x, y \in V(G_1 - H)$, then $i(G - \{x, y\}) < i(G) = i(G_1)$ by (d). If $x, y \in V(G_2 - H)$, then $i(G - \{x, y\}) < i(G) = i(G_1)$ by (b). Finally, if $x \in V(G_1 - H)$ and $y \in V(G_2)$, then $i(G - \{x, y\}) < i(G) = i(G_1)$ by (d), if $y \in V(H)$, and by (e) if $y \in V(G_2 - H)$. □

A graph G is called *strongly i-bicritical* if $i(G - \{x, y\}) = i(G) - 2$ for all pairs of non-adjacent vertices x, y. For example, for any $n \geq 2$, the complete bipartite graph $K_{n,n}$ is strongly i-bicritical.

Corollary 7.8. *Let G_1 and G_2 be graphs for which H is the subgraph of each one induced by $V(G_1) \cap V(G_2)$, and are such that $i(G_1) = i(G_2) - 1$. If G_1 is i-critical and i-bicritical, and G_2 is strongly i-bicritical, then $G_1(H)\widehat{\odot}G_2(H)$ is i-bicritical.*

The following is by way of analogy with Theorem 7.4. There is no analog of Corollary 7.6 when the graphs being operated on do not all have the same independent domination number.

Lemma 7.9. *Let H be a graph. Let $G_1, G_2, \ldots, G_k$ be graphs such that H is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \leq j < \ell \leq k$, and are such that $i(G_1) \leq i(G_2) \leq \cdots \leq i(G_k)$. If there exist subscripts j and ℓ such that $i(G_1) < i(G_j)$ and $i(G_1) < i(G_\ell)$, then, $G = G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is not i-bicritical.*

Proof. Note that $i(G) = i(G_1)$. Let $x \in V(G_j - H)$ and $y \in V(G_\ell - H)$. An independent dominating set of $G - \{x, y\}$ of size less than $i(G) = i(G_1)$ must be a subset of $V(G_j - x)$ or of $V(G_\ell - y)$. Since $i(G_1) \leq i(G_j) - 1 \leq i(G_j - x)$, and similarly for $G_\ell - x$, no such set exists. Therefore G is not i-bicritical. □

Theorem 7.10. *Let H be a graph. Let $G_1, G_2, \ldots, G_k$ be graphs such that G is the subgraph of each one induced by $V(G_j) \cap V(G_\ell)$, $1 \leq j < \ell \leq k$, and are such that $i(G_1) = i(G_2) = \cdots = i(G_{k-1}) < i(G_k)$. Then, $G = G_1(H)\widehat{\odot}G_2(H)\widehat{\odot}\cdots\widehat{\odot}G_k(H)$ is i-bicritical if and only if*

(a) either $i(G_k) = i(G_1) + 1$, or $|V(G_k - H)| = 1$;

(b) $E(G_k - H) = \emptyset$, and any pair of vertices $x, y \in V(G_k - H)$ is a strongly i-bicritical pair of G_k;

(c) if $x, y \in V(H)$, then either there exists $j, 1 \leq j \leq k - 1$ such that x, y is an i-bicritical pair of G_j, or x, y is a strongly i-bicritical pair of G_k;

(d) if $x \in V(H)$ and $y \in V(G_j - H)$ for $j < k$, then either x, y is an i-bicritical pair of G_j, or there exists $\ell \neq j$ such that $1 \leq \ell \leq k - 1$ and y is an i-critical vertex of G_ℓ; and

(e) for each $j \in \{1, 2, \ldots, k - 1\}$, every vertex in $V(G_j - H)$ is i-critical.

8 Construction of i-Bicritical Graphs via Wreath Product

Let G and H be disjoint graphs. The *wreath product of G with H*, also known as the *lexicographic product of G and H*, is the graph $G[H]$ with vertex set $V(G[H]) = \{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $E(G[H]) = \{(g_1, h_1)(g_2, h_2) : g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)\}$.

If D is an independent dominating set of $G[H]$, then we define

$$S_D = \{g \in V(G) : (g, h) \in D \text{ for some } h \in V(H)\}$$

and, for each $g \in S_D$,

$$T_g = \{h \in V(H) : (g, h) \in D\}.$$

The straightforward proof of the following proposition is omitted.

Proposition 8.1. *Let G and H be disjoint graphs. If D is an independent dominating set of $G[H]$, then S_D is an independent dominating set of G and, for each $g \in S_D$, T_g is an independent dominating set of H.*

Corollary 8.2. *For any disjoint graphs G and H, $i(G[H]) = i(G)i(H)$.*

Proof. Let D be an i-set of $G[H]$. By Proposition 8.1, $|S_D| \geq i(G)$ and, for each $g \in S_D$, $|T_g| \geq i(H)$. Hence $|D| \geq i(G)i(H)$.

On the other hand, if A is an i-set of G and B is an i-set of H, then the Cartesian product $A \times B$ is an independent dominating set of $G[H]$ with size $i(G)i(H)$. The result now follows. □

Theorem 8.3. *Let G and H be disjoint graphs that each have at least two vertices. Then $G[H]$ is i-bicritical if and only if H is both i-critical and i-bicritical, and either $|V(H)| = 2$ and G is i-critical, or $|V(H)| \geq 3$ and every vertex of G is in an i-set of G.*

Proof. We first consider the case where $E(G) = \emptyset$. The graph $G[H]$ is isomorphic to the disjoint union of $|V(G)|$ copies of H. By Theorem 4.1, $G[H]$ is bicritical if and only if H is both i-critical and i-bicritical. Also, the graph G is i-critical, so every vertex of G is in an i-set of G. Thus the statement holds when $E(G) = \emptyset$. Hence, in what follows, we assume $E(G) \neq \emptyset$.

Suppose $G[H]$ is i-bicritical.

We first show that H is i-critical. Let $g_1g_2 \in E(G)$, and $h \in V(H)$. Let D be an i-set of $G[H] - \{(g_1, h), (g_2, h)\}$. Since $|D| = i(G[H]) - 1$ and H has at least 2 vertices, by Proposition 8.1 and Corollary 8.2 either g_1 or g_2 belongs to S_D. Without loss of generality $g_1 \in S_D$. By Proposition 8.1 and Corollary 8.2 again, we must have that $|T_{g_1}| = i(H) - 1$. Therefore H is i-critical. Furthermore, since H has at least 2 vertices, $i(H) \geq 2$.

Next, we show that H is i-bicritical. If H has only 2 vertices, then since it is i-critical, it is isomorphic to the disjoint union of two copies of K_1, which is i-bicritical. Suppose, then, that H has at least 3 vertices. Let $h_1, h_2 \in V(H)$, and $g \in V(G)$. Let

D be an i-set of $G[H] - \{(g, h_1), (g, h_2)\}$. Since $|D| < i(G[H])$, by Proposition 8.1 and Corollary 8.2, we must have $g \in S_D$ and $|T_g| < i(H)$. Therefore H is i-bicritical.

Finally, we show that either $|V(H)| = 2$ and G is i-critical, or $|V(H)| \geq 3$ and every vertex of G is in an i-set of G. Let $h_1, h_2 \in V(H)$, and $g \in V(G)$. Consider $G[H] - \{(g, h_1), (g, h_2)\}$.

If $V(H) = \{h_1, h_2\}$, then $G[H] - \{(g, h_1), (g, h_2)\} \cong (G - g)[H]$. By Corollary 8.2, this graph has independent domination number $i(G - g)i(H)$. Since $G[H]$ is i-bicritical, $i(G - g)i(H) < i(G)i(H)$. Therefore g is an i-critical vertex of G, from which it follows that G is i-critical.

Now suppose that H has at least 3 vertices. Let D be an i-set of $G[H] - \{(g, h_1), (g, h_2)\}$. As above, we must have $g \in S_D$ and $|T_g| < i(H)$. By Proposition 8.1 and Corollary 8.2 we then have

$$i(G[H] - \{(g, h_1), (g, h_2)\}) \leq |S_D|i(H) - 1 \leq i(G)i(H) - 1,$$

from which it follows that $|S_D| \leq i(G)$. Therefore g is in an i-set of G.

We now prove the converse. Suppose that H is both i-critical and i-bicritical. Let $(g_1, h_1), (g_2, h_2) \in G[H]$, and consider $G[H] - \{(g_1, h_1), (g_2, h_2)\}$.

Suppose that $|V(H)| = 2$ and G is i critical. If $g_1 = g_2$, then

$$G[H] - \{(g_1, h_1), (g_2, h_2)\} \cong (G - g_1)[H].$$

Since G is i-critical, by Corollary 8.2 we have

$$i(G[H] - \{(g_1, h_1), (g_2, h_2)\}) = (i(G) - 1)i(H) < i(G)i(H) = i(G[H]).$$

Otherwise, $g_1 \neq g_2$. Since G is i-critical, there exists an i-set, S, of G such that $g_1 \in S$. Since H is i-critical, there exists an i-set, T', of $H - h_1$ such that, $T = T' \cup \{h_1\}$ is an i-set of H. Then $S \times T - \{(g_1, h_1)\}$ is an independent dominating set of $G[H]$ of size $i(G)i(H) - 1$. Therefore, $G[H]$ is i-bicritical.

Now suppose that $|V(H)| \geq 3$ and every vertex of g is in an i-set of G. By hypothesis, there exists an i-set, S, of G such that $g_1 \in S$.

Assume first that $g_1 = g_2$. Since H is i-bicritical, there exists an i-set, T', of $H - \{h_1, h_2\}$ which is a proper subset of an i-set T of H that contains h_1 or h_2, possibly both. Then $S \times T - \{(g_1, h_1), (g_2, h_2)\}$ is an independent dominating set of $G[H]$ of size $i(G)i(H) - 1$ or $i(G)i(H) - 2$.

Otherwise, $g_1 \neq g_2$. Suppose $g_1g_2 \in E(G)$. Since H is i-critical, there exists an i-set, T', of $H - h_1$ such that, $T = T' \cup \{h_1\}$ is an i-set of H. Then $S \times T - \{(g_1, h_1)\}$ is an independent dominating set of $G[H]$ of size $i(G)i(H) - 1$. Finally, suppose $g_1g_2 \notin E(G)$. Since H is i-critical, there exists an i-set, T_1', of $H - h_1$ such that $T_1 = T_1' \cup \{h_1\}$ is an i-set of H, and an i-set, T_2', of $H - h_2$ such that $T_2 = T_2' \cup \{h_2\}$ is an i-set of H. Let D be the set

$$D = \begin{cases} (S - \{g_1, g_2\}) \times T_1) \cup (\{g_1\} \times T_1') \cup (\{g_2\} \times T_2') & \text{if } g_2 \in S \\ (S - \{g_1\}) \times T_1) \cup (\{g_1\} \times T_1') & \text{if } g_2 \notin S. \end{cases}$$

Then D is an independent dominating set of $G[H] - \{(g_1, h_1), (g_2, h_2)\}$ of size $i(G)i(H) - 2$ or $i(G)i(H) - 1$. Therefore, $G[H]$ is i-bicritical. □

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