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Packing colouring of some classes of cubic graphs

Daouya Laïche

Faculty of Mathematics, Laboratory L'IFORCE
University of Sciences and Technology Houari Boumediene (USTHB)
B.P. 32 El-Alia, Bab-Ezzouar, 16111 Algiers
Algeria

Éric Sopena*

Univ. Bordeaux
Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence
France

Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two distinct vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$.

Recently, it was proved in [J. Balogh, A. Kostochka and X. Liu, *Discrete Math.* 341 (2018), 474–483] that χ_ρ is not bounded in the class of subcubic graphs, thus answering a question previously addressed in several papers. However, several subclasses of cubic or subcubic graphs have bounded packing chromatic number. In this paper, we determine the exact value of, or upper and lower bounds on, the packing chromatic number of some classes of cubic graphs, namely circular ladders, and so-called H-graphs and generalised H-graphs.

1 Introduction

All the graphs we consider are simple. For a graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The *distance* $d_G(u,v)$ between vertices u and v in G is the length (number of edges) of a shortest path joining u and v. The *diameter* of G is the maximum distance between two vertices of G. We denote by P_n, $n \geq 1$, the path of order n and by C_n, $n \geq 3$, the cycle of order n.

A *packing k-colouring* of G is a mapping $\pi : V(G) \to \{1, \ldots, k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d_G(u,v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a

* Corresponding author: `Eric.Sopena@labri.fr`

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packing k-colouring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets V_i, $1 \le i \le k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i, $1 \le i \le k$. A packing colouring of G is *optimal* if it uses exactly $\chi_\rho(G)$ colours.

The packing colouring of graphs was introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall in [13,14], under the name *broadcast colouring.* In their seminal paper [14], the question of determining the maximum packing chromatic number in the class of cubic graphs of a given order is posed. In [18], Sloper proved that the packing chromatic number is unbounded in the class of k-ary trees for every $k \ge 3$, from which it follows that the packing chromatic number is unbounded in the class of graphs with maximum degree 4.

In [12], Gastineau and Togni observed that each cubic graph of order at most 20 has packing chromatic number at most 10. They also observed that the largest cubic graph with diameter 4 (this graph has 38 vertices and is described in [1]) has packing chromatic number 13, and asked whether there exists a cubic graph with packing chromatic number larger than 13 or not. This question was answered positively by Brešar, Klavžar, Rall and Wash [9] who exhibited a cubic graph on 78 vertices with packing chromatic number at least 14. Recently, Balogh, Kostochka and Liu finally proved in [2] that the packing chromatic number is unbounded in the class of cubic graphs, and Brešar and Ferme gave in [5] an explicit infinite family of subcubic graphs with unbounded packing chromatic number.

On the other hand, the packing chromatic number is known to be bounded above in several classes of graphs with maximum degree 3, for instance in complete binary trees [18], hexagonal lattices [6, 10, 15], base-3 Sierpiński graphs [7] or particular Sierpiński-type graphs [4], subdivisions of subcubic graphs [8, 12] and of cubic graphs [3], or several subclasses of outerplanar subcubic graphs [11].

In this paper we prove that the packing chromatic number is bounded in other classes of cubic graphs, in particular extending partial results given in [19]. More precisely, we determine the exact value of, or upper and lower bounds on, the packing chromatic number of circular ladders (in Section 3), H-graphs (in Section 4) and generalised H-graphs (in Section 5).

2 Preliminary results

In this section we give a few results that will be useful in the sequel.

Let G be a graph. A subset S of $V(G)$ is an *i-packing*, for some integer $i \ge 1$, if any two vertices in S are at distance at least $i+1$ in G. Note that such a set S is a 1-packing if and only if S is an independent set. A packing colouring of G is thus a partition of $V(G)$ into k disjoint subsets $V_1, \dots, V_k$, such that V_i is an i-packing for every i, $1 \le i \le k$.

For every integer $i \ge 1$, we denote by $\rho_i(G)$ the maximum cardinality of an i-packing in G. Since at most $\rho_i(G)$ vertices can be assigned colour i in any packing colouring of G, we have the following result.

Proposition 2.1 *If G is a graph with $\chi_\rho(G) = k$, then*

$$\sum_{i=1}^{i=k} \rho_i(G) \geq |V(G)|.$$

Let H be a subgraph of G. Since $d_G(u,v) \leq d_H(u,v)$ for any two vertices $u, v \in V(H)$, the restriction to $V(H)$ of any packing colouring of G is a packing colouring of H. Hence, having packing chromatic number at most k is a hereditary property:

Proposition 2.2 (Goddard, Hedetniemi, Hedetniemi, Harris & Rall [14]) *Let G and H be two graphs. If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.*

In particular, Proposition 2.2 gives a lower bound on the packing chromatic number of a graph G whenever G contains a subgraph H whose packing chromatic number is known. As we will see later, all the cubic graphs we consider in this paper contain a corona of a cycle as a subgraph. Recall that the *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbour to every vertex of G. In [17], we have determined with I. Bouchemakh the packing chromatic number of the corona of cycles.

Theorem 2.3 (Laïche, Bouchemakh, Sopena [17])
The packing chromatic number of the corona graph $C_n \odot K_1$ is given by:

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3,4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

This result will thus provide a lower bound on the packing chromatic number of each cubic graph considered in this paper.

3 Circular ladders

Recall that the Cartesian product $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, two vertices (u, u') and (v, v') being adjacent if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

The *circular ladder* CL_n of length $n \geq 3$ is the Cartesian product $CL_n = C_n \,\square\, K_2$. Note that CL_n is a bipartite graph if and only if n is even.

For every circular ladder CL_n, we let

$$V(CL_n) = \{u_0, \ldots, u_{n-1}\} \;\cup\; \{v_0, \ldots, v_{n-1}\},$$

and

$$E(CL_n) = \{u_i v_i \mid 0 \leq i \leq n-1\} \;\cup\; \{u_i u_{i+1}, v_i v_{i+1} \mid 0 \leq i \leq n-1\}$$

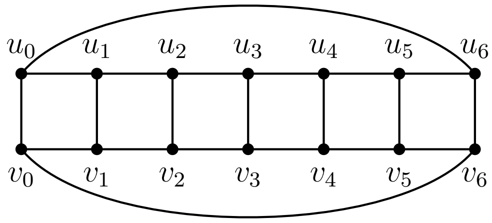


Figure 1: The circular ladder CL_7.

(subscripts are taken modulo n). Figure 1 depicts the circular ladder CL_7.

Note that for every $n \geq 3$, the corona graph $C_n \odot K_1$ is a subgraph of the circular ladder CL_n. Therefore, by Proposition 2.2, Theorem 2.3 provides a lower bound on the packing chromatic number of circular ladders. More precisely, $\chi_\rho(CL_n) \geq 4$ if $n \in \{3, 4\}$, and $\chi_\rho(CL_n) \geq 5$ if $n \geq 5$.

William and Roy [19] proved that the packing chromatic number of a circular ladder of length $n = 6q$, $q \geq 1$, is at most 5. In Theorem 3.4 below, we extend this result and determine the packing chromatic number of every circular ladder.

We first need the following technical lemma, which will also be useful in Section 5.

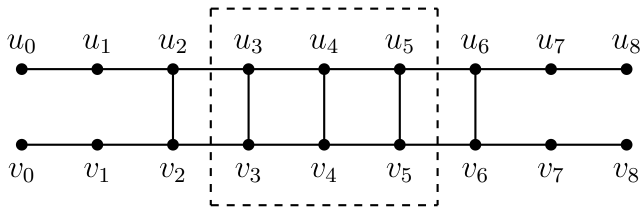


Figure 2: The graph X.

Lemma 3.1 *Let X be the graph depicted in Figure 2, and π be a packing 5-colouring of X. If $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$ for some integer i, $3 \leq i \leq 5$, then either u_i or v_i has colour 2, and its three neighbours have colours 3, 4 and 5 (the three corresponding edges are the vertical edges surrounded by the dashed box).*

Proof. The proof is done by case analysis and is given in Appendix A. □

Observe now that for every integer $n \geq 9$, the subgraph of CL_n induced by the set of vertices $\{u_i, v_i \mid 0 \leq i \leq 8\}$ contains the graph X of Figure 2 as a subgraph. Moreover, every packing 5-colouring π of CL_n, $6 \leq n \leq 8$, can be "unfolded" to produce a packing 5-colouring π' of X, by setting $\pi'(u_i) = \pi(u_i)$ and $\pi'(v_i) = \pi(v_i)$ for every i, $0 \leq i \leq n-1$, and $\pi'(u_{n-1+j}) = \pi(u_{j-1})$ and $\pi'(v_{n-1+j}) = \pi(v_{j-1})$ for every j, $1 \leq j \leq 9-n$. This follows from the fact that vertices u_j and u_{n+j}, as well as vertices v_j and v_{n+j}, are at distance $n \geq 6$ from each other, while the largest colour used by π' is 5. Therefore, thanks to the symmetries of CL_n for every $n \geq 6$, Proposition 2.2 and Lemma 3.1 give the following corollary.

Corollary 3.2 *Let CL_n, $n \geq 6$, be a circular ladder with $\chi_\rho(CL_n) \leq 5$, and π be a packing 5-colouring of CL_n. For every integer i, $0 \leq i \leq n-1$, if $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$, then either u_i or v_i has colour 2, and its three neighbours have colours 3, 4 and 5.*

Let CL_n be a circular ladder satisfying the hypothesis of Corollary 3.2, and π be a packing 5-colouring of CL_n. From Corollary 3.2, it follows that if $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$ for some edge u_iv_i of CL_n, then the colour 2 has to be used on the edge u_iv_i and, since the neighbours of the 2-coloured vertex are coloured with 3, 4 and 5, the colour 2 can be replaced by colour 1. Therefore, we get the following corollary.

Corollary 3.3 *If CL_n, $n \geq 6$, is a circular ladder with $\chi_\rho(CL_n) \leq 5$, then there exists a packing 5-colouring of CL_n such that the colour 1 is used on each edge of CL_n.*

Note that from Corollary 3.3, it follows that for every integer $n \geq 6$, $\chi_\rho(CL_n) \leq 5$ implies that CL_n is a bipartite graph. Hence, $\chi_\rho(CL_n) \geq 6$ for every odd $n \geq 6$.

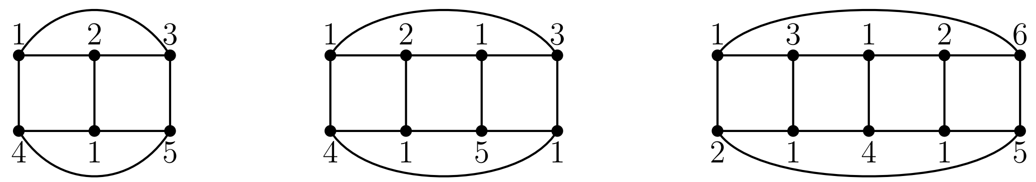


Figure 3: Optimal packing colouring of CL_3, CL_4 and CL_5.

We are now able to prove the main result of this section.

Theorem 3.4 *For every integer $n \geq 3$,*

$$\chi_\rho(CL_n) = \begin{cases} 5 & \text{if } n = 3, \text{ or } n \text{ is even and } n \notin \{8, 14\}, \\ 7 & \text{if } n \in \{7, 8, 9\}, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We first consider the case $n \leq 5$. Figure 3 describes a packing 5-colouring of CL_3 and CL_4, and a packing 6-colouring of CL_5. We claim that these three packing colourings are optimal. To see that, observe that $\rho_1(CL_3) = 2$, $\rho_i(CL_3) = 1$ for every $i \geq 2$, $\rho_1(CL_4) = \rho_1(CL_5) = 4$, $\rho_2(CL_4) = \rho_2(CL_5) = 2$, and $\rho_i(CL_4) = \rho_i(CL_5) = 1$ for every $i \geq 3$. The optimality for CL_3 and CL_5 then follows from Proposition 2.1. The optimality for CL_4 also follows, with the additional observation that colour 2 can be used at most once if colour 1 is used four times.

Assume now $n \geq 6$. Since $n \geq 6$ and every circular ladder CL_n contains the corona graph $C_n \odot K_1$ as a subgraph, we get $\chi_\rho(CL_n) \geq \chi_\rho(C_n \odot K_1) \geq 5$ by Theorem 2.3 and Proposition 2.2. Moreover, by Corollary 3.3, we have $\chi_\rho(CL_n) \geq 6$ if n is odd.

We now consider two general cases.

1. *n is even* and $n \notin \{8, 14\}$.
   As observed above, in that case, it is enough to exhibit a packing 5-colouring of CL_n to prove $\chi_\rho(CL_n) = 5$.

   If $n \equiv 0 \pmod 6$, a packing 5-colouring of CL_n is obtained by repeating the following circular pattern (the first row gives the colours of vertices u_i, $0 \leq i \leq n-1$, the second row gives the colours of vertices v_i, $0 \leq i \leq n-1$, according to the value of ($i \mod 6$)):

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\|$$

   If $n \equiv 2 \pmod 6$, which implies $n \geq 20$, a packing 5-colouring of CL_n is obtained by repeating the previous circular pattern $\frac{n-20}{6}$ times and adding a pattern of length 20, as illustrated below:

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 3\ 1\ 4\ 1\ 5\ 1\ 3\ 1\ 2\ 1\ 3\ 1\ 4\ 1\ 5 \\ 2\ 1\ 4\ 1\ 5\ 1\ 2\ 1\ 3\ 1\ 2\ 1\ 4\ 1\ 5\ 1\ 2\ 1\ 3\ 1 \end{matrix}$$

   Finally, if $n \equiv 4 \pmod 6$, which implies $n \geq 10$, a packing 5-colouring of CL_n is obtained by repeating the same circular pattern $\frac{n-10}{6}$ times and adding a pattern of length 10:

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 3\ 1\ 4\ 1\ 5 \\ 2\ 1\ 4\ 1\ 5\ 1\ 2\ 1\ 3\ 1 \end{matrix}$$

2. *n is odd* and $n \geq 11$.
   As observed above, in that case, it is enough to exhibit a packing 6-colouring of CL_n to prove $\chi_\rho(CL_n) = 6$.

   Similarly as in the previous case, if $n \equiv 1, 3$ or $5 \pmod 6$, a packing 6-colouring of CL_n is obtained by repeating the previous circular pattern $\frac{n-7}{6}$, $\frac{n-9}{6}$ or $\frac{n-5}{6}$ times, respectively, and adding a pattern of length 7, 9 or 5, respectively, as illustrated below:

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\| \begin{matrix} 1\ 3\ 1\ 4\ 1\ 2\ 6 \\ 6\ 1\ 2\ 1\ 3\ 1\ 5 \end{matrix}$$

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\| \begin{matrix} 1\ 4\ 1\ 2\ 3\ 1\ 4\ 1\ 6 \\ 2\ 1\ 6\ 1\ 5\ 2\ 1\ 3\ 1 \end{matrix}$$

$$\left\| \begin{matrix} 1\ 3\ 1\ 2\ 1\ 5 \\ 2\ 1\ 4\ 1\ 3\ 1 \end{matrix} \right\| \begin{matrix} 1\ 3\ 1\ 2\ 6 \\ 2\ 1\ 4\ 1\ 5 \end{matrix}$$

It remains to consider four cases, namely $n = 7, 8, 9, 14$, which we consider separately.

1. $n = 7$.
   We first claim that $\chi_\rho(CL_7) \geq 7$. Note that $\rho_1(CL_7) = 6$, $\rho_2(CL_7) = 3$, $\rho_3(CL_7) = 2$, and $\rho_i(CL_7) = 1$ for every $i \geq 4$. However, if we use six times colour 1, colour 2 can be used at most twice. Hence, at most 13 vertices of CL_7 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

   A packing 7-colouring of CL_7 is given by the following pattern:

   1 3 1 2 1 4 5
   2 1 6 1 3 1 7

2. $n = 8$.
   We first claim that $\chi_\rho(CL_8) \geq 7$. Note that $\rho_1(CL_8) = 8$, $\rho_2(CL_8) = 4$, $\rho_3(CL_8) = \rho_4(CL_8) = 2$, and $\rho_i(CL_8) = 1$ for every $i \geq 5$. However, if we use eight times colour 1, colour 2 can be used at most twice, and then colour 4 at most once. On the other hand, if we use seven times colour 1, then, either colour 2 is used thrice, and then colour 4 can be used at most once, or colour 2 is used at most twice, and then colour 4 can be used at most twice. Hence, at most 15 vertices of CL_8 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

   A packing 7-colouring of CL_8 is given by the following pattern:

   1 3 1 2 1 5 1 7
   2 1 4 1 3 1 6 1

3. $n = 9$.
   We first claim that $\chi_\rho(CL_9) \geq 7$. Note that $\rho_1(CL_9) = 8$, $\rho_2(CL_9) = 4$, $\rho_3(CL_9) = \rho_4(CL_9) = 2$, and $\rho_i(CL_9) = 1$ for every $i \geq 5$. However, if we use eight times colour 1, colour 2 can be used at most thrice. Hence, at most 17 vertices of CL_9 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

   A packing 7-colouring of CL_9 is given by the following pattern:

   1 3 1 2 1 5 1 4 6
   2 1 4 1 3 1 2 1 7

4. $n = 14$.
   We first claim that $\chi_\rho(CL_{14}) \geq 6$. Note that $\rho_1(CL_{14}) = 14$, $\rho_2(CL_{14}) = 6$, $\rho_3(CL_{14}) = 4$, $\rho_4(CL_{14}) = 3$ and $\rho_5(CL_{14}) = 2$. However, if we use 14 times colour 1, colour 2 can be used at most four times. On the other hand, if we use 13 times colour 1, colour 2 can be used at most five times. Hence, at most 27 vertices of CL_{14} can be coloured with a colour in $\{1, \dots, 5\}$ and the claim follows.

   A packing 6-colouring of CL_{14} is given by the following pattern:

1 3 1 2 1 5 1 2 1 4 1 3 1 6
2 1 4 1 3 1 6 1 3 1 2 1 5 1

This completes the proof of Theorem 3.4. □

4 H-graphs

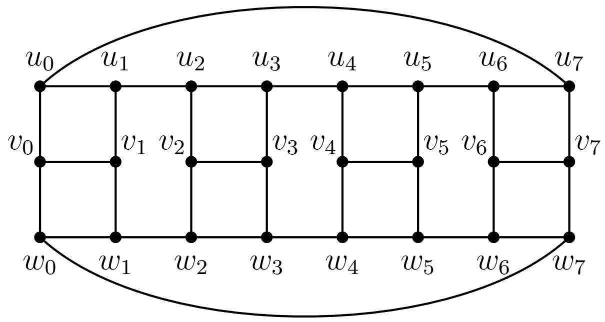


Figure 4: The H-graph $H(4)$.

The *H-graph* $H(r)$, $r \geq 2$, is the 3-regular graph of order $6r$, with vertex set

$$V(H(r)) = \{u_i, v_i, w_i : 0 \leq i \leq 2r-1\},$$

and edge set (subscripts are taken modulo $2r$)

$$\begin{aligned} E(H(r)) &= \{(u_i, u_{i+1}),\ (w_i, w_{i+1}),\ (u_i, v_i),\ (v_i, w_i) : 0 \leq i \leq 2r-1\} \\ &\quad \cup \{(v_{2i}, v_{2i+1}) : 0 \leq i \leq r-1\}. \end{aligned}$$

Figure 4 depicts the H-graph $H(4)$. These graphs have been introduced and studied by William and Roy in [19], where it is proved that $\chi_\rho(H(r)) \leq 5$ for every H-graph $H(r)$ with even $r \geq 4$. We complete their result in Theorem 4.3 below.

We first prove a technical lemma. For every pair of integers $r \geq 2$ and $0 \leq i \leq r-1$, we denote by $G_i(r)$ the subgraph of $H(r)$ induced by the set of vertices $\{u_{2i}, u_{2i+1}, v_{2i}, v_{2i+1}, w_{2i}, w_{2i+1}\}$. Observe that for every $r \geq 2$, all the subgraphs $G_i(r)$ are isomorphic to the graph depicted in Figure 5(a), and thus $\chi_\rho(G_i(r)) = \chi_\rho(P_2 \square P_3) = 4$ [14].

For a given packing 5-colouring π of $H(r)$, we denote by $\pi(G_i(r))$ the set of colours assigned to the vertices of $G_i(r)$. We then have the following result.

Lemma 4.1 *For every integer $r \geq 3$ and every packing 5-colouring π of $H(r)$, $\pi(G_i(r)) \cap \pi(G_{i+1}(r)) = \{1, 2, 3\}$ for every i, $0 \leq i \leq r-1$.*

Proof. Since $\chi_\rho(P_2 \square P_3) = 4$, every packing 5-colouring of $H(r)$ must use colour 4 or colour 5 on every $G_i(r)$, $0 \leq i \leq r-1$. We now prove that if colour 4 (respectively, colour 5) is used on $G_i(r)$, then colour 4 (respectively, colour 5) cannot be used on $G_{i+1}(r)$. Observe first that every vertex of $G_i(r)$ is at distance at most 5 from every vertex of $G_{i+1}(r)$. Therefore, colour 5 cannot be used on both $G_i(r)$ and $G_{i+1}(r)$. Suppose now that colour 4 is used on both $G_i(r)$ and $G_{i+1}(r)$. Up to symmetries, we necessarily have one of the two following cases.

1. $\pi(u_{2i}) = \pi(w_{2i+3}) = 4$ (see Figure 5(b)).
   Since every vertex of $G_{i-1}(r)$ is at distance at most 4 from u_{2i}, it follows that $G_{i-1}(r)$ does not contain the colour 4. This implies that $G_{i-1}(r)$ contains the colour 5 since $\chi_\rho(G_{i-1}(r)) > 3$. By symmetry, $G_{i+2}(r)$ must also contain the colour 5. Furthermore, since two consecutive $G_i(r)$s cannot both use colour 5, neither $G_i(r)$ nor $G_{i+1}(r)$ contains the colour 5.

   Now, on the remaining uncoloured vertices of $G_i(r)$, colour 1 can be used at most thrice, colour 2 at most twice and colour 3 at most once. If colour 1 is used thrice, then we necessarily have $\pi(u_{2i+1}) = \pi(v_{2i}) = \pi(w_{2i+1}) = 1$, so that $\{\pi(v_{2i+1}), \pi(w_{2i})\} = \{2, 3\}$, and no colour is available for w_{2i+2} (recall that colour 5 is not used on $G_{i+1}(r)$). If colour 1 is used twice, then we necessarily have, up to symmetry, $\pi(v_{2i}) = \pi(w_{2i+1}) = 1$, $\pi(u_{2i+1}) = \pi(w_{2i}) = 2$, and $\pi(v_{2i+1}) = 3$, and no colour is available for w_{2i+2}.

2. $\pi(v_{2i}) = \pi(v_{2i+3}) = 4$ (see Figure 5(c)).
   Similarly as before, since every vertex of $G_{i-1}(r)$ is at distance at most 4 from v_{2i} and two consecutive $G_i(r)$'s cannot both use colour 5, it follows from the first item of Lemma 4.1 that colour 5 is used neither on $G_i(r)$, nor, by symmetry, on $G_{i+1}(r)$.

   Again, on the remaining uncoloured vertices of $G_i(r)$, colour 1 can be used at most thrice, colour 2 at most twice and colour 3 at most once. If colour 1 is used thrice, then we necessarily have $\pi(u_{2i}) = \pi(v_{2i+1}) = \pi(w_{2i}) = 1$, so that $\{\pi(u_{2i+1}), \pi(w_{2i+1})\} = \{2, 3\}$. Up to symmetry, we may assume $\pi(u_{2i+1}) = 2$ and $\pi(w_{2i+1}) = 3$, which implies $\pi(u_{2i+2}) = 1$, and no colour is available for v_{2i+2} (recall that colour 5 is not used on $G_{i+1}(r)$). If colour 1 is used twice, then we necessarily have, up to symmetry, $\pi(u_{2i+1}) = \pi(w_{2i}) = 1$, $\pi(u_{2i}) = \pi(w_{2i+1}) = 2$, and $\pi(v_{2i+1}) = 3$, and no colour is available for u_{2i+2}.

This completes the proof. □

From Lemma 4.1, it follows that every $G_i(r)$ must use colour 4 or 5, and that no two consecutive $G_i(r)$'s can use the same colour from $\{4, 5\}$. Therefore, $H(r)$ does not admit any packing 5-colouring when r is odd.

Corollary 4.2 *For every odd integer r, $r \geq 3$, $\chi_\rho(H(r)) > 5$.*

We are now able to prove the main result of this section.

Theorem 4.3 *For every integer $r \geq 2$, $\chi_\rho(H(r)) = 5$ if r is even, and $6 \leq \chi_\rho(H(r)) \leq 7$ if r is odd.*

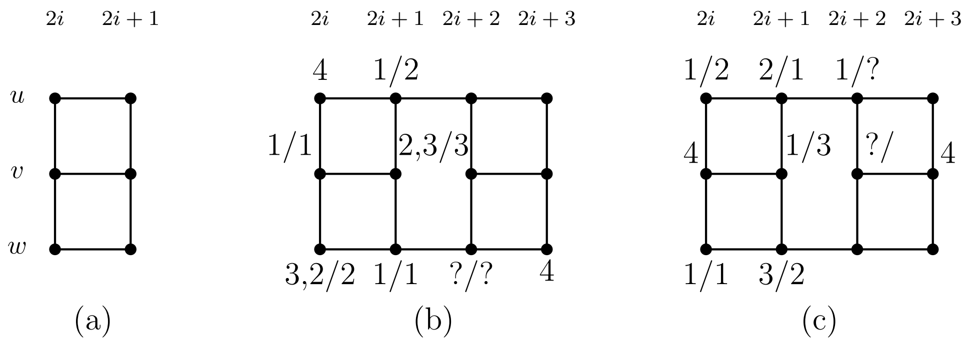


Figure 5: The subgraph $G_i(r)$ and two configurations for the proof of Lemma 4.1.

Proof. We consider two cases, according to the parity of r.

1. *r is even.*
   Since $H(r)$ contains the corona graph $C_6 \odot K_1$ as a subgraph (consider for instance the 6-cycle $u_1v_1w_1w_2v_2u_2$), we get $\chi_\rho(H(r)) \geq 5$ by Theorem 2.3 and Proposition 2.2. A packing 5-colouring of $H(r)$ is then obtained by repeating the pattern depicted in Figure 6(a), and thus $\chi_\rho(H(r)) = 5$.

2. *r is odd.*
   From Corollary 4.2, we get $\chi_\rho(H(r)) \geq 6$. A packing 7-colouring of $H(r)$ is described in Figure 6(b), where the circular pattern (surrounded by the dashed box) is repeated $\frac{r-3}{2}$ times. This gives $\chi_\rho(H(r)) \leq 7$.

This concludes the proof. □

5 Generalised H-graphs

We now consider a natural extension of H-graphs. For every integer $r \geq 2$, the *generalised H-graph* $H^\ell(r)$ with ℓ levels, $\ell \geq 1$, is the 3-regular graph of order $2r(\ell+2)$, with vertex set

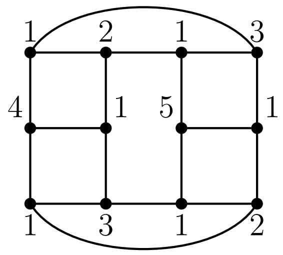
$$V(H^\ell(r)) = \{u_j^i : 0 \leq i \leq \ell + 1, 0 \leq j \leq 2r - 1\}$$

and edge set (subscripts are taken modulo $2r$)

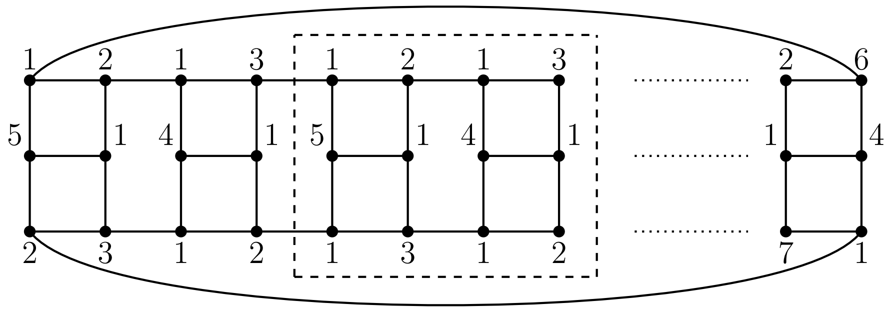
$$\begin{aligned} E(H^\ell(r)) \quad = \quad & \{(u_j^0, u_{j+1}^0),\ (u_j^{\ell+1}, u_{j+1}^{\ell+1}) : 0 \leq j \leq 2r - 1\} \\ & \cup \{(u_{2j}^i, u_{2j+1}^i) : 1 \leq i \leq \ell,\ 0 \leq j \leq r - 1\} \\ & \cup \{(u_j^i, u_j^{i+1}) : 0 \leq i \leq \ell,\ 0 \leq j \leq 2r - 1\}. \end{aligned}$$

Figure 7 depicts the generalised H-graph with three levels $H^3(4)$. Note that generalised H-graphs with one level are precisely H-graphs.

The three following lemmas will be useful for determining the packing chromatic number of generalised H-graphs.



(a) A packing 5-colouring pattern for $H(r)$, r even, $r \geq 2$



(b) A packing 7-colouring pattern for $H(r)$, r odd, $r \geq 3$

Figure 6: Packing colouring patterns for H-graphs.

Lemma 5.1 *For every pair of integers $\ell \geq 3$ and $r \geq 3$, let $H^\ell(r)$ be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$ and let π be a packing 5-colouring of $H^\ell(r)$. For every edge $u^i_{2j}u^i_{2j+1}$, $1 \leq i \leq \ell$, $0 \leq j \leq r-1$, with $\pi(u^i_{2j}) \neq 1$ and $\pi(u^i_{2j+1}) \neq 1$, either u^i_{2j} or u^i_{2j+1} has colour 2 and its three neighbours have colours 3, 4 and 5.*

Proof. We first claim that every such edge $u^i_{2j}u^i_{2j+1}$ belongs to a subgraph of $H^\ell(r)$ isomorphic to the graph X depicted in Figure 2, in such a way that $u^i_{2j}u^i_{2j+1}$ corresponds to one of the edges u_3v_3, u_4v_4 or u_5v_5 of X. Indeed, consider first the "extremal" case of $H^3(3)$, and observe that X is a subgraph of the subgraph of $H^3(3)$ induced by the set of vertices

$$\{u^0_0, \ldots, u^0_5\} \ \cup \ \{u^4_0, \ldots, u^4_5\} \ \cup \ \{u^1_2, u^1_3, u^2_2, u^2_3, u^3_2, u^3_3\}.$$

Our claim then follows for $H^3(3)$ thanks to its symmetries.

It is now easy to see that our claim holds for every generalised H-graph $H^\ell(r)$ with $\ell, r \geq 3$. The result then follows by Lemma 3.1. □

From Lemma 5.1, it follows that if $\pi(u^i_{2j}) \neq 1$ and $\pi(u^i_{2j+1}) \neq 1$ for some edge $u^i_{2j}u^i_{2j+1}$ of $H^\ell(r)$, $1 \leq i \leq \ell$, $0 \leq j \leq r-1$, then the colour 2 has to be used on this edge and, since the neighbours of the 2-coloured vertex are coloured with 3, 4 and 5, the colour 2 can be replaced by colour 1. Therefore, we get the following corollary.

Corollary 5.2 *For every pair of integers $\ell \geq 3$ and $r \geq 3$, if $H^\ell(r)$ is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring of $H^\ell(r)$ such*

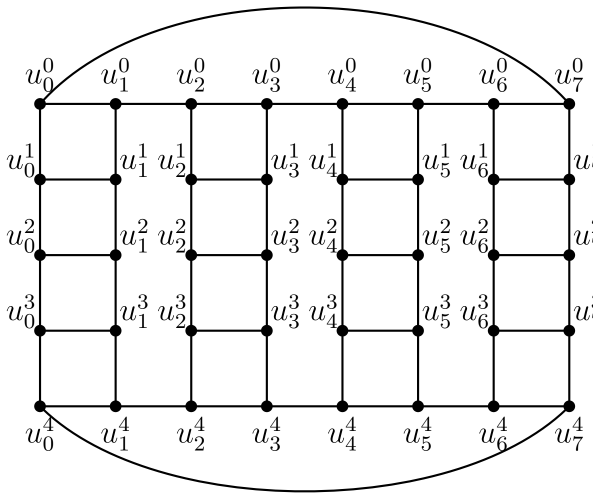


Figure 7: The generalised H-graph $H^3(4)$.

that, for every pair of integers i and j, $1 \leq i \leq \ell$, $0 \leq j \leq r-1$, the colour 1 is used on the edge $u_{2j}^i u_{2j+1}^i$ of $H^\ell(r)$.

Lemma 5.3 *For every pair of integers $\ell \geq 3$ and $r \geq 3$, let $H^\ell(r)$ be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$ and π be a packing 5-colouring of $H^\ell(r)$. For every j, $0 \leq j \leq 2r-1$, π must assign colour 1 to one vertex of each of the edges $u_j^0 u_{j+1}^0$ and $u_j^{\ell+1} u_{j+1}^{\ell+1}$ (subscripts are taken modulo $2r$).*

Proof. The proof is done by case analysis and is given in Appendix B. □

Let $H^\ell(r)$ be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$. From Corollary 5.2 and Lemma 5.3, it follows that one can always produce a packing 5-colouring of $H^\ell(r)$ that uses colour 1 on each edge $u_{2j}^i u_{2j+1}^i$ of $H^\ell(r)$, $0 \leq i \leq \ell+1$, $0 \leq j \leq r-1$. Since adjacent vertices cannot be assigned the same colour and $H^\ell(r)$ is a bipartite graph, we get the following corollary.

Corollary 5.4 *For every pair of integers $\ell \geq 3$ and $r \geq 3$, if $H^\ell(r)$ is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring of $H^\ell(r)$ such that the colour 1 is used on each edge of $H^\ell(r)$.*

Lemma 5.5 *For every pair of integers $\ell \geq 3$ and $r \geq 3$, if $H^\ell(r)$ is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring π of $H^\ell(r)$ such that $\pi(u_j^0) \notin \{4,5\}$ and $\pi(u_j^{\ell+1}) \notin \{4,5\}$ for every j, $0 \leq j \leq 2r-1$.*

Proof. Let π be a packing 5-colouring of $H^\ell(r)$ such that colour 1 is used on each edge of $H^\ell(r)$ (the existence of such a colouring is ensured by Corollary 5.4). Thanks to the symmetries of $H^\ell(r)$, it suffices to prove the result for any vertex u_{2j}^0, $0 \leq j \leq r-1$. Suppose to the contrary that $\pi(u_{2j}^0) \in \{4,5\}$ for some j, $0 \leq j \leq r-1$. We have two cases to consider.

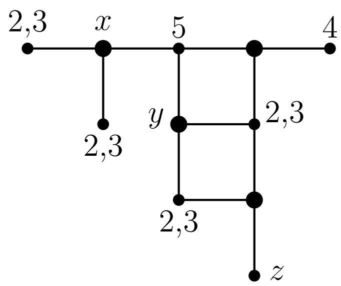


Figure 8: The subgraph Y of $H^\ell(r)$.

1. $\pi(u^0_{2j}) = 4$.
   Let Y be the subgraph of $H^\ell(r)$ depicted in Figure 8, where the vertex u^0_{2j} is the unique vertex with colour 4, and vertices with colour 1 are drawn as "big vertices". Observe that the three neighbours of x, as well as the three neighbours of y, must use colours 2, 3 and 5. Therefore, the common neighbour of x and y must be assigned colour 5. It then follows that no colour is available for z.

2. $\pi(u^0_{2j}) = 5$.
   The proof is similar to the proof of the previous case, by switching colours 4 and 5.

This completes the proof. □

Let $H^\ell(r)$ be a generalised H-graph satisfying the hypothesis of Lemma 5.5, and π be a packing 5-colouring of $H^\ell(r)$. From Lemma 5.5, it follows that the restriction of π to the $2r$-cycle induced by the set of vertices $\{u^0_j \mid 0 \le j \le 2r-1\}$ is a packing 3-colouring. It is not difficult to check (see [17]) that a $2r$-cycle admits a packing 3-colouring if and only if r is even. Therefore, we get the following corollary.

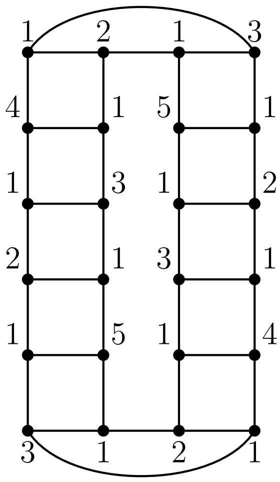
Corollary 5.6 *For every pair of integers $\ell \ge 3$ and $r \ge 3$, r odd, $\chi_\rho(H^\ell(r)) \ge 6$.*

We are now able to prove the main results of this section. We first consider the case of generalised H-graphs $H^\ell(r)$ with $\ell \notin \{2, 5\}$.

Theorem 5.7 *For every pair of integers $\ell \ge 3$, $\ell \ne 5$, and $r \ge 2$,*

$$\chi_\rho(H^\ell(r)) = \begin{cases} 5 & \text{if } r \text{ is even,} \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We consider two cases, according to the parity of r.



12 13
41 51
13 12
21 31
15 14
31 21

Figure 9: A packing 5-colouring of $H^4(2)$ and its corresponding colouring pattern.

1. *r is even.*
   Since the corona graph $C_{2\ell+4} \odot K_1$ is a subgraph of $H^\ell(r)$ for every $r \geq 2$ (consider the cycle of length $2\ell+4$ induced by the set of vertices $\{u_1^i | 0 \leq i \leq \ell+1\} \cup \{u_2^i | 0 \leq i \leq \ell+1\}$), we get $\chi_\rho(H^\ell(r)) \geq \chi_\rho(C_{2\ell+4} \odot K_1) = 5$ by Theorem 2.3 and Proposition 2.2.

   We now prove $\chi_\rho(H^\ell(r)) \leq 5$. Figure 9 depicts a packing 5-colouring of $H^4(2)$, together with its corresponding colouring pattern. It can easily be checked that this (6×4)-pattern is periodic, that is, can be repeated, both vertically and horizontally, to produce a packing 5-colouring of any generalised H-graph of the form $H^{6i+4}(2j)$, with $i \geq 0$ and $j \geq 1$.

   If $\ell \not\equiv 4 \pmod 6$, we use the colouring patterns depicted in Figure 10, depending on the value of ℓ modulo 6. The upper six rows of each colouring pattern, surrounded by double lines, can be repeated as many times as required, or even deleted when $\ell \equiv 1, 2, 3 \pmod 6$. Therefore, these colouring patterns give us a packing 5-colouring of any generalised H-graph of the form $H^\ell(2)$, for every $\ell \geq 3$, $\ell \neq 5$. It is again easy to check that each of these colouring patterns is "horizontally periodic", that is, can be horizontally repeated in order to get a packing 5-colouring of any generalised H-graph of the form $H^\ell(r)$, for every $\ell \geq 3$, $\ell \neq 5$, $\ell \not\equiv 4 \pmod 6$, and even r.

2. *r is odd.*
   The inequality $\chi_\rho(H^\ell(r)) \geq 6$ directly follows from Corollary 5.6. Therefore, we only need to prove the inequality $\chi_\rho(H^\ell(r)) \leq 6$ (recall that $\ell \geq 3$ and $\ell \neq 5$).

   We first consider a few particular cases. A packing 6-colouring of $H^3(3)$ is depicted in Figure 11(a), and a packing 6-colouring of $H^3(r)$, for every odd $r \geq 5$, is depicted in Figure 11(b) (the first four columns, surrounded by a

$\ell \equiv 0 \pmod 6$	$\ell \equiv 1 \pmod 6$	$\ell \equiv 2 \pmod 6$	$\ell \equiv 3 \pmod 6$	$\ell \equiv 5 \pmod 6$
12 13	12 13	12 13	12 13	12 13
41 51	41 51	41 51	41 51	41 51
13 12	13 12	13 12	13 12	13 12
21 31	21 31	21 31	21 31	21 31
15 14	15 14	15 14	15 14	15 14
31 21	31 21	31 21	31 21	31 21
14 15	12 13	12 13	12 13	14 15
21 31	41 51	41 51	41 51	21 31
	13 12	13 12	13 12	13 12
		51 41	51 41	51 41
		12 13	12 13	12 13
		31 21		41 51
		14 15		13 12
		21 31		
		15 14		
		31 21		

Figure 10: Colouring patterns for $H^\ell(r)$, r even.

double line, are repeated $\frac{r-5}{2}$ times, and thus do not appear if $r = 5$). A packing 6-colouring of $H^4(3)$ is depicted in Figure 12(a), and a packing 6-colouring of $H^4(r)$, for every odd $r \geq 5$, is depicted in Figure 12(b) (the first four columns are repeated $\frac{r-5}{2}$ times). A packing 6-colouring of $H^6(r)$, for every odd $r \geq 3$, is depicted in Figure 12(c) (the four columns surrounded by a double line are repeated $\frac{r-3}{2}$ times, and thus do not appear if $r = 3$). A packing 6-colouring of $H^7(3)$ is depicted in Figure 12(d), and a packing 6-colouring of $H^7(r)$, for every odd $r \geq 5$, is depicted in Figure 12(e) (the four columns surrounded by a double line, are repeated $\frac{r-3}{2}$ times).

In order to produce a packing 6-colouring of $H^\ell(r)$, with $\ell \geq 8$, $r \geq 3$, and r odd, we use the colouring patterns depicted in Figures 13 and 14. In both these figures, the four columns surrounded by double lines must be repeated $\frac{r-3}{2}$ times (and thus do not appear if $r = 3$) or $\frac{r-5}{2}$ times when $\ell = 9$ and $r \geq 5$ (and thus do not appear if $r = 5$). In Figure 14, the six rows surrounded by double lines must be repeated $\frac{\ell-6-(\ell \bmod 6)}{6}$ times (and thus do not appear if $\ell = 8$).

This completes the proof. □

The last two theorems of this section deal with the cases not covered by Theorem 5.7, that is, $\ell = 2$ and $\ell = 5$, respectively.

13	12	16
21	51	41
14	23	12
31	14	31
12	61	25

(a)

12	13	12	15	12	61	23
41	51	41	31	31	14	51
13	12	13	14	14	23	12
51	41	21	21	21	51	41
12	13	15	16	13	12	13

(b)

Figure 11: Colouring patterns for $H^3(3)$ and for $H^3(r)$, $r \geq 5$, r odd.

Theorem 5.8 *For every integer $r \geq 2$,*

$$\chi_\rho(H^2(r)) = \begin{cases} 7 & \textit{if } r \in \{2,4,7,8,11\}, \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. The fact that $H^2(r)$ does not admit a packing 6-colouring for every $r \in \{2,4,7,8,11\}$ has been checked by a computer program, using brute-force search. Packing 7-colourings for each of these graphs are depicted in Figure 15.

Assume now $r \notin \{2,4,7,8,11\}$. We checked by a computer program, again using brute-force search, that the subgraph of such a generalised H-graph induced by three successive ladders, that is, by the set of vertices $\{u_i^j \mid 0 \leq i \leq 5,\ 0 \leq j \leq 3\}$, does not admit a packing 5-colouring. Packing 6-colourings of such generalised H-graphs are depicted in Figure 16, according to the value of r, r modulo 3, or r modulo 6 (periodic patterns, made of 6 or 12 columns, are surrounded by double lines). $\square$

Theorem 5.9 *For every integer $r \geq 2$, $\chi_\rho(H^5(r)) = 6$.*

Proof. Again, we checked by a computer program, using brute-force search, that both $H^5(2)$ and the subgraph of $H^5(r)$, $r \geq 5$, induced by three successive ladders, that is, by the set of vertices $\{u_i^j \mid 0 \leq i \leq 5,\ 0 \leq j \leq 6\}$, do not admit a packing 5-colouring. Packing 6-colourings of $H^5(r)$, $r \in \{2,3,5\}$, are depicted in Figure 17, while packing 6-colourings of $H^5(r)$, $r = 4$ or $r \geq 6$, are depicted in Figure 18 according to the value of r modulo 4, or r modulo 6 (periodic patterns, made of eight or twelve columns, are surrounded by double lines and are repeated at least once when $r \equiv 0 \pmod 4$ or $r \equiv 3 \pmod 6$). $\square$

6 Discussion

In this paper, we have studied the packing chromatic number of some classes of cubic graphs, namely circular ladders, H-graphs and generalised H-graphs. We have determined the exact value of this parameter for every such graph, except for the case of H-graphs $H(r)$ with $r \geq 3$, r odd (see Theorem 4.3), for which we proved $6 \leq \chi_\rho(H(r)) \leq 7$. Using a computer program, we have checked that $\chi_\rho(H(r)) = 7$ for every odd r up to $r = 13$. We thus propose the following question.

$$\begin{array}{l}
13\ 12\ 16 \\
21\ 51\ 31 \\
14\ 14\ 15 \\
31\ 31\ 21 \\
15\ 12\ 14 \\
21\ 61\ 31
\end{array}$$

(a) $H^4(3)$

$$\begin{array}{||l||l}
12\ 13 & 12\ 14\ 13\ 16\ 13 \\
51\ 41 & 51\ 31\ 21\ 21\ 21 \\
13\ 12 & 13\ 12\ 15\ 13\ 14 \\
21\ 31 & 21\ 51\ 31\ 51\ 31 \\
14\ 15 & 14\ 13\ 12\ 14\ 15 \\
31\ 21 & 31\ 21\ 61\ 31\ 21
\end{array}$$

(b) $H^4(r)$, $r \geq 5$, r odd

$$\begin{array}{l||l||l}
12\ 13 & 12\ 13 & 16 \\
41\ 51 & 41\ 51 & 21 \\
13\ 12 & 13\ 12 & 13 \\
21\ 31 & 21\ 31 & 51 \\
15\ 14 & 15\ 14 & 12 \\
31\ 21 & 31\ 21 & 31 \\
12\ 15 & 14\ 15 & 14 \\
61\ 31 & 21\ 31 & 21
\end{array}$$

(c) $H^6(r)$, $r \geq 3$, r odd

$$\begin{array}{l}
12\ 13\ 14 \\
31\ 51\ 21 \\
16\ 12\ 15 \\
21\ 31\ 31 \\
14\ 14\ 16 \\
31\ 21\ 21 \\
15\ 13\ 13 \\
21\ 61\ 51 \\
13\ 12\ 14
\end{array}$$

(d) $H^7(3)$

$$\begin{array}{||l||l}
12\ 13 & 15\ 12\ 16 \\
51\ 41 & 21\ 31\ 31 \\
13\ 12 & 13\ 14\ 12 \\
21\ 31 & 61\ 21\ 51 \\
14\ 15 & 14\ 15\ 13 \\
31\ 21 & 31\ 31\ 21 \\
12\ 13 & 12\ 12\ 14 \\
51\ 41 & 51\ 41\ 31 \\
13\ 12 & 13\ 16\ 12
\end{array}$$

(e) $H^7(r)$, $r \geq 5$, r odd

Figure 12: Colouring patterns for $H^4(r)$, $H^6(r)$ and $H^7(r)$, $r \geq 3$, r odd.

Question 1 *Is it true that $\chi_\rho(H(r)) = 7$ for every H-graph $H(r)$ with $r \geq 3$, r odd?*

In [16, 17], we have extended the notion of packing colouring to the case of digraphs. If D is a digraph, the (weak) directed distance between two vertices u and v in D is defined as the length of a shortest directed path between u and v, in either direction. Using this notion of distance in digraphs, the packing colouring readily extends to digraphs. Recall that an orientation of an undirected graph G is any antisymmetric digraph obtained from G by giving to each edge of G one of its two possible orientations. It then directly follows from the definition that $\chi_\rho(D) \leq \chi_\rho(G)$ for any orientation D of G. A natural question for oriented graphs, related to this work, is then the following.

Question 2 *Is it true that the packing chromatic number of any oriented graph with maximum degree 3 is bounded by some constant?*

$$\begin{array}{l}
13\ 12\ 16\\
21\ 51\ 41\\
14\ 13\ 12\\
31\ 21\ 31\\
12\ 14\ 15\\
51\ 31\ 21\\
13\ 12\ 13\\
21\ 51\ 41\\
14\ 23\ 12\\
31\ 14\ 31\\
12\ 61\ 25
\end{array}$$

$\ell = 9,\ r = 3$

$$\begin{array}{||l||l}
12\ 13 & 12\ 15\ 12\ 61\ 23\\
41\ 51 & 41\ 31\ 31\ 14\ 51\\
13\ 12 & 13\ 14\ 14\ 23\ 12\\
21\ 31 & 21\ 21\ 21\ 51\ 31\\
15\ 14 & 15\ 13\ 13\ 12\ 14\\
31\ 21 & 31\ 51\ 51\ 31\ 21\\
12\ 13 & 12\ 12\ 12\ 14\ 13\\
41\ 51 & 41\ 31\ 31\ 21\ 51\\
13\ 12 & 13\ 14\ 14\ 13\ 12\\
51\ 41 & 21\ 21\ 21\ 51\ 41\\
12\ 13 & 15\ 16\ 13\ 12\ 13
\end{array}$$

$\ell = 9,\ r \geq 5$

$$\begin{array}{l||l||l}
12\ 13 & 12\ 13 & 16\\
41\ 51 & 41\ 51 & 21\\
13\ 12 & 13\ 12 & 13\\
21\ 31 & 21\ 31 & 41\\
15\ 14 & 15\ 14 & 12\\
31\ 21 & 31\ 21 & 31\\
12\ 13 & 12\ 13 & 15\\
41\ 51 & 41\ 51 & 21\\
13\ 12 & 13\ 12 & 14\\
21\ 31 & 21\ 31 & 31\\
15\ 14 & 15\ 14 & 12\\
31\ 21 & 31\ 21 & 61
\end{array}$$

$\ell = 10,\ r \geq 3$

$$\begin{array}{l}
13\ 12\ 16\\
41\ 51\ 31\\
12\ 13\ 12\\
51\ 41\ 51\\
13\ 12\ 13\\
21\ 31\ 21\\
14\ 15\ 14\\
31\ 21\ 61\\
15\ 14\ 12\\
21\ 31\ 51\\
13\ 12\ 13\\
61\ 51\ 21\\
12\ 13\ 14
\end{array}$$

$\ell = 11,\ r = 3$

$$\begin{array}{l||l||l}
13\ 12 & 13\ 12 & 16\\
41\ 51 & 41\ 51 & 31\\
12\ 13 & 12\ 13 & 12\\
51\ 41 & 51\ 41 & 51\\
13\ 12 & 13\ 12 & 13\\
21\ 31 & 21\ 31 & 21\\
14\ 15 & 14\ 15 & 14\\
31\ 21 & 31\ 21 & 61\\
15\ 14 & 15\ 14 & 12\\
21\ 31 & 21\ 31 & 51\\
13\ 12 & 13\ 12 & 13\\
61\ 51 & 41\ 51 & 21\\
12\ 13 & 12\ 13 & 14
\end{array}$$

$\ell = 11,\ r \geq 5$

Figure 13: Colouring patterns for $H^\ell(r)$, $9 \leq \ell \leq 11$, $r \geq 3$, r odd.

A Proof of Lemma 3.1

The configurations used in the proof correspond to partial colourings of the graph X and are depicted in Figures 19 and 20, with the following drawing convention. If $\{a, b\}$ is the set of colours assigned to two distinct vertices, then the "colour" of both these vertices is denoted "a, b". If the same configuration describes two partial colourings of X and the colours assigned to some vertex by these two colourings are respectively a and b, then the "colour" of this vertex is denoted "$a|b$". Finally, if a vertex has no available colour, its "colour" is denoted "?".

Suppose that for some i, $3 \leq i \leq 5$, $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$. We first prove the

12 13	12 13	16
41 51	41 51	21
13 12	13 12	13
21 31	21 31	41
15 14	15 14	12
31 21	31 21	31
12 13	12 13	15
41 51	41 51	21
13 12	13 12	13
21 41	51 41	41
15 13	12 13	12
31 21	31 21	61
12 15	14 15	13
41 31	21 31	21

$\ell \equiv 0 \pmod 6$

$\ell \geq 12$

12 13	12 13	16
41 51	41 51	21
13 12	13 12	13
21 31	21 31	41
15 14	15 14	12
31 21	31 21	31
12 13	12 13	15
41 51	41 51	21
13 12	13 12	16
21 31	21 31	41
15 14	15 14	13
31 21	31 21	21
12 13	12 13	15
61 51	41 51	31
13 12	13 12	14

$\ell \equiv 1 \pmod 6$

$\ell \geq 13$

12 13	12 13	16
41 51	41 51	21
13 12	13 12	13
21 31	21 31	41
15 14	15 14	12
31 21	31 21	31
12 13	12 13	15
41 51	41 51	21
13 12	13 12	16
51 41	51 41	31
12 13	12 13	14
31 21	31 21	21
14 15	14 15	13
21 31	21 31	51
15 14	15 14	12
31 21	31 21	61

$\ell \equiv 2 \pmod 6$

$\ell \geq 8$

12 13	12 13	16
41 51	41 51	21
13 12	13 12	13
21 31	21 31	41
15 14	15 14	12
31 21	31 21	31
12 13	12 13	15
41 51	41 51	61
13 12	13 12	12
51 41	51 41	41
12 13	12 13	13
31 21	31 21	21
14 15	14 15	15
21 31	21 31	31
13 12	13 12	14
51 41	51 41	21
12 13	12 13	16

$\ell \equiv 3 \pmod 6$

$\ell \geq 15$

12 13	12 13	16
41 51	41 51	21
13 12	13 12	15
21 31	21 31	31
15 14	15 14	12
31 21	31 21	41
12 13	12 13	13
41 51	41 51	21
13 12	13 12	15
21 31	21 31	61
15 14	15 14	12
31 21	31 21	31
12 13	12 13	14
41 51	41 51	21
13 12	13 12	15
21 31	21 31	31
15 14	15 14	12
31 21	31 21	61

$\ell \equiv 4 \pmod 6$

$\ell \geq 16$

13 12	13 12	16
41 51	41 51	31
12 13	12 13	12
31 21	31 21	41
15 14	15 14	13
21 31	21 31	21
13 12	13 12	15
41 51	41 51	31
12 13	12 13	12
51 41	51 41	61
13 12	13 12	14
21 31	21 31	21
14 15	14 15	13
31 21	31 21	51
15 14	15 14	12
21 31	21 31	41
13 12	13 12	13
41 51	41 51	21
12 13	12 13	16

$\ell \equiv 5 \pmod 6$

$\ell \geq 17$

Figure 14: Colouring patterns for $H^\ell(r)$, $\ell = 8$ or $\ell \geq 12$, $r \geq 3$, r odd.

13 16 21 21 14 17 31 51	13 16 12 15 21 21 31 31 14 13 14 12 31 51 21 71	13 16 14 12 17 14 15 21 21 21 51 31 21 31 14 13 13 13 12 13 12 31 51 71 21 41 51 61
$r = 2$	$r = 4$	$r = 7$

13 16 13 12 17 13 12 15 21 21 21 54 31 21 41 31 14 13 14 21 14 15 13 12 31 51 71 36 21 31 21 61	13 16 13 12 14 13 16 14 12 17 15 21 21 21 51 31 21 21 21 31 31 31 14 13 14 13 12 15 13 13 15 12 12 31 51 71 21 61 31 41 71 21 41 61
$r = 8$	$r = 11$

Figure 15: Packing 7-colourings of $H^2(r)$, $r \in \{2, 4, 7, 8, 11\}$.

following claim.

Claim 1 $2 \in \{\pi(u_i), \pi(v_i)\}$.

Proof. Assume to the contrary that this is not the case, that is, $\{\pi(u_i), \pi(v_i)\} \subseteq \{3, 4, 5\}$. Thanks to the symmetry exchanging u_i and v_i, we may assume $\pi(u_i) < \pi(v_i)$, without loss of generality. Recall that there is no edge $u_{i-2}v_{i-2}$ (respectively, $u_{i+2}v_{i+2}$) in X when $i = 3$ (respectively, $i = 5$). We consider the following cases (subscripts are taken modulo n).

1. $\pi(u_i) = 3$ and $\pi(v_i) = 4$.
   In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 2, 5\}$.

   If $\pi(u_{i+1}) = 1$, then $\{\pi(v_{i+1}), \pi(u_{i+2})\} = \{2, 5\}$. If $\pi(v_{i+1}) = 2$ (and $\pi(u_{i+2}) = 5$), then $\pi(v_{i-1}) = 1$, so that $\pi(u_{i-1}) = 2$ and no colour is available for v_{i-2} (see Figure 19(a)). If $\pi(u_{i+2}) = 2$ (and $\pi(v_{i+1}) = 5$), then $\{\pi(u_{i-1}), \pi(v_{i-1})\} = \{1, 2\}$, and no colour is available either for u_{i-2} or for v_{i-2} (see Figure 19(b)).

   If $\pi(u_{i+1}) = 2$, then $\pi(v_{i+1}) \in \{1, 5\}$. If $\pi(v_{i+1}) = 5$, then $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 2$ and no colour is available for u_{i-2} (see Figure 19(c)). If $\pi(v_{i+1}) = 1$, then either $\pi(u_{i-1}) = 5$, so that no colour is available for v_{i+2} (see Figure 19(d)), or $\pi(u_{i-1}) = 1$, which implies $\{\pi(u_{i-2}), \pi(v_{i-1})\} = \{2, 5\}$, so that again no colour is available for v_{i+2} (see Figure 19(e)).

   Finally, if $\pi(u_{i+1}) = 5$, then $\{\pi(u_{i-1}), \pi(v_{i-1})\} = \{1, 2\}$, and no colour is available either for u_{i-2} or for v_{i-2} (see Figure 19(f)).

2. $\pi(u_i) = 3$ and $\pi(v_i) = 5$.
   Observe that the proof is similar to the proof of the previous case, by switching colours 4 and 5, in all cases illustrated in Figure 19(b), (c), (d) and (f).

$$\begin{array}{l}
13\ 12\ 13\ 12\ 16\\
21\ 51\ 41\ 51\ 31\\
14\ 13\ 12\ 14\ 15\\
31\ 21\ 61\ 31\ 21
\end{array}
\qquad
\begin{array}{||l||}
13\ 12\ 15\\
21\ 41\ 31\\
16\ 13\ 14\\
31\ 51\ 21
\end{array}
\qquad
\begin{array}{||l||l}
13\ 12\ 15 & 13\ 14\ 12\ 13\ 14\ 12\ 15\\
21\ 41\ 31 & 61\ 21\ 31\ 51\ 21\ 61\ 31\\
16\ 13\ 14 & 12\ 13\ 16\ 12\ 15\ 13\ 12\\
31\ 51\ 21 & 31\ 51\ 21\ 41\ 31\ 21\ 41
\end{array}$$

$r = 5$ $\qquad$ $r \equiv 0 \pmod 3$ $\qquad$ $r \equiv 1 \pmod 3,\ r \geq 10$

$$\begin{array}{l}
13\ 12\ 14\ 13\ 16\ 12\ 13\ 12\ 13\ 16\ 12\ 13\ 12\ 16\\
21\ 51\ 31\ 21\ 21\ 31\ 41\ 51\ 41\ 21\ 31\ 41\ 51\ 31\\
14\ 13\ 12\ 15\ 13\ 15\ 12\ 13\ 15\ 14\ 15\ 12\ 23\ 15\\
31\ 21\ 61\ 31\ 41\ 21\ 31\ 61\ 21\ 31\ 21\ 36\ 14\ 21
\end{array}$$

$r = 14$

$$\begin{array}{l}
13\ 16\ 12\ 13\ 14\ 12\ 16\ 13\ 12\ 15\ 13\ 16\ 12\ 13\ 14\ 12\ 16\ 13\ 12\ 15\\
21\ 21\ 31\ 51\ 21\ 31\ 31\ 21\ 41\ 31\ 21\ 21\ 31\ 51\ 21\ 31\ 31\ 21\ 41\ 31\\
14\ 13\ 14\ 12\ 13\ 15\ 12\ 15\ 13\ 12\ 14\ 13\ 14\ 12\ 13\ 15\ 12\ 15\ 13\ 12\\
31\ 51\ 21\ 31\ 61\ 21\ 41\ 31\ 21\ 61\ 31\ 51\ 21\ 31\ 61\ 21\ 41\ 31\ 21\ 61
\end{array}$$

$r = 20$

$$\begin{array}{l||l||}
13\ 12\ 14\ 13\ 12\ 41\ 63\ 12\ 14\ 13\ 16\ 12\ 13\ 12\ 13\ 16\ 12\ 13\ 12\ 14 & 13\ 12\ 14\ 13\ 12\ 14\\
21\ 51\ 31\ 21\ 51\ 32\ 21\ 51\ 31\ 21\ 21\ 31\ 41\ 51\ 41\ 21\ 31\ 41\ 51\ 31 & 21\ 51\ 31\ 61\ 51\ 31\\
16\ 13\ 15\ 16\ 13\ 15\ 14\ 13\ 12\ 15\ 13\ 15\ 12\ 13\ 15\ 14\ 15\ 12\ 23\ 15 & 16\ 13\ 15\ 12\ 13\ 15\\
31\ 41\ 21\ 31\ 41\ 21\ 31\ 21\ 61\ 31\ 41\ 21\ 31\ 61\ 21\ 31\ 21\ 36\ 14\ 21 & 31\ 41\ 21\ 31\ 41\ 21
\end{array}$$

$r \equiv 2 \pmod 6,\ r \geq 26$

$$\begin{array}{l}
13\ 12\ 14\ 13\ 16\ 12\ 13\ 12\ 14\ 13\ 12\ 41\ 63\ 12\ 13\ 12\ 16\\
21\ 51\ 31\ 21\ 21\ 31\ 41\ 51\ 31\ 21\ 51\ 32\ 21\ 51\ 41\ 51\ 31\\
14\ 13\ 12\ 15\ 13\ 15\ 12\ 23\ 15\ 16\ 13\ 15\ 14\ 13\ 12\ 14\ 15\\
31\ 21\ 61\ 31\ 41\ 21\ 36\ 14\ 21\ 31\ 41\ 21\ 31\ 21\ 61\ 31\ 21
\end{array}$$

$r = 17$

$$\begin{array}{l||l||}
13\ 12\ 41\ 63\ 12\ 14\ 13\ 16\ 12\ 13\ 12\ 13\ 16\ 12\ 13\ 12\ 14 & 13\ 12\ 14\ 13\ 12\ 14\\
21\ 51\ 32\ 21\ 51\ 31\ 21\ 21\ 31\ 41\ 51\ 41\ 21\ 31\ 41\ 51\ 31 & 21\ 51\ 31\ 61\ 51\ 31\\
16\ 13\ 15\ 14\ 13\ 12\ 15\ 13\ 15\ 12\ 13\ 15\ 14\ 15\ 12\ 23\ 15 & 16\ 13\ 15\ 12\ 13\ 15\\
31\ 41\ 21\ 31\ 21\ 61\ 31\ 41\ 21\ 31\ 61\ 21\ 31\ 21\ 36\ 14\ 21 & 31\ 41\ 21\ 31\ 41\ 21
\end{array}$$

$r \equiv 5 \pmod 6,\ r \geq 23$

Figure 16: Colouring patterns for $H^2(r)$, $r \notin \{2, 4, 7, 8, 11\}$.

$r=2$	$r=3$	$r=5$
13 12	13 16 15	13 12 41 25 14
41 61	21 21 21	21 51 16 31 21
15 13	14 13 14	16 13 23 12 13
31 21	31 51 31	31 21 51 41 61
12 14	15 12 12	14 16 12 13 12
61 51	21 31 51	21 31 41 51 41
13 12	16 14 13	15 12 13 12 13

Figure 17: Packing 6-colourings of $H^5(r)$, $r \in \{2,3,5\}$.

‖		‖
‖	13 16 13 15	‖
‖	21 21 41 21	‖
‖	14 13 12 14	‖
‖	31 41 51 31	‖
‖	15 15 13 12	‖
‖	21 21 21 51	‖
‖	16 13 14 13	‖

$r \equiv 0 \pmod 4$, $r \geq 4$

‖		‖	
‖	13 16 13 15	‖	13 16 13 14 13 15
‖	21 21 41 21	‖	21 21 51 21 21 21
‖	14 13 12 14	‖	14 13 12 15 16 14
‖	31 41 51 31	‖	31 51 31 31 51 31
‖	15 15 13 12	‖	15 14 14 42 13 12
‖	21 21 21 51	‖	21 21 21 16 21 51
‖	16 13 14 13	‖	16 13 15 23 14 13

$r \equiv 2 \pmod 4$, $r \geq 6$

‖		‖	
‖	21 51 21 31 51 31	‖	21 51 21 61 21 52 16
‖	14 13 16 14 12 16	‖	13 13 14 13 13 13 43
‖	31 21 31 21 31 21	‖	41 41 31 21 41 41 21
‖	12 14 15 13 14 15	‖	12 12 15 14 12 12 15
‖	51 31 21 51 21 31	‖	51 31 21 51 51 31 31
‖	16 15 14 16 15 14	‖	16 15 13 13 13 15 12
‖	31 21 31 21 31 21	‖	31 21 41 21 61 21 41

$r \equiv 1 \pmod 6$, $r \geq 7$

‖		‖	
‖	21 51 21 31 51 31	‖	21 52 16
‖	14 13 16 14 12 16	‖	13 13 43
‖	31 21 31 21 31 21	‖	41 41 21
‖	12 14 15 13 14 15	‖	12 12 15
‖	51 31 21 51 21 31	‖	51 31 31
‖	16 15 14 16 15 14	‖	16 15 12
‖	31 21 31 21 31 21	‖	31 21 41

$r \equiv 3 \pmod 6$, $r \geq 9$

‖		‖	
‖	21 51 21 31 51 31	‖	21 51 21 31 51 31 41 31 21 41 31
‖	14 13 16 14 12 16	‖	14 13 16 14 12 12 12 16 15 12 16
‖	31 21 31 21 31 21	‖	31 21 31 21 31 61 31 51 31 31 21
‖	12 14 15 13 14 15	‖	12 14 15 13 14 15 15 12 12 15 15
‖	51 31 21 51 21 31	‖	51 31 21 51 21 31 21 41 41 61 31
‖	16 15 14 16 15 14	‖	16 15 14 16 15 14 16 13 13 13 12
‖	31 21 31 21 31 21	‖	31 21 31 21 31 21 31 21 51 21 41

$r \equiv 5 \pmod 6$, $r \geq 11$

Figure 18: Colouring patterns for $H^5(r)$, $r = 4$ or $r \geq 6$.

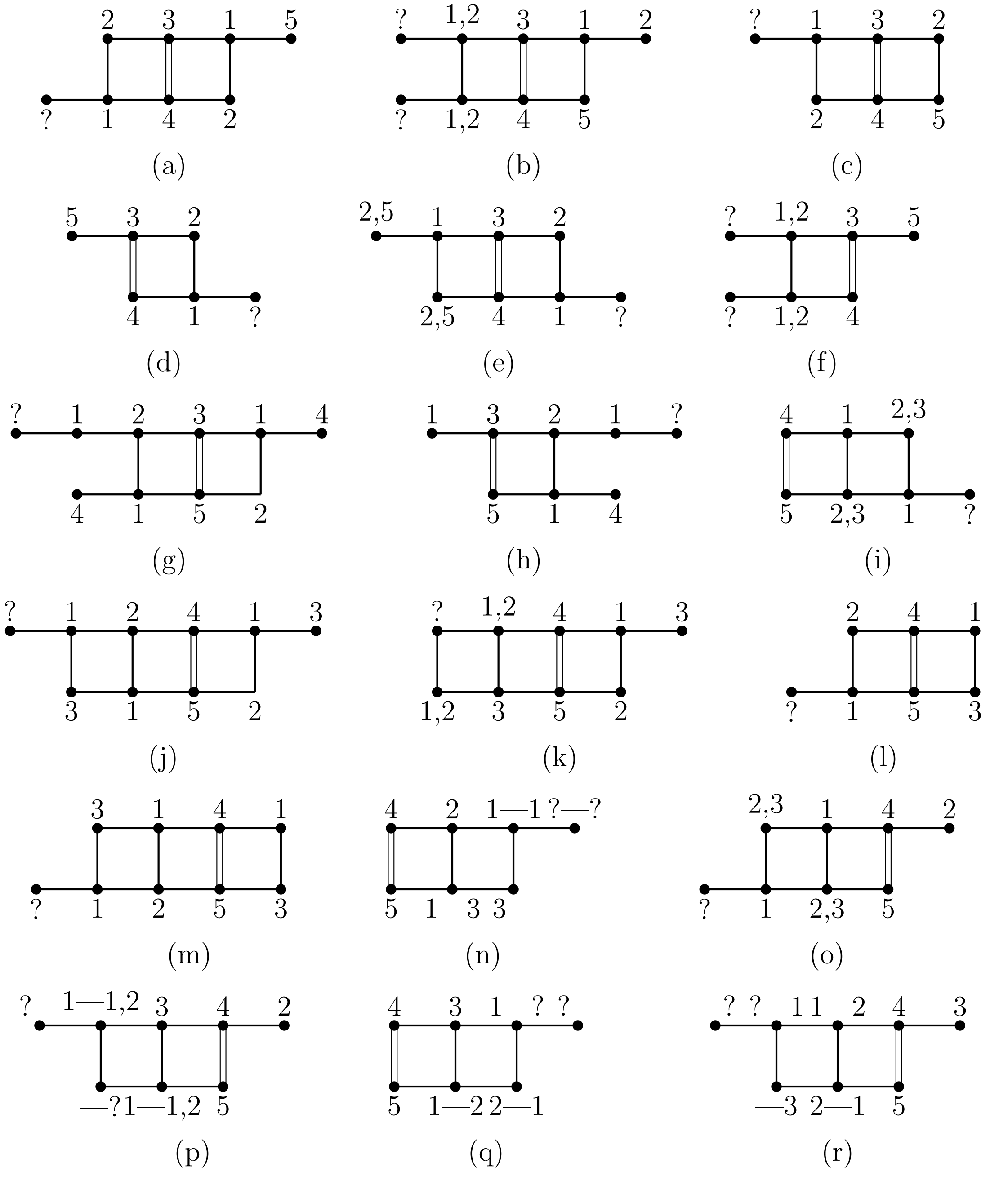


Figure 19: Configurations for the proof of Lemma 3.1 (the double edge is the edge u_iv_i).

Therefore, only two cases remain to be considered, which were illustrated in Figure 19(a) and (e), respectively.

(a) $\pi(u_{i+1}) = 1$ and $\pi(u_{i+2}) = 4$.
In that case, we have $\pi(v_{i+1}) = 2$, and thus $\pi(v_{i-1}) = 1$, which implies $\pi(v_{i-2}) = 4$ and thus $\pi(u_{i-1}) = 2$, so that $\pi(u_{i-2}) = 1$, $\pi(v_{i-2}) = 4$, and no colour is available for u_{i-3} (see Figure 19(g)).

(b) $\pi(u_{i+1}) = 2$, $\pi(u_{i-1}) = 1$ and $\pi(v_{i+1}) = 1$.
In that case, we necessarily have $\pi(v_{i+2}) = 4$, so that $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3} (see Figure 19(h)).

3. $\pi(u_i) = 4$ and $\pi(v_i) = 5$.
In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 2, 3\}$. We consider six subcases, depending on the value of $\pi(u_{i+1})$ and i.

(a) $\pi(u_{i+1}) = 1$ and $i \in \{3, 4\}$.
In that case, we have $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{2, 3\}$, which implies $\pi(v_{i+2}) = 1$, and no colour is available for v_{i+3} (see Figure 19(i)).

(b) $\pi(u_{i+1}) = 1$ and $i = 5$.
In that case, we have $\pi(v_6) \in \{2, 3\}$. If $\pi(v_6) = 2$, then we necessarily have $\pi(u_7) = 3$, and thus $\pi(v_4) \in \{1, 3\}$. If $\pi(v_4) = 1$, we get successively $\pi(u_4) = 2$, $\pi(v_3) = 3$, $\pi(u_3) = 1$, and no colour is available for u_2 (see Figure 19(j)). If $\pi(v_4) = 3$, then $\{\pi(u_4), \pi(v_3)\} = \{1, 2\}$ and no colour is available for u_3 (see Figure 19(k)).

If $\pi(v_6) = 3$, then $\{\pi(u_4), \pi(v_4)\} = \{1, 2\}$. If $\pi(v_4) = 1$ and $\pi(u_4) = 2$, then no colour is available for v_3 (see Figure 19(l)). If $\pi(u_4) = 1$ and $\pi(v_4) = 2$, then we necessarily have $\pi(v_3) = 1$ and $\pi(u_3) = 3$, and no colour is available for v_2 (see Figure 19(m)).

(c) $\pi(u_{i+1}) = 2$ and $i \in \{3, 4\}$.
In that case, we necessarily have $\pi(v_{i+1}) \in \{1, 3\}$. If $\pi(v_{i+1}) = 1$, then $\pi(v_{i+2}) = 3$, which implies $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3}. If $\pi(v_{i+1}) = 3$, then $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3} (see Figure 19(n)).

(d) $\pi(u_{i+1}) = 2$ and $i = 5$.
In that case, we necessarily have $\pi(u_4) \in \{1, 3\}$. If $\pi(u_4) = 1$, then $\{\pi(u_3), \pi(v_4)\} = \{2, 3\}$, so that $\pi(v_3) = 1$, and no colour is available for v_2 (see Figure 19(o)). If $\pi(u_4) = 3$, then either $\pi(u_3) = \pi(v_4) = 1$, which implies $\pi(v_3) = 2$ and no colour is available for u_2, or $\{\pi(u_3), \pi(v_4)\} = \{1, 2\}$, and no colour is available for v_3 (see Figure 19(p)).

(e) $\pi(u_{i+1}) = 3$ and $i \in \{3, 4\}$.
In that case, either $\pi(v_{i+1}) = 1$, so that $\pi(v_{i+2}) = 2$, $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3}, or $\pi(v_{i+1}) = 2$, so that $\pi(v_{i+2}) = 1$ and no colour is available for u_{i+2} (see Figure 19(q)).

(f) $\pi(u_{i+1}) = 3$ and $i = 5$.
In that case, $\pi(u_4) \in \{1, 2\}$. If $\pi(u_4) = 1$, then $\pi(v_4) = 2$ and no colour is available for u_3. If $\pi(u_4) = 2$, then $\pi(v_4) = 1$, so that $\pi(u_3) = 1$ and $\pi(v_3) = 3$, and no colour is available for u_2 (see Figure 19(r)).

This completes the proof of Claim 1. □

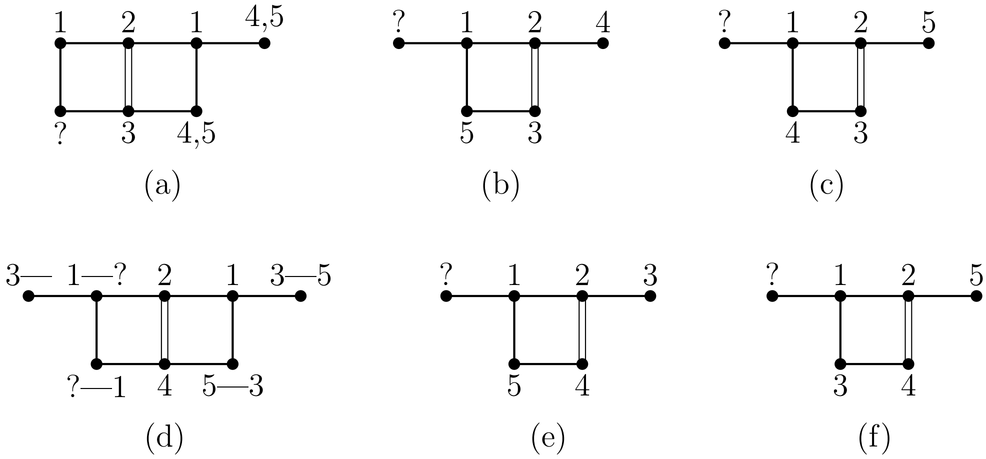


Figure 20: Configurations for the proof of Lemma 3.1 (cont.).

By Claim 1, we can thus assume $\pi(u_i) = 2$, without loss of generality (again, thanks to the symmetry exchanging u_i and v_i), so that $\pi(v_i) \in \{3, 4, 5\}$. To finish the proof of Lemma 3.1, we need to prove that $\{\pi(u_{i-1}), \pi(u_{i+1})\} = \{3, 4, 5\} \setminus \{\pi(v_i)\}$. Suppose that this is not the case. We consider the following cases, according to the value of $\pi(v_i)$.

1. $\pi(v_i) = 3$.
   In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 4, 5\}$.

   If $\pi(u_{i+1}) = 1$, then $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{4, 5\}$, so that $\pi(u_{i-1}) = 1$, and no colour is available for v_{i-1} (see Figure 20(a)).

   If $\pi(u_{i+1}) = 4$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 5$, and no colour is available for u_{i-2} (see Figure 20(b)), or $\pi(u_{i-1}) = 5$, which contradicts our assumption since it would imply $\{\pi(u_{i-1}), \pi(u_{i+1})\} = \{3, 4, 5\} \setminus \{\pi(v_i)\}$.

   Similarly, if $\pi(u_{i+1}) = 5$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 4$, and no colour is available for u_{i-2} (see Figure 20(c)), or $\pi(u_{i-1}) = 4$, which again contradicts our assumption.

2. $\pi(v_i) = 4$ (the case $\pi(v_i) = 5$ is similar, by switching colours 4 and 5).
   In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 3, 5\}$.

   If $\pi(u_{i+1}) = 1$, then $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{3, 5\}$. If $\pi(u_{i+2}) = 3$ and $\pi(v_{i+1}) = 5$, then $\pi(u_{i-1}) = 1$, so that $\pi(u_{i-2}) = 3$, and no colour is available for v_{i-1}. If

$\pi(u_{i+2}) = 5$ and $\pi(v_{i+1}) = 3$, then $\pi(v_{i-1}) = 1$, and no colour is available for u_{i-1} (see Figure 20(d)).

If $\pi(u_{i+1}) = 3$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 5$, and no colour is available for u_{i-2}, or $\pi(u_{i-1}) = 5$, which contradicts our assumption (see Figure 20(e)).

Finally, if $\pi(u_{i+1}) = 5$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 3$, and no colour is available for u_{i-2}, or $\pi(u_{i-1}) = 3$, which contradicts our assumption (see Figure 20(f)).

This completes the proof of Lemma 3.1.

B Proof of Lemma 5.3

We first prove the following claim.

Claim 2 *For every integer j, $0 \leq j < r$, either $\pi(u^0_{2j}) = 1$ or $\pi(u^0_{2j+1}) = 1$.*

Proof. Thanks to the symmetries of $H^\ell(r)$, it is enough to prove the claim for the edge $u^0_2 u^0_3$. Suppose to the contrary that $\pi(u^0_2) \neq 1$ and $\pi(u^0_3) \neq 1$. Thanks to the symmetries of $H^\ell(r)$, we can assume $\pi(u^0_2) < \pi(u^0_3)$, without loss of generality.

We consider four cases. The corresponding configurations are depicted in Figure 21, using the same drawing convention as for the proof of Lemma 3.1 (see Appendix A).

1. $\pi(u^0_2) = 2$ and $\pi(u^0_3) = 3$.
   In that case, $\pi(u^1_2) \in \{1, 4, 5\}$. If $\pi(u^1_2) = 1$, then $\{\pi(u^2_2), \pi(u^1_3)\} = \{4, 5\}$, which implies $\pi(u^0_1) = 1$, and no colour is available for u^0_0 (see Figure 21(a)). If $\pi(u^1_2) = 4$, then either $\pi(u^0_1) = 1$, which implies $\pi(u^0_0) = 5$, and no colour is available for u^1_1, or $\pi(u^0_1) = 5$, which implies $\pi(u^0_4) = 1$, $\pi(u^0_5) = 2$, and no colour is available for u^1_4 (see Figure 21(b)). The case $\pi(u^1_2) = 5$ is similar, by switching colours 4 and 5.

2. $\pi(u^0_2) = 2$ and $\pi(u^0_3) = 4$ (the case $\pi(u^0_2) = 2$ and $\pi(u^0_3) = 5$ is similar, by switching colours 4 and 5).
   In that case, $\pi(u^1_2) \in \{1, 3, 5\}$. If $\pi(u^1_2) = 1$, then $\{\pi(u^2_2), \pi(u^1_3)\} = \{3, 5\}$, which implies $\pi(u^0_1) = 1$, $\pi(u^0_0) = 3$, and no colour is available for u^1_1 (see Figure 21(c)). If $\pi(u^1_2) = 3$, then either $\pi(u^0_1) = 1$, which implies $\pi(u^0_0) = 5$, and no colour is available for u^1_1, or $\pi(u^0_1) = 5$, which implies $\pi(u^2_2) = \pi(u^1_3) = 1$, so that $\pi(u^2_3) = 2$, and no colour is available for u^3_2 (see Figure 21(d)). Finally, if $\pi(u^1_2) = 5$, then either $\pi(u^0_1) = 1$, which implies $\pi(u^0_0) = 3$, and no colour is available for u^1_1, or $\pi(u^0_1) = 3$, which implies $\pi(u^1_1) = 1$, $\pi(u^2_1) = 2$, and no colour is available for u^1_0 (see Figure 21(e)).

3. $\pi(u^0_2) = 3$ and $\pi(u^0_3) = 4$.
   In that case, $\pi(u^1_2) \in \{1, 2, 5\}$. If $\pi(u^1_2) = 1$, then $\{\pi(u^2_2), \pi(u^1_3)\} = \{2, 5\}$,

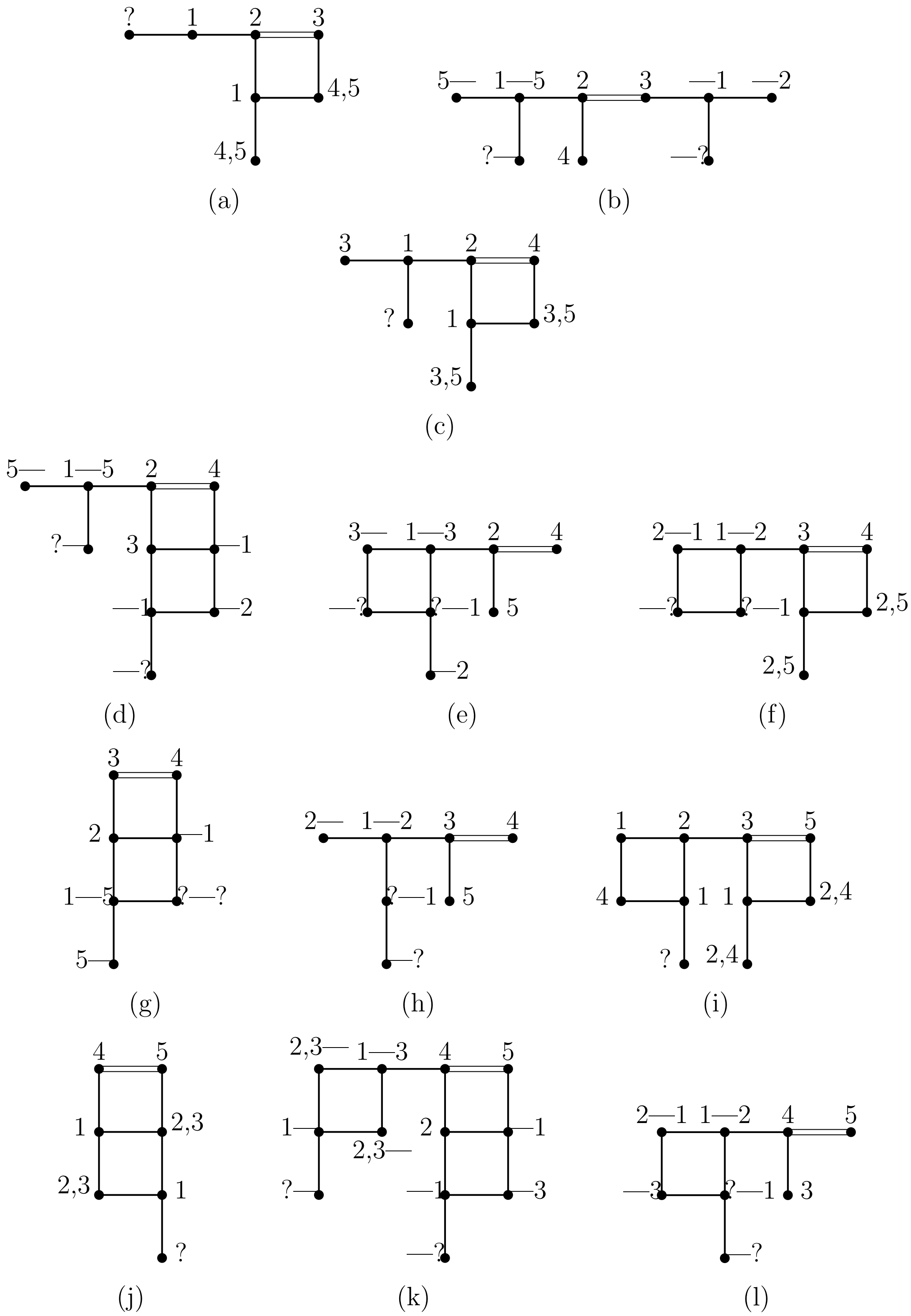


Figure 21: Configurations for the proof of Claim 2 (the double edge is the edge $u_2^0u_3^0$).

and thus either $\pi(u_1^0) = 1$, so that $\pi(u_0^0) = 2$, and no colour is available for u_1^1, or $\pi(u_1^0) = 2$, so that $\pi(u_0^0) = 1$, and no colour is available for u_0^1 (see Figure 21(f)). If $\pi(u_2^1) = 2$, then either $\pi(u_2^2) = 1$, which implies $\pi(u_2^3) = 5$, and no colour is available for u_3^2, or $\pi(u_2^2) = 5$, which implies $\pi(u_3^1) = 1$, and no colour is available for u_3^2 (see Figure 21(g)). Finally, if $\pi(u_2^1) = 5$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 2$, and no colour is available for u_1^1, or $\pi(u_1^0) = 2$, which implies $\pi(u_1^1) = 1$, and no colour is available for u_1^2 (see Figure 21(h)).

4. $\pi(u_2^0) = 3$ and $\pi(u_3^0) = 5$.
   This case is similar to the previous one, by switching colours 4 and 5, except when $\pi(u_2^1) = 1$ (which implies $\{\pi(u_2^2), \pi(u_3^1)\} = \{2, 4\}$) and $\pi(u_1^0) = 2$. In that case, we necessarily have $\pi(u_0^0) = \pi(u_1^1) = 1$, which implies $\pi(u_0^1) = 4$, and no colour is available for u_1^2 (see Figure 21(i)).

5. $\pi(u_2^0) = 4$ and $\pi(u_3^0) = 5$.
   In that case, $\pi(u_2^1) \in \{1, 2, 3\}$. If $\pi(u_2^1) = 1$, then $\{\pi(u_2^2), \pi(u_3^1)\} = \{2, 3\}$, which implies $\pi(u_3^2) = 1$, and no colour is available for u_3^3 (see Figure 21(j)). If $\pi(u_2^1) = 2$, then either $\pi(u_1^0) = 1$, which implies $\{\pi(u_0^0), \pi(u_1^1)\} = \{2, 3\}$, so that $\pi(u_0^1) = 1$, and no colour is available for u_0^2, or $\pi(u_1^0) = 3$, which implies $\pi(u_2^2) = \pi(u_3^1) = 1$, so that $\pi(u_3^2) = 3$, and no colour is available for u_2^3 (see Figure 21(k)). Finally, if $\pi(u_2^1) = 3$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 2$, and no colour is available for u_1^1, or $\pi(u_1^0) = 2$, which implies $\pi(u_0^0) = \pi(u_1^1) = 1$, so that $\pi(u_0^1) = 3$, and no colour is available for u_1^2 (see Figure 21(l)).

This completes the proof of Claim 2. □

Since the cycle induced by the set of vertices $\{u_0^0, u_1^0, \dots, u_{2r-1}^0\}$ has even length, and adjacent vertices cannot be assigned the same colour, it follows from Claim 2 that colour 1 must be used on each edge $u_j^0 u_{j+1}^0$, $0 \le j \le 2r - 1$ (subscripts are taken modulo $2r$). By symmetry, colour 1 must also be used on each edge $u_j^{\ell+1} u_{j+1}^{\ell+1}$, $0 \le j \le 2r - 1$. This concludes the proof of Lemma 5.3.

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