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Existence of t strongly symmetric pairwise orthogonal diagonal Latin squares for $t \in \{3, 4, 5\}$

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Abstract

A Latin square is said to possess the diagonal property if both its main and back diagonals are transversals. A Latin square, L of order v with entries and row and column indices from the set $\{0, 1, 2, \ldots, v-1\}$ is called strongly symmetic if $L(v-1-i, v-1-j) = v-1-L(i, j)$ for all i, j. In this paper we investigate existence of t pairwise orthogonal Latin squares of order v with both the diagonal and strongly symmetric properties (i.e. t SSPODLS(v)) for $t = 3, 4, 5$. It is not possible for a strongly symmetric Latin square of order v to exist for $v \equiv 2 \pmod 4$, but for other v, we obtain 3 SSPODLS(v) for $v \geq 7$, 4 SSPODLS(v) for $v \geq 7$, $v \notin \{20, 21\}$, and 5 SSPODLS(v) for $v \geq 301$.

1 Introduction

A Latin square of order v is a $v \times v$ array L with entries from a set S of size v such that each element of S appears once in each row and once in each column of L. A transversal in a Latin square of order v is a set of v cells, one from each row and column, containing a different symbol in each cell.

Two Latin squares $A = (a_{i,j})$ and $B = (b_{i,j})$ of order v are said to be orthogonal if the v^2 ordered pairs $(a_{i,j}, b_{i,j})$ are all distinct. Also, t Latin squares of order v are said to form a set of t MOLS(v), or t mutually orthogonal Latin squares of order v if every pair of them is orthogonal.

A Latin square L of order v with entries and row and column indices from $S = \{0, 1, \ldots, v-1\}$ is called strongly symmetric if, for all $i, j \in S$, it satisfies the condition $L(v-1-i, v-1-j) = v-1-L(i, j)$. A Latin square is said to possess the diagonal property if its main and back diagonals are both transversals.

Not a lot is currently known about 3 or more MOLS(v) each satisfying the strongly symmetric condition. In 2002, Cao et al. [12] proved existence of 2 MOLS(v) satisfying the self-orthogonal property (i.e. the squares are of the form A_1 and $A_2 = A_1^T$),

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the strongly symmetric property and the diagonal property for all $v \equiv 0, 1, 3$ (mod 4), $v \notin \{3, 15\}$. In [13], existence of two such squares for $v = 15$ was established.

Two strongly symmetric MOLS(3) exist, but two MOLS(3) cannot possess the self-orthogonal property or the diagonal property. In addition, from Lemma 2.4 in [13], a strongly symmetric Latin square of order v cannot exist for any $v \equiv 2$ (mod 4). (The authors of [13] stated this result only for strongly symmetric Latin squares also satisfying the diagonal and self-orthogonal conditions, but these last two conditons are not used in the proof.)

More is known about existence of t MOLS(v) satisfying the diagonal property (denoted as t PODLS(v), or t pairwise orthogonal diagonal Latin squares of order v). Several authors have investigated existence of 2 PODLS(v). The full spectrum (2 PODLS(v) exist for all positive integers v except $2, 3, 6$) was eventually obtained in [11]. Existence of t PODLS(v) was investigated in [15, 19] for $t = 3$, and in [16, 17] respectively, for $t = 4, 5$. For $t = 3, 4, 5$ further improvements were made in [10]. These results are summarised in following theorem:

Theorem 1.1 *Let $D_3 = D_4 = \{2, 3, 4, 5, 6\}$ and $D_5 = D_4 \cup \{7\}$. For $t = 3, 4, 5$, there do not exist t PODLS(v) for $v \in D_t$. For other orders of v and $t \in \{3, 4, 5\}$, there always exist t PODLS(v) except possibly when $v \in P_t$, where*

$$\begin{aligned} P_3 &= \{10\}; \\ P_4 &= P_3 \cup \{18, 20, 21, 22, 26, 30, 34, 38, 42, 54\}; \\ P_5 &= P_4 \cup \{12, 14, 15, 28, 33, 35, 39, 44, 45, 46, 50, 51, 52, 60, 66, 68, 70, 74, 82, 84, 98\}. \end{aligned}$$

In this paper, for $t = 3, 4, 5$, we look at existence of t MOLS(v), each of which also satisfies both the strongly symmetric and diagonal conditions, that is, t strongly symmetric pairwise orthogonal diagonal Latin squares of order v. Any t such squares will be denoted as t SSPODLS(v). More specifically, we prove the following theorem.

Theorem 1.2 *If $v \equiv 2$ (mod 4), then a strongly symmetric Latin square of order v cannot exist. For $v \equiv 0, 1$ or 3 (mod 4), there do not exist t SSPODLS(v) if $t = 2$ and $v = 3$, if $t \in \{3, 4\}$ and $3 \leq v \leq 5$, or if $t = 5$ and $3 \leq v \leq 7$. For larger $v \equiv 0, 1$ or 3 (mod 4), and $t \in \{2, 3, 4\}$, t SSPODLS(v) exist, except possibly when $t = 4$ and $v \in \{20, 21\}$. Also 5 SSPODLS(v) exist if $v \equiv 0, 1$ or 3 (mod 4), and $v \geq 301$.*

2 Using quasi-difference matrices to obtain SSPODLS

Let G be an additive abelian group of order v. A $(v, k; \lambda_1, \lambda_2; u)$-QDM or quasi-difference matrix over G is an array $Q = (q_{i,j})$ with k rows and $\lambda_1(v - 1 + 2u) + \lambda_2$ columns such that each entry of Q either is empty or contains an element of G. (We shall usually assume we have labelled the rows of Q as $0, 1, 2, \ldots, k - 1$ and the columns as $0, 1, 2, \ldots, \lambda_1(v - 1 + 2u) + \lambda_2 - 1$). In addition:

1. Each row of Q contains $\lambda_1 u$ empty entries and each column of Q contains at most one empty entry;

2. For any two distinct rows i_1 and i_2 of Q, the multiset of differences $q_{i_1,j} - q_{i_2,j}$ ($j = 0, 1, \ldots, \lambda_1(v-1+2u) + \lambda_2 - 1$, with $q_{i_1,j}, q_{i_2,j}$ both non-empty) contains each nonzero element of G exactly λ_1 times, and the zero element of G exactly λ_2 times.

A $(v, k; \lambda_1, \lambda_1; 0)$-QDM is more commonly denoted as a $(v, k; \lambda_1)$-DM or difference matrix.

In [10] a construction for PODLS using quasi-difference matrices was given. In this section, we show how this construction can produce SSPODLS when a few extra conditions are added. First we need to define incomplete MOLS (or IMOLS), IPODLS, and ISSPODLS.

Let A be a $(v + u) \times (v + u)$ array, let V be a set of size $v + u$ and let U be a subset of V of size u. The array A is said to be an incomplete Latin square over (V, U) or an ILS$(v + u, u)$ if

1. A has an empty $u \times u$ subarray (called a hole) and each cell of A which is not part of the hole contains an element of V;

2. Each row and column of A that is part of the hole contains each element of $V \setminus U$ once, and no element of U. Every other row and column of A contains each element of V exactly once.

A holey transversal in an ILS$(v + u, u)$ over (V, U) is a set of v cells, no two of which lie in the same row or column, such that each element of $V \setminus U$ lies in one of the v cells, and none of the v cells lies in any row or column which is part of the hole. Two incomplete Latin squares ILS$(v + u, u)$ over (V_1, U_1) and (V_2, U_2) are said to be orthogonal if their superposition produces each pair from $(V_1 \times V_2) \setminus (U_1 \times U_2)$ exactly once. Any t ILS$(v + u, u)$, each pair of which is orthogonal, are said to form a set of t IMOLS$(v + u, u)$. Two or more IMOLS$(v + u, u)$ are said to be idempotent if they possess a common holey transversal.

Suppose we have t IMOLS$(v+u, u)$, (denoted as A_ℓ for $\ell = 1, 2, \ldots, t$) over (V, U) with row and column indices from $\{0, 1, 2, \ldots, v+u-1\}$. If these IMOLS satisfy two extra conditions: (1) Whenever row i is part of the hole, row $v+u-1-i$, column i and column $v+u-1-i$ are also, (2) the main and back diagonals of each A_ℓ contain each element of $V \setminus U$ eactly once. Then these t IMOLS are said to be t IPODLS$(v+u, u)$. Suppose further, three extra conditions are satisfied (3) $V = \{0, 1, 2, \ldots, v + u - 1\}$, (4) Whenever $y \in U$, $v+u-1-y \in U$ and (5) For all $\ell = 1, 2, \ldots, t$, and all i, j such that $A_\ell(i, j)$ is non-blank, we have $A_\ell(v+u-1-i, v+u-1-j) = v+u-1-A_\ell(i, j)$. In this case, the squares $A_\ell, (\ell = 1, 2, \ldots, t)$ are called t ISSPODLS$(v + u, u)$.

We also need to define the concept of pairwise matching columns in a QDM. If $Q = (q_{i,j})$ (where $0 \le i \le k-1, 0 \le j \le v + 2u - 1$) is a $(v, k; 1, 1; u)$-QDM over an additive abelian group G, and H is a subgroup of G of order $v/2$, then two columns

$(y_0, y_1, \ldots, y_{k-1})^T$ and $(z_0, z_1, \ldots, z_{k-1})^T$ of Q are said to be *pairwise matching* with respect to H if for all $i = 0, 1, \ldots, k-1$, the values y_i and z_i are either both blank, both in H or both in $G \setminus H$.

Now suppose there exist an element α of order 2 in G and a 2×2 submatrix of Q (in rows i_1, i_2 and columns j_1, j_2) satisfying two extra conditions: (1) columns j_1, j_2 contain no blank entries (2) for some $a \in G$, $q_{i_2,j_1} - q_{i_1,j_1} = a$ and $q_{i_2,j_2} - q_{i_1,j_2} = a + \alpha$. In this case we say the QDM satifies the $*$ condition (in rows i_1, i_2) and denote it by QDM*.

We now prove the main theorem in this section:

Theorem 2.1 *Let G be an additive abelian group of even order v. Let α be an order 2 element of G, and suppose a $(v, k; 1, 1; u)$-QDM* over G exists. Then:*

1. *If $u \in \{0, 1\}$, then $k-2$ ISSPODLS$(v+u, u)$ and k SSPODLS$(v+u)$ exist.*
2. *Suppose $u \geq 2$, and either $u = 2m$ or $u = 2m+1$. Let H be a subgroup of G of order $v/2$ such that $\alpha \notin H$. Suppose also, that for each row i in the QDM*, $2m$ of the u columns with a blank entry in row i can be partitioned into m sets of pairwise matching columns with respect to H. Then $k-2$ ISSPODLS$(v+u, u)$ exist. If also, $k-2$ SSPODLS(u) exist, then $k-2$ SSPODLS$(v+u)$ exist.*

Proof: Let $Q1$ be a $(v, k; 1, 1; u)$-QDM* over G with the appropriate pairwise matching columns, with row indices from $\{0, 1, \ldots, k-1\}$, and column indices from $\{0, 1, \ldots, v+2u-1\}$. Without loss of generality, we can also assume that the rows of $Q1$ in which the * condition is satisfied are $i_1 = 0$ and $i_2 = 1$. Replace the u blank entries in each row by the elements of $S = \{\infty_0, \infty_1, \ldots, \infty_{u-1}\}$ in such a way that for any integer s with $0 \leq s \leq (u-2)/2$, and any row i with $0 \leq i \leq k-1$, the two columns containing ∞_s and ∞_{u-1-s} in row i are pairwise matching. Also, let $G = \{g_0 = 0, g_1, \ldots, g_{v-1}\}$, where for $0 \leq i \leq v-1$, $g_{v-1-i} = g_i + \alpha$. Note that this implies $g_{v-1} = \alpha$. In addition, if $u \geq 2$, the elements of G should be chosen so that $H = \{g_0 = 0, g_1, \ldots, g_{(v-2)/2}\}$ (possible since $\alpha \notin H$).

For convenience, we now relabel the elements of $G \cup S$ as $x_0, x_1, \ldots, x_{v+u-1}$, where $x_i = g_i$ for $0 \leq i \leq (v-2)/2$, $x_{v/2+i} = \infty_i$ for $0 \leq i \leq u-1$, and $x_{v+u-1-i} = g_{v-1-i} = \alpha + g_i$ for $0 \leq i \leq (v-2)/2$.

We can also assume, without loss of generality that two columns of $Q1$ satisfying the $*$ condition in rows 0 and 1 are $j_1 = 0$ and $j_2 = v+u-1$. (Thus for some $a \in G$, $Q1(1, j_1) - Q1(0, j_1) = a$ and $Q1(1, j_2) - Q1(0, j_2) = a + \alpha$). We shall also assume that the columns of $Q1$ are ordered so that the following conditions are satisfied: (1) $Q1(1, j) - Q1(0, j) = a + x_j \in G$ for $0 \leq j \leq v/2 - 1$ and $v/2 + u \leq j \leq v+u-1$, (2) $Q1(1, v/2 + j) = x_{v/2+j} = \infty_j = Q1(0, v+u+j)$ for $0 \leq j \leq u-1$.

Now let $Q2$ be the matrix obtained by subtracting a from all non-infinite elements in row 1 of $Q1$. Next, normalise each column j of $Q2$ by adding $-Q2(0, j)$ to column j if $Q2(0, j)$ is not an infinite element or by adding $-Q2(1, j)$ to column j if $Q2(0, j)$ is an infinite element. Note that if j_3 and j_4 are any two pairwise matching columns, then the amounts added to columns j_3, j_4 here are either both in H or both in $G \setminus H$ and therefore these columns remain pairwise matching even after they've been normalised. Let $Q = (q_{i,j})$ be the resulting normalised QDM*. Note that (1) if $0 \leq j \leq v+u-1$, then $q_{0,j} = 0$ and $q_{1,j} = x_j$, and (2) if $0 \leq j \leq u-1$, then $q_{0,v+u+j} = \infty_j = x_{j+v/2}$ and $q_{1,v+u+j} = 0$.

Now we define $k-2$ incomplete Latin squares ILS$(v+u,u)$ with row and column indices from $\{0,1,\ldots,v+u-1\}$ as follows. These squares will be denoted as A_ℓ (for $\ell = 1,2,\ldots,k-2$), where (1) $A_\ell(0,j) = q_{\ell+1,j}$ for $0 \leq j \leq v+u-1$, (2) if $x_i \in G$ and $x_j \in S$, then $A_\ell(i,j) = A_\ell(0,j)+x_i$, (3) if $x_i \in G$, $x_j \in G$ and $x_j - x_i = x_n$, then $A_\ell(i,j) = A_\ell(0,n) + x_i$, (4) if $x_i \in S$ and $x_j \in G$, then $A_\ell(i,j) = q_{\ell+1,v+u+i} + x_j$, (5) if $x_i \in S$ and $x_j \in S$, then $A_\ell(i,j)$ is empty.

This is a standard construction for IMOLS from a QDM, i.e. the squares A_ℓ (for $\ell = 1,2,\ldots,k-2$) are $k-2$ IMOLS$(v+u,u)$. It is also not hard to verify that each element of G appears appears exactly once on both the main and back diagonals of each A_ℓ, since whenever $x_i \in G$, $A_\ell(i,i) = A_\ell(0,0)+x_i$, and (as $x_{v+u-1-i} - x_{v+u-1} = (\alpha + x_i) - \alpha = x_i$), $A_l(i, v+u-1-i) = A_l(0, v+u-1) + x_i$.

Thus the squares A_ℓ (for $\ell = 1,2,\ldots,k-2$) are also $k-2$ IPODLS$(v+u,u)$. We now need to look at the strongly symmetric condition and convert these squares to $k-2$ ISSPODLS$(v+u,u)$, namely B_ℓ (for $\ell = 1,2,\ldots k-2$). This will be done by making some adjustments to the infinite elements plus the rows and columns which are part of the hole. We also need to rename the symbols. Initially, if symbol x_t appears in any (i,j) cell of A_ℓ, then symbol $t \in \{0,1,2,\ldots,v+u-1\}$ will be placed in the (i,j) cell of B_ℓ. Some further adjustments to an (i,j) cell of B_ℓ are needed however, when one of x_i, x_j or $A_\ell(i,j)$ is an infinite element.

If all of x_i, x_j, and $A_\ell(i,j)$ are elements of G, then let $A_\ell(i,j) = x_m$. Here, $x_{v+u-1-i} - x_i = \alpha = x_{v+u-1-j} - x_j$. Thus $A_\ell(v+u-1-i, v+u-1-j) = \alpha + A_\ell(i,j) = \alpha + x_m = x_{v+u-1-m}$. So $B_\ell(i,j) = m$ and $B_\ell(v+u-1-i, v+u-1-j) = v+u-1-m$. Thus in this case, cells (i,j) and $(v+u-1-i, v+u-1-j)$ of B_ℓ satisfy the strongly symmetric condition. If $u = 1$, then $\infty_0 = x_{(v+u-1)/2}$; here, the strongly symmetric condition for cells (i,j) and $(v+u-1-i, v+u-1-j)$ in B_ℓ is also satisfied when one of x_i, x_j, $A_\ell(i,j)$ equals ∞_0.

When $u \geq 2$, any infinite point $\infty_t = x_{v/2+t}$ and its pairwise matching infinite point $\infty_{u-1-t} = x_{v/2+u-1-t}$ will both appear in $v/2$ cells in each of the top left and bottom right $v/2 \times v/2$ subsquares of A_ℓ or in each of the top right and bottom left $v/2 \times v/2$ subsquares of A_ℓ. Here, for $0 \leq t \leq (u-2)/2$, we now interchange the positions of $v/2+t$ and $v/2+u-1-t$ in either the bottom left $v/2 \times v/2$ subsquare of B_ℓ (if they appear there), or else in the bottom right $v/2 \times v/2$ subsquare of B_ℓ. Note that in any square A_n with $n \neq \ell$, the $v/2$ cells occupied by each of these infinite points in the corresponding $v/2 \times v/2$ subsquare of A_n will either both contain all elements of H once or all elements of $G \setminus H$ once. This is because of the pairwise matching condition, and the fact that $H = \{x_0 = 0, x_1, \ldots, x_{(v-2)/2}\}$. Thus making these interchanges after relabelling does not affect the orthogonality condition.

In addition, within the bottom central $v/2 \times u$ subsquare of B_ℓ, we interchange columns $v/2+t$ and $v/2+u-1-t$ for $0 \le t \le (u-2)/2$. Similarly, within the central right $u \times v/2$ subsquare of B_ℓ we interchange rows $v/2+t$ and $v/2+u-1-t$ for $0 \le t \le (u-2)/2$. Because the two columns of Q containing $x_{v/2+t}$ and $x_{v/2+u-1-t}$ in row i for $i \in \{0,1\}$ are pairwise matching, making these interchanges still leaves each element of G once in each row and in each column of B_ℓ. Also, these interchanges don't affect the orthogonality condition.

After these interchanges are made, the (i,j) and $(v+u-i, v+u-j)$ cells in B_ℓ satisfy the strongly symmetric condition even when x_i, x_j or $A_\ell(i,j)$ is an infinite element. For instance if x_j is an infinite element, $i \le (v-2)/2$ and $A_\ell(i,j) = x_m$, then $A_\ell(v+u-1-i,j) = A_\ell(0,j)+x_{v+u-1-i} = A_\ell(i,j)-x_i+x_{v+u-1-i} = A_\ell(i,j)+\alpha = x_m+\alpha = x_{v+u-1-m}$. Thus $B_\ell(i,j) = m$ and $B_\ell(v+u-1-i, v+u-1-j) = v+u-1-m$, so here, the (i,j) and $(v+u-1-i, v+u-1-j)$ cells in B_ℓ again satisfy the strongly symmetric condition.

Making all these changes ensures that for any $y \in \{0,1,\ldots,v+u-1\}$, if y lies in any cell (i,j) of any B_ℓ, then $v+u-1-y$ lies in the $(v+u-1-i, v+u+1-j)$ cell. Thus the squares B_ℓ (for $\ell = 1,2,\cdots k-2$) are $k-2$ ISSPODLS(v,u).

Finally, if $k-2$ SSPODLS(u) exist, we add $v/2$ to all entries in these SSPODLS(u) and then use them to fill the empty $u \times u$ subsquares of the squares B_ℓ. When this is done, the squares B_ℓ (for $\ell = 1,2,\ldots,k-2$) are $k-2$ SSPODLS$(v+u)$. □

The next two theorems give two important special cases of Theorem 2.1 (when $u = 0$ or 1):

Theorem 2.2 *Suppose that v is an even integer, and there exists a $(v,k;1,1;1)$-QDM* over an abelian group of order v. Then $k-2$ SSPODLS$(v+1)$ also exist.*

Theorem 2.3 *Suppose that v is an even integer, and there exists a $(v,k;1)$-DM* over an abelian group of order v. Then $k-2$ SSPODLS(v) also exist.*

We remark that if u is even and $k \ge 3$, then Theorem 2.1 can give $k-2$ SSPODLS$(v+u)$ only if $v \equiv u \equiv 0 \pmod 4$. If $u \equiv 2 \pmod 4$, then $k-2$ SSPODLS(u) can't exist, so the theorem doesn't give $k-2$ SSPODLS$(v+u)$. If $v \equiv 2 \pmod 4$, then it is sufficient to assume $k = 3$ (if $k > 3$, delete all but 3 rows of the $(v,k;1,1;u)$-QDM*, $Q = (q_{i,j})$). In this case, consider the multiset C of differences $q_{1,j}-q_{0,j}, q_{2,j}-q_{0,j}, q_{2,j}-q_{1,j}$ for $j = 0,1,\ldots,v+2u-1$ (but differences that include a blank entry are not included).

Since Q is a QDM, C must contain exactly $3v/2$ elements from H and $3v/2$ elements from $G \setminus H$. This is an odd number since $v \equiv 2 \pmod 4$. However, within the set of columns of Q containing a blank entry there is an even number of such differences in $G \setminus H$, since these columns can be partitioned into 'matching pairs'. It is also not hard to see that any column with no blank entry (i.e. 3 non-blank entries) contains 0 or 2 such differences in $G \setminus H$, again an even number. The total number of such differences over all columns therefore can't equal the odd number $3v/2$, a contradiction.

It is also worth mentioning that if $G = \mathbb{Z}_v$ (where $v \equiv 0 \pmod 4$ and $u \geq 2$), then Theorem 2.1 cannot give $k-2$ SSPODLS$(v+u)$ or even $k-2$ ISSPODLS$(v+u, u)$. This is because the only order 2 element in G is $\alpha = v/2$, and α lies inside the only subgroup H of order $v/2$ in G (here, $H = \{0, 2, 4, \ldots, v-2\}$), which is not allowed in Theorem 2.1 when $u \geq 2$.

Example 2.1 The following is a normalised $(10, 4; 1, 1; 3)$-QDM* over $\mathbb{Z}_{10}$ (with the blank entries replaced by $x_5 = x$, $x_6 = z$, and $x_7 = y$). It also satisfies the pairwise matching condition: for any $i \in \{0, 1, 2, 3\}$, the two columns containing x and y in row i are pairwise matching. The elements of $G = \mathbb{Z}_{10}$ are $x_t = 2t$ and $x_{12-t} = 5 + 2t$ for $0 \leq t \leq 4$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & z & y \\ 0 & 2 & 4 & 6 & 8 & x & z & y & 3 & 1 & 9 & 7 & 5 & 0 & 0 & 0 \\ 0 & 5 & 1 & y & x & 3 & 4 & 7 & 9 & z & 8 & 2 & 6 & 2 & 4 & 8 \\ 0 & y & x & 8 & 6 & 1 & 5 & 3 & z & 7 & 2 & 4 & 9 & 1 & 9 & 5 \end{pmatrix}$$

This QDM* gives the following 2 IPODLS$(13, 3)$ (A_1 and A_2) using Theorem 2.1. The columns $(0*, 0*, 0, 0)^T$ and $(0*, 5*, 6, 9)^T$ satisfy the * condition. To make checking easier, the elements x_t corresponding to row and column t (for $t = 0, 1, 2, \ldots, 12$) of each square are given in the first column and row.

	0	2	4	6	8	x	z	y	3	1	9	7	5
0	0	5	1	y	x	3	4	7	9	z	8	2	6
2	x	2	7	3	y	5	6	9	z	0	4	8	1
4	y	x	4	9	5	7	8	1	2	6	0	3	z
6	7	y	x	6	1	9	0	3	8	2	5	z	4
8	3	9	y	x	8	1	2	5	4	7	z	6	0
x	2	4	6	8	0				5	3	1	9	7
z	4	6	8	0	2				7	5	3	1	9
y	8	0	2	4	6				1	9	7	5	3
3	5	1	z	2	9	6	7	0	3	x	y	4	8
1	9	z	0	7	3	4	5	8	6	1	x	y	2
9	z	8	5	1	7	2	3	6	0	4	9	x	y
7	6	3	9	5	z	0	1	4	y	8	2	7	x
5	1	7	3	z	4	8	9	2	x	y	6	0	5

	0	2	4	6	8	x	z	y	3	1	9	7	5
0	0	y	x	8	6	1	5	3	z	7	2	4	9
2	8	2	y	x	0	3	7	5	9	4	6	1	z
4	2	0	4	y	x	5	9	7	6	8	3	z	1
6	x	4	2	6	y	7	1	9	0	5	z	3	8
8	y	x	6	4	8	9	3	1	7	z	5	0	2
x	1	3	5	7	9				4	2	0	8	6
z	9	1	3	5	7				2	0	8	6	4
y	5	7	9	1	3				8	6	4	2	0
3	7	5	0	z	2	4	8	6	3	9	1	x	y
1	3	8	z	0	5	2	6	4	y	1	7	9	x
9	6	z	8	3	1	0	4	2	x	y	9	5	7
7	z	6	1	9	4	8	2	0	5	x	y	7	3
5	4	9	7	2	z	6	0	8	1	3	x	y	5

To convert these to 2 ISSPODLS$(13, 3)$ we first interchange the first and third rows in the central right 3×5 subarray, and the first and third columns in the bottom central 5×3 subarray. We also interchange the values x and y in the bottom right 5×5 subarray. Finally, we replace the values $0, 2, 4, 6, 8$ by $0, 1, 2, 3, 4$, the values $5, 7, 9, 1, 3$ by $12, 11, 10, 9, 8$ and x, z, y by $5, 6, 7$. We then obtain the following 2 ISSPODLS$(13, 3)$, B_1 and B_2.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	12	9	7	5	8	2	11	10	6	4	1	3
1	5	1	11	8	7	12	3	10	6	0	2	4	9
2	7	5	2	10	12	11	4	9	1	3	0	8	6
3	11	7	5	3	9	10	0	8	4	1	12	6	2
4	8	10	7	5	4	9	1	12	2	11	6	3	0
5	1	2	3	4	0				9	10	11	12	8
5	2	3	4	0	1				11	12	8	9	10
7	4	0	1	2	3				12	8	9	10	11
8	12	9	6	1	10	0	11	3	8	7	5	2	4
9	10	6	0	11	8	4	12	2	3	9	7	5	1
10	6	4	12	9	11	3	8	1	0	2	10	7	5
11	3	8	10	12	6	2	9	0	5	4	1	11	7
12	9	11	8	6	2	1	10	4	7	5	3	0	12

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	7	5	4	3	9	12	8	6	11	1	2	10
1	4	1	7	5	0	8	11	12	10	2	3	9	6
2	1	0	2	7	5	12	10	11	3	4	8	6	9
3	5	2	1	3	7	11	9	10	0	12	6	8	4
4	7	5	3	2	4	10	8	9	11	6	12	0	1
5	9	8	12	11	10				4	3	2	1	0
6	10	9	8	12	11				1	0	4	3	2
7	12	11	10	9	8				2	1	0	4	3
8	11	12	0	6	1	3	4	2	8	10	9	7	5
9	8	4	6	0	12	2	3	1	5	9	11	10	7
10	3	6	4	8	9	1	2	0	7	5	10	12	11
11	6	3	9	10	2	0	1	4	12	7	5	11	8
12	2	10	11	1	6	4	0	3	9	8	7	5	12

The next lemma gives some known difference matrices which will be useful for obtaining some examples of SSPODLS:

Lemma 2.4 *If* $(t, v) \in S = \{(4, 12), (4, 28), (4, 44), (4, 52), (4, 60), (6, 24), (6, 40), (6, 56), (7, 36), (7, 48), (4, 60), (8, 80)\}$ *then* t *SSPODLS*(v) *exist.*

Proof: A $(v, 6; 1)$-DM* is given in [1] when $v \in \{28, 44, 52\}$. A $(24, 8; 1)$-DM* and a $(36, 9; 1)$-DM* are given in [7], while a $(40, 8; 1)$-DM* and an $(80, 10; 1)$-DM* are given in [6]. A $(12, 6; 1)$-DM* and a $(56, 8; 1)$-DM* are given in [18], a $(48, 9; 1)$-DM* is given in [5], and a $(60, 6; 1)$-DM* is given in [2]. These DM*s are all over additive abelian groups, therefore, by Theorem 2.3, there exist t SSPODLS(v) for each $(t, v) \in S$. □

A transversal T in a Latin square A of order q (with entries, row indices and column indices from $\{0, 1, 2, \ldots, q-1\}$) is said to be strongly symmetric if the $(q-1-i, q-1-j)$ cell of A is in T whenever the (i, j) cell of A is. Two disjoint transversals T_1 and T_2 in A are said to be pairwise strongly symmetric (or to form a strongly symmetric pair) if, whenever an (i, j) cell is in T_1, the $(q-1-i, q-1-j)$ cell is in T_2. For q a prime power we have the following important result:

Lemma 2.5 *If* q *is an even prime power, then there exist* $q-2$ *SSPODLS*(q) *with* q *common disjoint strongly symmetric transversals that include the main and back diagonals. Also, if* q *is an odd prime power, then there exist* $q-3$ *SSPODLS*(q) *with* q *common disjoint transversals, such that one is the main diagonal, and the others can be partitioned into* $(q-1)/2$ *strongly symmetric pairs.*

Proof: Let the elements of GF(q) be $x_0 = 0, x_1, \ldots, x_{q-1}$, where $x_{q-1-s} = 1 - x_s$ for $0 \leq s \leq (q-1)/2$. (Note that this implies $x_{q-1} = 1$ and, if q is odd, $x_{(q-1)/2} = 2^{-1}$).

Now we define $q-1$ Latin squares A_ℓ for $\ell = 0, 1, \ldots, q-2$. Their row and column indices are ordered as $0, 1, 2 \ldots, q-1$, and their entries are $A_0(i, j) = x_i - x_j$,

$A_\ell(i,j) = x_\ell(x_i) + (1 - x_\ell)x_j$ if $1 \le \ell \le q-2$. It is well known that if $m \neq n$, then A_m is orthogonal to A_n. The q symbols in square A_0 give us q common disjoint transversals in each of the other squares; also, for any $i,j \in \{0,1,\ldots,q-1\}$, $A_0(q-1-i, q-1-j) = x_{q-1-i} - x_{q-1-j} = (1-x_i) - (1-x_j) = x_j - x_i$. When q is even, this also equals $x_i - x_j = A_0(i,j)$. Thus for even q, the q common disjoint transversals in $A_1, A_2, \ldots, A_{q-2}$ determined by the elements of A_0 will be strongly symmetric if we replace symbol x_t by t for $t = 0,1,\ldots,q-1$. Likewise, if q is odd, $x_{q-1-i} - x_{q-1-j} = x_j - x_i =$ -$(x_i - x_j)$, so here, for any $g \in GF(q)$, $g \neq 0$, the transversals in the squares $A_1, A_2, \ldots, A_{q-2}$ determined by elements $-g$ and g in A_0 will form a strongly symmetric pair if we replace x_t by t for $t = 0,1,\ldots,q-1$.

For both q even and q odd, the transversal determined by element zero in A_0 is the main diagonal in the other squares, since for any $i \in \{0,1,\ldots,q-1\}$, $x_i - x_i = 0$. When q is even, the transversal determined by element 1 in A_0 is the back diagonal in the other squares, since here for any $i \in \{0,1,\ldots,q-1\}$, $x_i - x_{q-1-i} = x_i - (1-x_i) = 1$. If q is odd, then the element 2^{-1} in $A_{(q-1)/2}$ determines a transversal in the other squares which is the back diagonal, since the back diagonal entries in $A_{(q-1)/2}$ are $2^{-1}x_i + 2^{-1}x_{q-1-i} = 2^{-1}(x_i + (1-x_i)) = 2^{-1}$. Thus the squares A_ℓ (for $\ell = 1,2,\ldots,q-2$, $\ell \neq (q-1)/2$ if q is odd) are diagonal squares.

If $\ell > 0$ and $A_\ell(i,j) = x_t$, then $A_\ell(q-1-i, q-1-j) = x_\ell x_{q-1-i} + (1-x_\ell)x_{q-1-j} = x_\ell(1-x_i) + (1-x_\ell)(1-x_j) = 1 - (x_\ell x_i + (1-x_\ell)x_j) = 1 - A_\ell(i,j) = x_{q-1-t}$.

Therefore if we let B_ℓ be the Latin squares obtained from A_ℓ by replacing symbol x_t by t for $t = 0,1,\ldots,q-1$, then the squares B_ℓ (for $\ell = 1,2,\ldots,q-2$, $\ell \neq (q-1)/2$ if q is odd) form a set of $q-2$ SSPODLS(q) if q is even, or $q-3$ SSPODLS(q) if q is odd. This completes the proof. □

As an example, when $q = 4$ and GF(4) $= \{x_0 = 0,\ x_1 = y,\ x_2 = y^2 = y+1,\ x_3 = 1\}$, the squares A_ℓ, B_ℓ (for $\ell = 0,1,2$) are displayed below. Here, B_1 and B_2 are 2 SSPODLS(4). For the squares A_ℓ, the elements x_n are given as the first element in row n and column n for each $n \in \{0,1,2,3\}$. For the squares B_ℓ, the value n appears in these positions.

	A_0				A_1				A_2			
	0	y	y^2	1	0	y	y^2	1	0	y	y^2	1
0	0	y	y^2	1	0	1	y	y^2	0	y^2	1	y
y	y	0	1	y^2	y^2	y	1	0	1	y	0	y^2
y^2	y^2	1	0	y	1	0	y^2	y	y	1	y^2	0
1	1	y^2	y	0	y	y^2	0	1	y^2	0	y	1

	B_0				B_1				B_2			
	0	1	2	3	0	1	2	3	0	1	2	3
0	0	1	2	3	0	3	1	2	0	2	3	1
1	1	0	3	2	2	1	3	0	3	1	0	2
2	2	3	0	1	3	0	2	1	1	3	2	0
3	3	2	1	0	1	2	0	3	2	0	1	3

We also give the squares A_ℓ and B_ℓ (for $\ell = 0, 1, 3$) when $q = 5$. The elements of $GF(5)$ are ordered as $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 1$. Here, B_1 and B_3 are 2 SSPODLS(5). Again, for the squares A_ℓ, the elements x_n are given as the first element in row n and column n for $n \in \{0, 1, 2, 3, 4\}$, while for the squares B_ℓ, the value n appears in these positions.

	A_0					A_1					A_3				
	0	2	3	4	1	0	2	3	4	1	0	2	3	4	1
0	0	3	2	1	4	0	3	2	1	4	0	4	1	3	2
2	2	0	4	3	1	4	2	1	0	3	3	2	4	1	0
3	3	1	0	4	2	1	4	3	2	0	2	1	3	0	4
4	4	2	1	0	3	3	1	0	4	2	1	0	2	4	3
1	1	4	3	2	0	2	0	4	3	1	4	3	0	2	1

	B_0					B_1					B_3				
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	0	2	1	4	3	0	2	1	4	3	0	3	4	2	1
1	1	0	3	2	4	3	1	4	0	2	2	1	3	4	0
2	2	4	0	3	1	4	3	2	1	0	1	4	2	0	3
3	3	1	4	0	2	2	4	0	3	1	4	0	1	3	2
4	4	3	2	1	0	1	0	3	2	4	3	2	0	1	4

We finish this section with several examples of t SSPODLS (with $t \in \{3, 4, 5\}$). They are all obtainable by Theorem 2.1 or one of its special cases in Theorem 2.2 or 2.3. For these SSPODLS, we give the relevant QDM*, and indicate the columns and entries within them that guarantee the * condition.

Lemma 2.6 *There exist* 3 *SSPODLS*(v) *for* $v \in \{20, 21\}$ *and* 4 *SSPODLS*(v) *for* $v = 100$.

Proof: For $v = 20$, 3 SSPODLS(v) are obtainable by Theorem 2.3, since a $(20, 5; 1)$-DM* over $Z_2 \times Z_{10}$ exists. This difference matrix can be obtained by replacing each column in the array C below by its five cyclic shifts. Take $\alpha = (1, 0)$; here, the columns $((0,0)*, (0,0)*, (0,1), (1,1), (0,5))^T$ and $((0,1)*, (1,1)*, (0,5), (0,0), (0,0))^T$ satisfy the * condition.

$$C: \quad \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (0,4) & (0,0) & (0,2) & (0,6) \\ (1,2) & (0,1) & (0,0) & (0,3) \\ (0,1) & (1,1) & (1,5) & (0,2) \\ (1,4) & (0,5) & (1,8) & (1,9) \end{array}$$

For $v = 21$, we can apply Theorem 2.2, since a $(20, 5; 1, 1; 1)$-QDM* over $\mathbb{Z}_{20}$ exists. This QDM* is displayed below. The columns $(0*, 1*, 2, 3, 4)^T$ and $(0*, 11*, 3, 2, 19)^T$ satisfy the * condition.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–	10	15	5	0
1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	5	0	–	10	15	0
2	1	6	13	4	11	14	17	3	19	9	7	18	8	12	16	15	5	0	–	10	0
3	9	19	17	18	8	12	6	2	1	16	13	14	11	4	7	10	15	5	0	–	0
4	18	12	1	17	14	6	8	19	13	7	16	2	9	11	3	–	10	15	5	0	0

For $v = 100$, we apply Theorem 2.1, using an $(84, 6; 1, 1; 16)$-QDM* over $G = \mathbb{Z}_2 \times \mathbb{Z}_{42}$. For the arrays A_1, A_2 below, each column of $[A_1|A_2]$ (except the last column of A_2) generates five columns in this QDM* by cyclically permuting the entries in its first five rows, whilst leaving the sixth entry unaltered. The last column of A_2 generates no columns in the QDM* other than itself. We take $\alpha = (1, 0)$, and the subgroup H of order $|G|/2 = 42$ as $\{0\} \times \mathbb{Z}_{42}$. Here, the columns $((0, 15)*, (0, 32)*, (0, 28), (0, 30), (1, 35), (0, 0))^T$ and $((0, 40)*, (1, 15)*, (0, 9), (1, 36), (0, 6), (0, 0))^T$ satisfy the * condition.

We now verify that the required condition for pairwise matching columns is satisfied. Consider the six columns generated by the last columns of A_1 and A_2. For all these columns, the first five entries lie in H and the sixth entry is blank, thus any two of these columns are pairwise matching. In addition for $j = 1, 2, \ldots, 9$, the j'th columns of A_1 and A_2 are pairwise matching.

A_1 :

(0, 0)	(1, 3)	(0, 14)	(1, 41)	(0, 27)	(0, 21)	(0, 18)	(1, 0)	(1, 39)	(0, 0)	(0, 10)	(0, 0)
(1, 3)	(1, 38)	(0, 24)	(1, 18)	(0, 39)	(1, 11)	(1, 16)	(0, 7)	(0, 13)	(0, 5)	(1, 19)	(0, 33)
(1, 18)	(1, 5)	(0, 19)	(0, 20)	(1, 10)	(0, 4)	(1, 40)	(0, 1)	(0, 12)	(1, 29)	(0, 2)	(0, 18)
(1, 29)	(1, 13)	(1, 30)	(1, 26)	(1, 31)	(1, 33)	(0, 36)	(1, 9)	(1, 32)	(0, 33)	(1, 12)	(0, 32)
(0, 17)	–	–	–	–	–	–	–	–	(0, 11)	(0, 3)	(0, 38)
–	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	–

A_2 :

(0, 0)	(1, 21)	(0, 16)	(1, 8)	(0, 26)	(0, 29)	(0, 41)	(1, 14)	(1, 1)	(0, 28)	(0, 40)	(0, 0)
(1, 37)	(1, 22)	(0, 34)	(1, 34)	(0, 23)	(1, 28)	(1, 20)	(0, 37)	(0, 35)	(0, 30)	(1, 15)	(0, 0)
(1, 35)	(1, 2)	(0, 22)	(0, 31)	(1, 37)	(0, 17)	(1, 7)	(0, 8)	(0, 38)	(1, 35)	(0, 9)	(0, 0)
(1, 9)	(1, 25)	(1, 6)	(1, 4)	(1, 23)	(1, 17)	(0, 25)	(1, 27)	(1, 24)	(0, 15)	(1, 36)	(0, 0)
(0, 10)	–	–	–	–	–	–	–	–	(0, 32)	(0, 6)	(0, 0)
–	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	–

□

Lemma 2.7 *There exist* 4 *SSPODLS*(v) *for* $v \in \{15, 33, 35, 39, 45, 51, 68\}$.

Proof: These SSPODLS can be obtained by Theorem 2.1. For $v \neq 68$, a $(v - 1, 6; 1, 1; 1)$-QDM* over $\mathbb{Z}_{v-1}$ exists, and for $v = 68$, a $(v - 8, 6; 1, 1; 8)$-QDM* over $\mathbb{Z}_2 \times \mathbb{Z}_{30}$ exists. For $v \in \{15,\ 33,\ 35\}$, these QDM*s can be found in [10]. For each

$v \in \{39, 45, 51\}$, take $\alpha = (v-1)/2$, and for $v = 68$, take $\alpha = (1, 0)$. Now we give two arrays A_1, A_2; each column $(a, b, c, d, e, f)^T$ of A_1 should be replaced by the six columns in the array C below. (The array C is isomorphic to the multiplication table of the dihedral group of order 6). Each column $(a, b, c, d, e, f)^T$ of A_2 (that doesn't consist entirely of zeros) is replaced by the first three columns of C. A column consisting entirely of zeros generates no columns other than itself. In each case the first two columns of A_2 satisfy the $*$ condition (in rows 1 and 4). For $v = 68$, the subgroup H of order 30 is $\{0\} \times Z_{30}$; here for $j = 1, 3, 5, 7$, the j'th and $(j+1)$'th columns of A_1 are pairwise matching.

$$C: \begin{array}{cccccc} a & b & c & d & e & f \\ b & c & a & e & f & d \\ c & a & b & f & d & e \\ d & f & e & a & c & b \\ e & d & f & b & a & c \\ f & e & d & c & b & a \end{array}$$

$v = 39$:

$$A_1: \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 36 & 11 & 6 & 1 & 35 \\ 23 & 26 & 15 & 8 & 28 & 22 \\ 12 & 28 & 24 & 32 & 36 & 37 \\ 32 & 4 & 23 & 11 & 5 & 18 \\ 16 & 25 & 31 & 37 & 18 & - \end{array} \qquad A_2: \begin{array}{cc} 0 & 0 \\ 0 & 5 \\ 0 & 29 \\ 0 & 19 \\ 0 & 24 \\ 0 & 10 \end{array}$$

$v = 45$:

$$A_1: \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 33 & 5 & 11 & 6 & 36 & 18 & 3 \\ 23 & 35 & 31 & 4 & 6 & 19 & 18 \\ 10 & 7 & 14 & 42 & 16 & 8 & 17 \\ 38 & 17 & 7 & 15 & 18 & 32 & 39 \\ 42 & 24 & 15 & 3 & 37 & 23 & - \end{array} \qquad A_2: \begin{array}{cc} 0 & 0 \\ 0 & 31 \\ 0 & 3 \\ 0 & 22 \\ 0 & 9 \\ 0 & 25 \end{array}$$

$v = 51$:

$$A_1: \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 31 & 5 & 24 & 42 & 6 & 11 & 3 \\ 27 & 3 & 13 & 4 & 12 & 41 & 5 & 18 \\ 10 & 14 & 45 & 44 & 34 & 32 & 13 & 21 \\ 31 & 5 & 43 & 37 & 23 & 22 & 47 & 46 \\ 48 & 4 & 29 & 15 & 27 & 5 & 49 & - \end{array} \qquad A_2: \begin{array}{cc} 0 & 0 \\ 0 & 13 \\ 0 & 31 \\ 0 & 25 \\ 0 & 38 \\ 0 & 6 \end{array}$$

$v = 68$:

A_1 :

$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,18)$	$(0,21)$	$(0,4)$	$(0,25)$	$(1,3)$	$(1,25)$	$(0,15)$	$(0,26)$	$(1,18)$	$(0,6)$	$(0,24)$
$(1,11)$	$(1,25)$	$(1,1)$	$(1,15)$	$(1,23)$	$(1,14)$	$(1,21)$	$(1,6)$	$(0,1)$	$(0,20)$	$(0,22)$
$(0,9)$	$(0,28)$	$(0,24)$	$(0,29)$	$(0,12)$	$(0,8)$	$(1,9)$	$(1,4)$	$(1,5)$	$(0,4)$	$(1,27)$
$(1,1)$	$(1,12)$	$(0,3)$	$(0,22)$	$(1,20)$	$(1,19)$	$(0,11)$	$(0,5)$	$(1,7)$	$(0,1)$	$(1,14)$
$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$(0,7)$	$(1,22)$	$(0,10)$

A_2 :

$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,0)$	$(0,5)$	$(0,1)$	$(0,11)$
$(0,0)$	$(0,18)$	$(0,23)$	$(0,14)$
$(0,0)$	$(1,0)$	$(0,15)$	$(1,15)$
$(0,0)$	$(1,5)$	$(0,16)$	$(1,26)$
$(0,0)$	$(1,18)$	$(0,8)$	$(1,29)$

□

Lemma 2.8 *There exist* 5 *SSPODLS*(v) *for* $v \in \{55, 69, 75, 76, 85, 87, 93, 95\}$.

Proof: These SSPODLS are obtainable by Theorem 2.1. For each $(v, u) \in \{(55, 1), (69, 9), (75, 9), (76, 8), (85, 13), (87, 11), (93, 13), (95, 11)\}$, a $(v-u, 7; 1, 1; u)$-QDM* can be obtained from two arrays A_1, A_2 given below. For each pair (v, u), the required QDM* is obtained by replacing each column of $[A_1| - A_1|A_2]$ by its seven cyclic shifts. In addition, when $(v, u) \in \{(69, 9), (93, 13), (95, 11)\}$, add an extra column whose entries are all $(0, 0)$. When $(v, u) \in \{(55, 1), (75, 9)\}$, the value $(v - u)/2$ is odd, and the given QDM* is over $G = \mathbb{Z}_{v-u}$; here, $\alpha = (v - u)/2$ and $H = \{0, 2, 4, \ldots, v - u - 2\}$. For the other cases $(v - u)/2$ is even, the given QDM* is over $G = \mathbb{Z}_2 \times \mathbb{Z}_{(v-u)/2}$ and $\alpha = (1, 0)$. Here, the subgroup H of order $(v - u)/2$ (not containing α) is $\{0\} \times \mathbb{Z}_{(v-u)/2}$.

In each case, each column of A_1 containing a blank entry and its corresponding column in $-A_1$ are pairwise matching. When the first two columns of A_2 contain a blank entry, these two columns are also pairwise matching. For $(v, u) = (76, 8)$, the third and fourth columns of A_2 also both contain a blank entry and are pairwise matching.

$(v, u) = (55, 1)$:

A_1 :

0	0	0
28	31	30
22	6	38
31	40	28
10	37	45
26	30	46
11	18	41

A_2 :

0	0
32	4
34	39
7	25
5	25
27	39
$-$	4

The columns $(10*, 26*, 11, 0, 28, 22, 31)^T$ and $(11*, 0*, 28, 22, 31, 10, 26)^T$ satisfy the * condition.

$(v, u) = (75, 9)$:

$$A_1: \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 57 & 44 & 36 & 49 & 60 \\ 3 & 52 & 60 & 18 & 42 \\ 65 & 14 & 53 & 2 & 56 \\ 54 & 17 & 55 & 58 & 31 \\ 27 & 12 & 32 & 43 & 65 \\ 47 & - & - & - & - \end{array} \qquad A_2: \begin{array}{cc} 0 & 0 \\ 26 & 1 \\ 13 & 22 \\ 46 & 51 \\ 59 & 51 \\ 33 & 22 \\ - & 1 \end{array}$$

The columns $(57*, 3, 65, 54*, 27, 47, 0)^T$ and $(27*, 47, 0, 57*, 3, 65, 54)^T$ satisfy the * condition.

$(v, u) = (69, 9)$:

$$A_1: \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (0,18) & (0,25) & (1,4) & (0,16) \\ (0,12) & (1,8) & (1,26) & (1,14) \\ (0,22) & (0,14) & (1,7) & (0,23) \\ (1,2) & (0,21) & (1,10) & (1,12) \\ (1,1) & (1,3) & (1,6) & (1,25) \\ (0,9) & - & - & - \end{array} \qquad A_2: \begin{array}{ccc} (0,0) & (0,0) & (0,0) \\ (1,1) & (1,3) & (1,14) \\ (0,6) & (0,10) & (1,12) \\ (1,6) & (1,25) & (1,27) \\ (0,1) & (0,18) & (1,29) \\ (1,0) & (1,15) & (0,15) \\ - & - & - \end{array}$$

The columns $((0, 12)*, (0, 22)*, (1, 2), \ldots, (0, 18))^T$ and $((0, 22)*, (1, 2)*, (1, 1), \ldots, (0, 12))^T$ satisfy the * condition.

$(v, u) = (76, 8)$:

$$A_1: \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (0,30) & (0,24) & (0,18) & (0,23) \\ (1,26) & (0,18) & (0,13) & (1,31) \\ (1,18) & (1,28) & (1,8) & (0,12) \\ (1,20) & (1,27) & (0,14) & (0,25) \\ (0,7) & (1,13) & (0,26) & (0,22) \\ (1,23) & (1,22) & - & - \end{array} \qquad A_2: \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (1,7) & (1,1) & (0,27) & (0,15) \\ (0,32) & (0,4) & (1,29) & (1,1) \\ (1,32) & (1,21) & (1,12) & (1,1) \\ (0,7) & (0,18) & (0,10) & (0,15) \\ (1,0) & (1,17) & (0,17) & (0,0) \\ - & - & - & - \end{array}$$

The columns $((0, 0)*, (0, 30)*, (1, 26), \ldots, (1, 23))^T$ and $((0, 30)*, (1, 26)*, (1, 18), \ldots, (0, 0))^T$ satisfy the * condition.

$(v,u) = (85,13)$:

$$A_1:\ \begin{array}{ccccc} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,22) & (0,26) & (0,33) & (0,31) & (1,14) \\ (0,29) & (0,3) & (1,12) & (0,14) & (0,20) \\ (1,19) & (1,28) & (1,6) & (1,1) & (0,29) \\ (0,10) & (1,4) & (0,8) & (0,9) & (1,12) \\ (0,35) & (1,5) & (0,28) & (0,30) & (1,10) \\ - & - & - & - & - \end{array} \qquad A_2:\ \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (1,3) & (1,1) & (1,5) & (1,4) \\ (0,27) & (0,21) & (1,13) & (0,11) \\ (1,27) & (1,3) & (1,31) & (0,15) \\ (0,3) & (0,19) & (1,23) & (0,15) \\ (1,0) & (1,18) & (0,18) & (0,11) \\ - & - & - & (1,4) \end{array}$$

The columns $((0,0)*,(1,4)*,(0,11),\ldots,(1,4))^T$ and $((0,11)*,(0,15)*,(0,15),\ldots,(1,4))^T$ satisfy the * condition.

$(v,u) = (87,11)$:

$$A_1:\ \begin{array}{ccccc} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (1,10) & (0,21) & (0,26) & (0,22) & (0,36) \\ (1,24) & (0,20) & (0,33) & (0,30) & (0,7) \\ (0,4) & (0,24) & (0,36) & (1,37) & (0,25) \\ (1,18) & (0,37) & (1,30) & (0,32) & (1,33) \\ (1,33) & (1,11) & (1,2) & (1,28) & (0,24) \\ (0,11) & - & - & - & - \end{array} \qquad A_2:\ \begin{array}{cccc} (0,0) & (0,0) & (0,0) & (0,0) \\ (1,11) & (1,1) & (1,2) & (1,13) \\ (0,26) & (0,4) & (1,8) & (1,18) \\ (1,26) & (1,23) & (1,27) & (0,35) \\ (0,11) & (0,20) & (1,21) & (0,35) \\ (1,0) & (1,19) & (0,19) & (1,18) \\ - & - & - & (1,13) \end{array}$$

The columns $((1,10)*,(1,24)*,(0,4)\ldots,(0,0))^T$ and $((0,4)*,(1,18)*,(1,33),\ldots,(1,24))^T$ satisfy the * condition.

$(v,u) = (93,13)$:

$$A_1:\ \begin{array}{cccccc} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,24) & (0,22) & (1,17) & (1,5) & (1,18) & (1,7) \\ (1,39) & (0,19) & (1,38) & (1,30) & (0,12) & (1,37) \\ (1,1) & (1,31) & (0,36) & (1,37) & (1,22) & (1,2) \\ (1,10) & (0,12) & (1,28) & (1,29) & (0,26) & (0,13) \\ (0,23) & (0,25) & (0,2) & (1,28) & (0,37) & (1,29) \\ (0,6) & - & - & - & - & - \end{array} \qquad A_2:\ \begin{array}{ccc} (0,0) & (0,0) & (0,0) \\ (1,9) & (1,1) & (1,3) \\ (1,35) & (1,13) & (1,7) \\ (0,35) & (0,33) & (1,27) \\ (0,9) & (0,21) & (1,23) \\ (1,0) & (1,20) & (0,20) \\ - & - & - \end{array}$$

The columns $((0,24)*,(1,39),(1,1)*,\ldots,(0,0))^T$ and $((0,23)*,(0,6),(0,0)*,\ldots,(1,10))^T$ satisfy the * condition.

$(v,u) = (95,11)$:

$$A_1:\ \begin{array}{cccccc} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,35) & (0,25) & (1,14) & (0,40) & (1,17) & (0,27) \\ (0,3) & (1,3) & (0,6) & (0,28) & (0,32) & (0,8) \\ (1,7) & (1,27) & (1,16) & (1,5) & (1,30) & (1,39) \\ (1,36) & (0,34) & (0,32) & (0,23) & (1,10) & (1,2) \\ (0,41) & (0,23) & (0,33) & (1,32) & (1,18) & (0,38) \\ (1,12) & (0,14) & - & - & - & - \end{array} \qquad A_2:\ \begin{array}{ccc} (0,0) & (0,0) & (0,0) \\ (1,39) & (1,1) & (0,3) \\ (1,13) & (1,37) & (0,7) \\ (0,13) & (0,16) & (0,28) \\ (0,39) & (0,22) & (0,24) \\ (1,0) & (1,21) & (0,21) \\ - & - & - \end{array}$$

The columns $((0,0)*, (0,35), (0,3)*, \ldots, (1,12))^T$ and $((0,0)*, (0,25), (1,3)*, \ldots, (0,14))^T$ satisfy the * condition.

Lemma 2.9 *There exist* 5 *idempotent IMOLS*(v,u) *for* $(v,u) \in \{(31,4),\ (39,3),\ (64,6),\ (65,6)\}$.

Proof: In each case this follows from existence of a $(v-u, 7; 1, 1; u)$-QDM with at least one column that contains no blank entry. These QDMs are over Z_{v-u} and are obtained like those in the previous lemma. We construct two arrays A_1, A_2 (but no A_2 array is needed for $(v,u) = (65,6)$) and then replace each column of $[A_1|-A_1|A_2]$ by its seven cyclic shifts. For $(v,u) = (65,6)$, we also add an extra column of zeros. The arrays A_1, A_2 are given below.

$(v,u) = (31,4)$:

A_1:			A_2:
	0	0	0
	10	19	7
	26	14	21
	20	26	3
	18	25	3
	15	2	21
	-	-	7

$(v,u) = (64,6)$:

A_1:					A_2:	
	0	0	0	0	0	0
	7	40	46	52	33	8
	56	54	36	10	35	7
	39	32	41	14	6	7
	16	45	3	25	4	8
	55	19	24	28	29	0
	28	34	-	-	-	-

$(v,u) = (39,3)$:

A_1 :			A_2 :	
	0	0	0	0
	21	26	29	5
	10	35	6	24
	32	7	24	20
	26	27	11	20
	23	29	18	24
	24	-	-	5

$(v,u) = (65,6)$:

A_1 :					
	0	0	0	0	0
	22	40	38	44	54
	2	36	30	15	12
	28	45	41	40	2
	12	44	39	53	43
	39	13	25	41	19
	36	7	-	-	-

□

3 Some SSPODLS from recursive constructions

In this section, we prove our main theorems for existence of 3, 4 and 5 SSPODLS. These SSPODLS are mostly obtainable by the recursive constructions in Lemma 3.2. First in Lemma 3.1 we give some existence results for 5 IMOLS [14] which will be needed later. In Lemma 3.1.4, the required idempotent IMOLS(v,x) with $v = 56+x$ are obtainable by a variant of Wilson's construction which has been used in several papers, for instance [9]. The other sets of 5 idempotent IMOLS(v,x) in Lemma 3.1.4 are obtainable from $(v-x, 7; 1, 1; x)$-QDMs, which come from a V$(5,6)$ vector for $(v,x) = (37,6)$, and can otherwise be found either in [1], [3] or in Lemma 2.9.

Lemma 3.1 *1. If $v \geq 66$, there exist 5 IMOLS$(v,6)$, except possibly for $v \in \{80, 81, 82, 115, 116, 117, 166\}$.*

2. If $v \geq 66$, there exist 5 IMOLS$(v,8)$.

3. If $v \geq 66$, there exist 5 IMOLS$(v,10)$, except possibly for $v \in \{68, 69, 70, 75, 76, 77, 78, 79, 99\}$.

4. There exist 5 idempotent IMOLS(v,x) for each $(v,x) \in \{(19,3), (39,3), (31,4), (39,4), (37,6), (41,6), (43,7), (50,6), (64,6), (65,6)\}$ and also for $v = 56+x$ with $x \leq 8$.

Our main methods for recursive constructions of SSPODLS are given in Lemma 3.2. Note that in Lemmas 3.2.4 and 3.2.5, existence of t SSPODLS(q) with the required extra conditions follows from Lemma 2.5 when q is a prime power $\geq t+3$.

Lemma 3.2 *1. If t SSPODLS(m) and t SSPODLS(q) exist, then t SSPODLS(mq) exist.*

2. Suppose q is an even integer. If t SSPODLS(q) and t idempotent MOLS(m) both exist, then t SSPODLS(mq) exist.

3. Suppose q is an odd integer, at most one of m, $s_1, s_2, \ldots, s_{n+2}$ is odd, and $s = \sum_{z=1}^{n+2} s_z$. Suppose also, there exist $t+n$ SSPODLS(q), t SSPODLS$(m+s)$, t MOLS(m), t idempotent IMOLS$(m+s_z, s_z)$ for $z = 1,2$ and t IMOLS$(m+s_z, s_z)$ for $3 \leq z \leq n+2$. Then t SSPODLS$(mq+s)$ exist.

4. Suppose q is an even integer, $s = \sum_{z=1}^{q} s_z$, and at most one of $s_1, s_2, \ldots, s_q$ is odd. Suppose also, there exist t SSPODLS(q) with q disjoint strongly symmetric transversals (including the main and back diagonals), t SSPODLS(s), t idempotent IMOLS$(m+s_z, s_z)$, for $z = 1,2$ and t IMOLS$(m+s_z, s_z)$ for $3 \leq z \leq q$. Then t SSPODLS$(mq+s)$ exist.

5. Suppose q is an odd integer, $s = \sum_{z=0}^{q-1} s_z$ where $s_0 = 0$ or 1, $s_z = s_{q-z}$ for $1 \leq z \leq q-1$, and at most one of m, s_0 is odd. Suppose also, there exist t SSPODLS(q) with q disjoint transversals such that one is the main diagonal and the others can be partitioned into $(q-1)/2$ strongly symmetric pairs. If there exist t SSPODLS(s), t SSPODLS$(m+s_0)$ and t idempotent IMOLS$(m+s_z, s_z)$ for $z = 0,1,2,\ldots,q-1$, then t SSPODLS$(mq+s)$ exist.

Proof of Lemma 3.2.5: The resulting t SSPODLS$(mq+s)$ will be labelled as M_ℓ for $\ell = 1,2,\ldots,t$. They will have symbols, row indices and column indices from $N = \{0,1,2,\ldots,mq+s-1\}$. We partition N into q subsets G_i (for $i = 0,1,\ldots,q-1$) of size m, plus q subsets S_z where $|S_z| = s_z$ for $0 \leq z \leq q-1$. Also let $S = \bigcup_{z=0}^{q-1} S_z$. First, for $0 \leq i < (q-1)/2$, set $G_i = \{im, im+1, \ldots, (i+1)m-1\}$, and $G_{q-1-i} = \{(q-1-i)m+s, (q-1-i)m+s+1, \ldots, (q-i)m+s-1\}$. Thus if $x \in G_i$, then

$qm + s - 1 - x \in G_{q-1-i}$. The remaining elements of N can be partitioned into sets $G_{(q-1)/2}$ and S_z (for $z = 0, 1, 2, \dots q-1$) in any manner such that if $x \in G_{(q-1)/2}$, then $qm + s - 1 - x \in G_{(q-1)/2}$, if $x \in S_0$, then $qm + s - 1 - x \in S_0$, and if $x \in S_z$ for some $z \geq 1$, then $qm + s - 1 - x \in S_{q-z}$.

Now we start with t SSPODLS(q) which are labelled as A_ℓ for $\ell = 1, 2, \dots, t$. These SSPODLS(q) will contain q disjoint common transversals, $T_0, T_1, \dots, T_{q-1}$ where T_0 is the main diagonal and for $1 \leq z \leq (q-1)/2$, T_z and T_{q-z} form a strongly symmetric pair. If $(i, j) \neq ((q-1)/2, (q-1)/2)$ and $A_\ell(i, j)$ lies in T_z, then we construct t IMOLS($m + s_z, s_z$) on an $(m + s_z) \times (m + s_z)$ subsquare of M_ℓ for $\ell = 1, 2 \dots, t$. This subsquare will be on the rows with indices from from $G_i \cup S_z$, and the columns with indices from from $G_j \cup S_z$. The holey rows and columns will have indices from from S_z. Also, if $A_\ell(i, j) = x$, then within M_ℓ, this subsquare will have non-holey symbols from G_x and holey symbols from S_z. In addition, these subsquares should be idempotent with a common holey transversal that is part of the main (back) diagonal if $A_\ell(i, j)$ lies on the main (back) diagonal of A_ℓ. Because of the way the sets G_x, S_z have been constructed, it is not hard to verify that y is a symbol (holey symbol) in this subsquare for $A_\ell(i, j)$ if and only if $qm + s - 1 - y$ is a symbol (holey symbol) in the corresponding subsquare for $A_\ell(q-1-i, q-1-j)$. Thus these subsquares of M_ℓ can (and should) be constructed so that $M_\ell(qm + s - 1 - i, qm + s - 1 - j) = qm + s - 1 - M_\ell(i, j)$.

Next, for $(i, j) = ((q-1)/2, (q-1)/2)$, $A_\ell(i, j) = (q-1)/2$ for all $\ell = 1, 2, \dots, t$, and this (i, j) cell lies in T_0. Here we construct t SSPODLS($m + s_0$), and replace each pair of non-holey symbols of the form $(x, m + s_0 - 1 - x)$ by two symbols of the form $y, mq + s - 1 - y$ from $G_{(q-1)/2}$. The value $(m + s_0 - 1)/2$ is replaced by $(mq + s - 1)/2$ if either m or s_0 is odd. If $s_0 = 1$, then S_0 will consist of the value $(mq + s - 1)/2$, while if m is odd, then $(mq + s - 1)/2$ will lie in $G_{(q-1)/2}$. In both these cases, $M_\ell((mq + s - 1)/2, (mq + s - 1)/2)$ will equal $(mq + s - 1)/2$ for all $\ell = 1, 2, \dots, t$. Insert these squares into the $(m + s_0) \times (m + s_0)$ subsquares of M_ℓ (for $\ell = 1, 2, \dots, t$) with row and column indices from $G_{(q-1)/2} \cup S_0$.

Finally construct t SSPODLS(s), but for $0 \leq x \leq (s-2)/2$, replace symbols x and $s - 1 - x$ by two symbols from S of the form $y, mq + s - 1 - y$, and, if $s_0 = 1$, replace the symbol $(s-1)/2$ by $(mq + s - 1)/2$. Then insert these t squares into the $s \times s$ subsquares of M_ℓ (for $\ell = 1, 2, \dots, t$) indexed by the rows and columns of S.

When this is done, $M_\ell, (\ell = 1, 2, \dots, t)$ are t SSPODLS($mq + s$). □

The other cases in Lemma 3.2 are similar. In Lemma 3.2.4, the values in the sets S_z (for $z = 1, 2, \dots, q$) must be chosen so that if $y \in S_z$ then $mq + s - 1 - y \in S_z$. This is because the corresponding transversals T_z are strongly symmetric, not pairwise strongly symmetric. Also, here $|S_z| = s_z$ for $1 \leq z \leq q$. In Lemma 3.2.3 we start with $t + n$ SSPODLS(q), but retain just t of them; these t SSPODLS will then have $n + 2$ strongly symmetric transversals, any two of which intersect only in the $((q-1)/2, (q-1)/2)$ cell. The first two of these transversals, T_1 and T_2 will be the main and back diagonals; the others, $T_3, T_4, \dots, T_{n+2}$, will consist of the cells occupied by the value $(q-1)/2$ in each of the n deleted SSPODLS. Here also, for $z = 1, 2, \ldots, n + 2$, the transversals T_z are strongly symmetric; therefore the corresponding sets S_z must be chosen so that if $y \in S_z$, then $mq + s - 1 - y \in S_z$.

We now turn our attention to existence of 5 SSPODLS:

Lemma 3.3 *Suppose* $8 \leq v \leq 516$*, and* $v \not\equiv 2$ *(mod 4). If* $v \notin \{12, 15, 20, 21, 28, 33, 35, 39, 44, 45, 51, 52, 60, 63, 68, 77, 84, 91, 92, 100, 108, 111, 116, 119, 120, 123, 124, 132, 133, 135, 140, 141, 148, 156, 159, 164, 172, 175, 180, 183, 188, 196, 204, 205, 212, 215, 220, 228, 236, 260, 268, 276, 292, 300\}$*, then* 5 *SSPODLS*$(v)$ *exist.*

Proof: First we consider the case $v < 147$. For v a prime power, see Lemma 2.5. For $v \in \{24, 36, 40, 48, 56, 80\}$, see Lemma 2.4, and for $v \in \{55, 69, 75, 76, 85, 87, 93, 95\}$, see Lemma 2.8. For $v = 72 = 8.9$, $88 = 8.11$, $99 = 9.11$, $104 = 8.13$, $117 = 9.13$, $136 = 8.17$, $143 = 11.13$ and $144 = 9.16$, apply Lemma 3.2.1. For $57 = 7.8 + 1$, $65 = 8.8 + 1$, and $129 = 16.8 + 1$, apply Lemma 3.2.4 with $m \in \{7, 8, 16\}$, $q = 8$ and $s = s_1 = 1$. For $96 = 8.11 + (8 = 8.1)$, $105 = 8.13 + 1$, $112 = 8.13 + (8 = 8.1)$, $115 = 8.13 + (11 = 11.1)$ and $145 = 8.17 + (9 = 9.1)$, apply Lemma 3.2.5 with $m = 8$, $q \in \{11, 13, 17\}$ and $s_z \in \{0, 1\}$ for $0 \leq z \leq q - 1$.

Now we consider $v \geq 147$. Here also, if v is a prime power, we can apply Lemma 2.5, and if v can be written as a product of two prime powers ≥ 8, we can apply Lemma 3.2.1. We shall not include details of these constructions. For the remaining values, see Table 1. Here, if v is written as a product of two integers, we use Lemma 3.2.2 when $v = 252$ or Lemma 3.2.1 otherwise. In other cases, we write $v = mq + (s = \sum s_z)$ in Table 1 and apply either Lemma 3.2.4 if q is even, or Lemma 3.2.5 if q is odd. □

Lemma 3.4 *If* $v \geq 517$ *and* $v \not\equiv 2$ *(mod 4) then* 5 *SSPODLS*(v) *exist.*

Proof: Let $S = \{24, 25, 27, 29, 31, 36\}$. For all $s \in S$, 5 SSPODLS(s) exist (by Lemma 2.4 if s is even, or by Lemma 2.5 if s is odd). Note that S contains one element in each residue class (mod 8) except the 2 and 6 (mod 8) residue classes. Further, any $s \in S$ can be written as $s = s_1 + s_2 + \ldots, s_8$ where $s_1 = 0$ or 1, $s_2 = 0$, and $s_z \in \{0, 6, 8\}$ for $3 \leq z \leq 8$. If $v \not\equiv 2$ (mod 4) and $v \geq 517 = 61 \cdot 8 + 29$, then we can write $v = 8m^* + s$ where $m^* \geq 61$ and $s \in S$. From [14], when $m^* \geq 61$, and $x \in \{0, 1\}$, there exist 5 idempotent MOLS($m^* + x$) and hence also 5 idempotent IMOLS($m^* + x, x$). Also, by Lemma 3.1, there exist 5 IMOLS($m^* + x, x$) for $m^* \geq 61$ and $x \in \{6, 8\}$ except possibly when $x = 6$ and $m^* \in E = \{74, 75, 76, 109, 110, 111, 160\}$. Thus if $m^* \notin E$, 5 SSPODLS(v) can be obtained by Lemma 3.2.4 with $m = m^*$, $q = 8$, $s \in S$, $s_1 = 0$ or 1, $s_2 = 0$, and $s_z \in \{0, 6, 8\}$ for $3 \leq z \leq 8$.

If instead m^* is one of the exceptional values in E, then from [14], there exist 5 idempotent IMOLS($m^* + x, x$) for $x \in \{0, 1\}$. Also, by Lemma 3.1, there exist 5 IMOLS($m^* + x, x$) for $x \in \{8, 10\}$. Further, any $s \in S$ can be written as $s = s_1 + s_2 + \ldots, s_8$ where $s_1 = 0$ or 1, $s_2 = 0$, and $s_z \in \{0, 8, 10\}$ for $3 \leq z \leq 8$. Thus in this case, we can apply Lemma 3.2.4 similarly with $m = m^*$, $q = 8$, $s \in S$, $s_1 \in \{0, 1\}$, $s_2 = 0$ and $s_z \in \{0, 8, 10\}$ for $3 \leq z \leq 8$. □

Table 1: Constructions for 5 SSPODLS(v) in Lemma 3.3

$147 = 8.17 + (11 = 11.1)$	$155 = 16.9 + (11 = 5.1 + 2.3)$	$160 = 16.9 + (16 = 4.1 + 4.3)$
$161 = 8.19 + (9 = 9.1)$	$165 = 8.19 + (13 = 13.1)$	$168 = 16.9 + (24 = 8.3)$
$177 = 16.11 + 1$	$185 = 16.11 + (9 = 3.1 + 2.3)$	$189 = 16.11 + (13 = 1.1 + 4.3)$
$192 = 16.11 + (16 = 4.1 + 4.3)$	$195 = 16.11 + (19 = 1.1 + 6.3)$	$201 = 16.11 + (25 = 1.1 + 8.3)$
$203 = 16.11 + (27 = 3.1 + 8.3)$	$213 = 8.25 + (13 = 13.1)$	$217 = 16.13 + (9 = 3.1 + 2.3)$
$219 = 16.13 + (11 = 5.1 + 2.3)$	$224 = 8.25 + (24 = 24.1)$	$231 = 16.13 + (23 = 5.1 + 6.3)$
$235 = 16.13 + (27 = 3.1 + 8.3)$	$237 = 16.13 + (29 = 5.1 + 8.3)$	$240 = 16.13 + (32 = 2.1 + 10.3)$
$244 = 16.13 + (36 = 12.3)$	$245 = 16.13 + (37 = 1.1 + 12.3)$	$249 = 8.31 + 1$
$252 = 7.36$	$255 = 8.29 + (23 = 23.1)$	$259 = 27.9 + (16 = 4.4)$
$264 = 8.31 + (16 = 16.1)$	$265 = 8.31 + (17 = 17.1)$	$267 = 27.9 + (24 = 6.4)$
$273 = 16.17 + 1$	$280 = 16.17 + (8 = 2.1 + 2.3)$	$284 = 31.8 + (36 = 6.6)$
$285 = 16.17 + (13 = 1.1 + 4.3)$	$287 = 31.9 + (8 = 8.1)$	$288 = 16.17 + (16 = 4.1 + 4.3)$
$291 = 16.17 + (19 = 1.1 + 6.3)$	$295 = 16.17 + (23 = 5.1 + 6.3)$	$301 = 16.17 + (29 = 5.1 + 8.3)$
$303 = 16.17 + (31 = 1.1 + 10.3)$	$305 = 16.19 + 1$	$308 = 16.17 + (36 = 12.3)$
$309 = 8.37 + (13 = 13.1)$	$312 = 16.19 + (8 = 8.1)$	$315 = 16.19 + (11 = 11.1)$
$316 = 35.8 + (36 = 6.6)$	$321 = 16.19 + (17 = 17.1)$	$324 = 9.36$
$327 = 16.19 + (23 = 5.1 + 6.3)$	$329 = 16.19 + (25 = 1.1 + 8.3)$	$332 = 8.37 + (36 = 36.1)$
$335 = 16.19 + (31 = 1.1 + 10.3)$	$336 = 8.41 + (8 = 8.1)$	$339 = 8.41 + (11 = 11.1)$
$340 = 16.19 + (36 = 12.3)$	$345 = 8.43 + 1$	$348 = 36.9 + (24 = 8.3)$
$355 = 8.41 + (27 = 27.1)$	$356 = 36.9 + (32 = 4.1 + 4.7)$	$357 = 8.43 + (13 = 13.1)$
$360 = 8.41 + (32 = 32.1)$	$363 = 8.43 + (19 = 19.1)$	$364 = 8.41 + (36 = 36.1)$
$365 = 8.41 + (37 = 37.1)$	$371 = 8.43 + (27 = 27.1)$	$372 = 36.9 + (48 = 2.3 + 6.7)$
$375 = 8.43 + (31 = 31.1)$	$380 = 8.43 + (36 = 36.1)$	$381 = 8.43 + (37 = 37.1)$
$384 = 8.43 + (40 = 40.1)$	$385 = 8.47 + (9 = 9.1)$	$388 = 44.8 + (36 = 6.6)$
$393 = 8.49 + 1$	$395 = 8.47 + (19 = 19.1)$	$396 = 11.36$
$399 = 8.47 + (23 = 23.1)$	$404 = 16.23 + (36 = 12.3)$	$405 = 8.49 + (13 = 13.1)$
$408 = 8.49 + (16 = 16.1)$	$411 = 8.49 + (19 = 19.1)$	$412 = 36.11 + (16 = 2.1 + 2.7)$
$413 = 36.11 + (17 = 3.1 + 2.7)$	$415 = 8.49 + (23 = 23.1)$	$417 = 8.49 + (25 = 25.1)$
$420 = 36.11 + (24 = 8.3)$	$427 = 16.25 + (27 = 3.1 + 8.3)$	$428 = 8.49 + (36 = 36.1)$
$429 = 8.49 + (37 = 37.1)$	$435 = 8.49 + (43 = 43.1)$	$436 = 16.25 + (36 = 12.3)$
$440 = 8.49 + (48 = 48.1)$	$444 = 36.11 + (48 = 2.3 + 6.7)$	$445 = 16.27 + (13 = 13.1)$
$447 = 16.25 + (47 = 5.1 + 14.3)$	$448 = 56.8$	$452 = 36.11 + (56 = 8.7)$
$453 = 16.25 + (53 = 5.1 + 16.3)$	$455 = 16.27 + (23 = 23.1)$	$456 = 16.25 + (56 = 2.1 + 18.3)$
$459 = 16.27 + (27 = 3.1 + 8.3)$	$460 = 8.53 + (36 = 36.1)$	$465 = 56.8 + (17 = 1.1 + 2.8)$
$468 = 16.27 + (36 = 12.3)$	$469 = 16.27 + (37 = 1.1 + 12.3)$	$471 = 56.8 + (23 = 1.7 + 2.8)$
$476 = 36.13 + (8 = 8.1)$	$480 = 56.8 + (32 = 4.8)$	$483 = 8.59 + (11 = 11.1)$
$484 = 56.8 + (36 = 1.4 + 4.8)$	$485 = 56.8 + (37 = 1.5 + 4.8)$	$489 = 56.8 + (41 = 1.1 + 5.8)$
$492 = 36.13 + (24 = 10.1 + 2.7)$	$495 = 56.8 + (47 = 1.7 + 5.8)$	$497 = 56.8 + (49 = 1.1 + 6.8)$
$500 = 58.8 + (36 = 6.6)$	$501 = 56.8 + (53 = 1.5 + 6.8)$	$504 = 56.8 + (56 = 7.8)$
$505 = 56.8 + (57 = 1.1 + 7.8)$	$507 = 56.8 + (59 = 1.3 + 7.8)$	$508 = 59.8 + (36 = 6.6)$
$511 = 36.13 + (43 = 1.1 + 6.7)$	$515 = 36.13 + (47 = 5.1 + 6.7)$	$516 = 36.13 + (48 = 6.1 + 6.7)$

We now turn our attention to existence of 4 SSPODLS(v):

Lemma 3.5 *If $7 \le v \le 300$ and $v \not\equiv 2 \pmod 4$ then there exist 4 SSPODLS(v), except possibly for $v \in \{20, 21\}$.*

Proof: Here we only need to deal with the exceptional values of v in Lemma 3.3. First we consider the case $v < 111$. For v a prime power, this follows by Lemma 2.5.

For $v \in \{12, 28, 44, 52, 60\}$, see Lemma 2.4. For $v \in \{15, 33, 35, 39, 45, 51, 68\}$, see Lemma 2.7, and for $v = 100$, see Lemma 2.6. For $v = 63 = 7.9$, $77 = 7.11$, $84 = 7.12$, $91 = 7.13$ and $108 = 9.12$, apply Lemma 3.2.1. For $v = 92$, apply Lemma 3.2.3 with $m = 8$, $q = 11$, $n = 2$, $s_1 = s_2 = 0$ and $s_3 = s_4 = 2$.

Constructions for $v \geq 111$ are given in Table 2. When v is given as a product of two integers, we use Lemma 3.2.1. In all other cases, we write $v = mq + s$, and apply either Lemma 3.2.4 when q is even, or Lemma 3.2.5 when q is odd.

The required sets of 4 IMOLS$(m + x, x)$ with $(m, x) \neq (8, 2)$ and $x > 1$ were all obtained from $(m, 6; 1, 1; x)$-QDMs, and are idempotent whenever $m + x > 5x$. References for existence of these QDMs are given in Table 3. When $x \in \{0, 1\}$, existence of 4 idempotent IMOLS$(m+x, x)$ is equivalent to existence of 4 idempotent MOLS$(m + x)$. Finally, 4 IMOLS$(10, 2)$ can be found in [14, p194]. □

Table 2: Constructions for 4 SSPODLS(v) in Lemma 3.5

$111 = 13.8 + (7 = 1.1 + 3.2)$	$116 = 13.8 + (12 = 6.2)$	$119 = 7.17$
$120 = 15.8$	$123 = 16.7 + (11 = 5.1 + 2.3)$	$124 = 16.7 + (12 = 4.3)$
$132 = 11.12$	$133 = 7.19$	$135 = 9.15$
$140 = 8.16 + (12 = 6.2)$	$141 = 8.16 + (13 = 1.1 + 6.2)$	$148 = 17.8 + (12 = 6.2)$
$156 = 12.13$	$159 = 16.9 + (15 = 3.1 + 4.3)$	$164 = 19.8 + (12 = 3.4)$
$172 = 20.8 + (12 = 6.2)$	$175 = 7.25$	$180 = 19.8 + (28 = 7.4)$
$183 = 16.11 + (7 = 1.1 + 2.3)$	$188 = 16.11 + (12 = 4.3)$	$196 = 8.23 + (12 = 12.1)$
$204 = 12.17$	$205 = 12.17 + 1$	$212 = 8.25 + (12 = 12.1)$
$215 = 8.25 + (15 = 15.1)$	$220 = 16.13 + (12 = 12.1)$	$228 = 8.27 + (12 = 12.1)$
$236 = 16.13 + (28 = 4.1 + 8.3)$	$260 = 8.29 + (28 = 28.1)$	$268 = 15.16 + (28 = 14.2)$
$276 = 8.31 + (28 = 28.1)$	$292 = 12.23 + (16 = 16.1)$	$300 = 17.16 + (28 = 14.2)$

Table 3: References for existence of 4 idempotent IMOLS$(m + x, x)$ in Lemma 3.5

m	x	Reference	m	x	Reference	m	x	Reference
13	2	[8]	15	2	[1]	16	3	[1]
17	2	[9]	19	4	[8]	20	2	[4]

Finally there exist 3 SSPODLS(v), for the missing values of v in Lemma 3.5:

Lemma 3.6 *If $v \in \{20, 21\}$ then 3 SSPODLS(v) exist.*

Proof: See Lemma 2.6. □

4 Conclusion

In this paper, by using direct construction methods based on quasi-difference matrices together with certain recursive methods, some new existence results for existence of

t SSPODLS have been obtained for $t \in \{3, 4, 5\}$. These results are summarised in the following theorem:

Theorem 4.1 *If $v \equiv 2 \pmod 4$, then a strongly symmetric Latin square of order v cannot exist. Also, there do not exist 3 or 4 SSPODLS(v) for $v \leq 5$, or 5 SSPODLS(v) for $v \leq 7$. For larger values of $v \not\equiv 2 \pmod 4$, and $t \in \{3, 4, 5\}$, there exist t SSPODLS(v) in the following cases:*

$t = 3$ and $v \geq 7$;

$t = 4$, $v \geq 7$ and $v \notin \{20, 21\}$;

$t = 5$, $v \geq 8$, and $v \notin$ {12, 15, 20, 21, 28, 33, 35, 39, 44, 45, 51, 52, 60, 63, 68, 77, 84, 91, 92, 100, 108, 111, 116, 119, 120, 123, 124, 132, 133, 135, 140, 141, 148, 156, 159, 164, 172, 175, 180, 183, 188, 196, 204, 205, 212, 215, 220, 228, 236, 260, 268, 276, 292, 300}.

For 4 SSPODLS, only two unknown cases ($v = 20$ and 21) remain; however, both these values are relatively small and there is currently no obvious approach for finding these SSPODLS. For 5 SSPODLS, it seems likely that some improvements can be made. These may come from some new recursive methods; perhaps also Lemma 2.1 can be improved to produce new SSPODLS (or even PODLS) from QDMs over odd order groups when the number of infinite points is even. Also, it seems likely that 5 SSPODLS(92) can be obtained by constructing an $(84, 7; 1, 1; 8)$-QDM* over $Z_2 \times Z_{42}$ with appropriate pairwise matching columns using the method in Lemma 2.8. However, most of the largest unknown cases for 5 SSPODLS(v) are for $v \equiv 4 \pmod 8$; for this residue class, it is likely that more small examples of 5 SSPODLS(v) with $v \equiv 4 \pmod 8$ will be needed. Currently 36 is the only value of $v \leq 60$ in the 4 (mod 8) residue class for which 5 SSPODLS(v) (or even 5 idempotent MOLS(v)) are known.

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