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Constructions for well-covered graphs

A. S. Finbow

Saint Mary's University
Halifax, N.S. B3H 3C3
Canada
art.finbow@smu.ca

C. A. Whitehead

22 Leyfield Road
Sheffield, S17 3EE
U.K.
cawhitehead@uwclub.net

Abstract

A well-covered graph G is one in which every maximal independent set of vertices has the same size. Given a well-covered graph G (not necessarily connected), we establish necessary and sufficient conditions for both adding edges to and deleting edges from G in such a manner that the graph so obtained is also well-covered and has the same independence number as G. In addition, we establish a method for constructing a well-covered supergraph of G by adjoining a new vertex and its neighbourhood set to G.

1 Introduction

The independence number of a finite simple graph G with vertex set $V(G)$ and edge set $E(G)$ is the maximum cardinality of an independent set in $V(G)$ and denoted by $\beta(G)$. A graph G is called *well-covered* if every maximal independent set in G has cardinality $\beta(G)$. Clearly G is well-covered if and only if every connected component of G is well-covered. The girth of a graph containing a cycle is the length of its shortest cycle and is denoted by $g(G)$; when G is acyclic $g(G)$ is defined to be infinity. All graphs considered will be simple and finite.

The objective of this study is to establish a constructive procedure for obtaining all well-covered graphs starting from a collection of known well-covered graphs. In Section 3 we prove necessary and sufficient conditions for both adding edges to and deleting edges from a well-covered graph G (not necessarily connected) to give a well-covered graph H with $\beta(H) = \beta(G)$ and present a number of constructions

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based on these results. In Section 4 we establish a method of adjoining a new vertex and its neighbourhood set to a well-covered graph G to give a well-covered graph H with $\beta(H) = \beta(G) + 1$. Of fundamental importance to all our constructions in Section 3 is the recognition of certain distinguished sets of vertices in G, which we call *extendable sets*. Although some extendable sets are easily identified, for example the open neighbourhood set of any independent set of vertices in a graph is an extendable set, finding all the extendable sets in an arbitrary graph would seem to be a difficult problem. Properties of extendable sets are established in Section 2.

We have used the results in this paper to construct over 200 non-isomorphic well-covered connected graphs G with $\beta(G) \leq 4$ that are triangle-free, starting from a collection of K_1s and K_2s. Of these, just seventeen graphs G have $\beta(G) \leq 3$ and we illustrate these, together with those with $\beta(G) = 4$ and $|V(G)| \leq 9$, in an appendix. Each of these latter graphs can be produced in several different ways, either by using a construction established in Section 3 or by using the construction of Section 4.

Well-covered graphs were introduced by M. D. Plummer [19] in 1970. The corresponding recognition problem was found in the early '90's to be co-NP-complete independently by V. Chvátal and P. J. Slater [4] and by R. S. Sankaranarayana and L. K. Stewart [23]. Recognition is, however, known to be polynomial for certain classes of well-covered graphs, for instance, those that are bipartite [22], claw-free [25, 24], have girth at least 5 [6], have neither 4-cycles nor 5-cycles [7], are chordal [21], are of bounded degree [3], are planar quadrangulations [8] or are planar triangulations [9, 10, 11, 12]. There are surveys of early results regarding well-covered graphs by B. Hartnell [13] and M. D. Plummer [20].

An *extendable vertex* in a well-covered graph was first defined in [6].

Definition 1.1 A vertex $x \in V(G)$ in a well-covered graph G is called *extendable* if $G - x$ is well-covered and $\beta(G) = \beta(G - x)$.

Extendable vertices were used in [6] and [7] in the construction of families of well-covered graphs. The complementary notion, that of a vertex that is not extendable, was used by S. L. Gasquoine et al. in [15] in establishing constructions for well-covered graphs with no 4-cycles. The structure of these latter graphs was further investigated by B. Hartnell in [14]. In [16, 17, 18] M. Pinter established constructions for building 1-well-covered graphs (well-covered graphs such that each of their vertices is extendable). These constructions were generalized and additional constructions found for these graphs by A. Finbow and B. Hartnell in [5].

We use the following notation. For $u, v \in V(G)$, let $d_G(u, v)$ denote the length of a shortest u-v path in G. Then $N_G(v) := \{x \in V(G) : d_G(x, v) = 1\}$ and $N_G[v] := N_G(v) \cup \{v\}$. More generally, $N^r_G[v] := \{x \in V(G) : d_G(x, v) \leq r\}$. Similarly, for any set $S \subseteq V(G)$, $N_G(S) = (\cup_{v \in S} N_G(v)) \setminus S$ and $N^r_G[S] = \cup_{v \in S} N^r_G[v]$. Let $X, Y \subseteq V(G)$. Then $E_G(X, Y) := \{xy \in E(G) : x \in X, y \in Y\}$. Finally, we denote by $\langle X \rangle$ the subgraph of G generated by the vertices of the set $X \subseteq V(G)$.

We make repeated use of the following well-known observation (see for example [1]).

Lemma 1.2 *Let G be a well-covered graph and let $v \in V(G)$. Then $H := G - N_G[v]$ is also well-covered and $\beta(H) = \beta(G) - 1$. Suppose J is an independent set in G. Then by extension, $H' := G - N_G[J]$ is also well-covered and $\beta(H') = \beta(G) - |J|$.*

It is useful to expand the definition of *extendable vertex* by allowing such a vertex to reside in a non-well-covered graph.

Definition 1.3 A vertex $x \in V(G)$ in a graph G is called *extendable* if every maximal independent set I in $G - x$ contains a vertex of $N_G(x)$.

Note that $x \in V(G)$ is extendable in this sense if and only if $|N_G(x) \setminus N_G(I)| \geq 1$, for every maximal independent set I in $G - N_G[x]$.
We note that in the case when G is a well-covered graph, Definition 1.3 is equivalent to Definition 1.1.

Definition 1.4 A set $S \subseteq V(G)$ in a graph G is called an *extendable set* in G if $S \subseteq N_G(J)$ for every maximal independent set J in $G - S$.

Clearly, an extendable set in a component of a graph G is also an extendable set in G. Further, the empty set is an extendable set in any graph.

Example 1.1 Each of the 4-cycle C_4 and the 5-cycle C_5 is an example of a well-covered graph G with $\beta(G) = 2$. $C_5 := x_0x_1x_2x_3x_4$ contains 16 extendable sets: $\emptyset$, $\{x_i\}$, $\{x_i, x_{i+2}\}$ and $\{x_i, x_{i+2}, x_{i+3}\}$, for $i \in \{0, 1, 2, 3, 4\}$ (where addition on indices is modulo 5); while $C_4 := x_0x_1x_2x_3$ contains just three extendable sets: $\emptyset$, $\{x_0, x_2\}$ and $\{x_1, x_3\}$. The path $P_5 := x_0x_1x_2x_3x_4$ is not well-covered. It contains the following 9 extendable sets: $\emptyset$, $\{x_1\}$, $\{x_3\}$, $\{x_0, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_0, x_2, x_3\}$, $\{x_0, x_2, x_4\}$, $\{x_1, x_2, x_4\}$.

In the case when G is well-covered, Definition 1.4 implies the following.

Lemma 1.5 *Let G be a well-covered graph. A set $S \subseteq V(G)$ is an extendable set in G provided that $G - S$ is well-covered and $\beta(G - S) = \beta(G)$. Conversely, if $A \subseteq V(G)$ is not an extendable set in G, then $G - A$ contains a maximal independent set J with $|J| < \beta(G)$. In this case, either $G - A$ is not well-covered, or it is well-covered but $\beta(G - A) < \beta(G)$.*

2 Properties of extendable sets

In this section, we establish some properties of extendable sets and show how they may be constructed. In particular, our results show that every connected graph of order at least 4 contains many extendable sets.

Lemma 2.1 *Let G be a graph containing a set S of vertices with the property that $S = N_G(J)$ for some independent set J in G. Then S is an extendable set in G.*

Proof. Note that in $G - S$, every vertex of J is an isolate. Let I be any maximal independent set in $G - S$. Then $J \subseteq I$ and the result follows. $\square$

Lemma 2.2 *Let S be an extendable set in a graph G and let J be an independent set in $G - S$. Then $S \setminus N_G(J)$ is an extendable set in $H := G - N_G[J]$.*

Proof. Let I be a maximal independent set in $H - (S \setminus N_G(J))$. We show that $S \setminus N_G(J) \subseteq N_H(I)$. Note first that $I \cup J$ is a maximal independent set in $G - S$. But S is an extendable set in G and hence $S \subseteq N_G(I \cup J)$, implying $S \setminus N_G(J) \subseteq N_G(I)$. Then $V(H) \cap (S - N_G(J)) \subseteq V(H) \cap N_G(I)$. However $S \setminus N_G(J) \subseteq V(H)$ and so we have $S \setminus N_G(J) \subseteq V(H) \cap N_G(I) \subseteq N_H(I)$, as required. $\square$

Lemma 2.3 *Let S_1, S_2 be extendable sets in a graph G with $N_G[S_1] \cap S_2 = \emptyset$. Then $S_1 \cup S_2$ is an extendable set in G.*

Proof. Let J be a maximal independent set in $G - (S_1 \cup S_2)$. We show that $S_1 \cup S_2 \subseteq N_G(J)$. Let I be a maximal independent set in $\langle S_2 \setminus N_G(J) \rangle$, where $I = \emptyset$ if $S_2 \subseteq N_G(J)$. Then $J \cup I$ is a maximal independent set in $G - S_1$. Hence $S_1 \subseteq N_G(J \cup I)$ implying $S_1 \subseteq N_G(J)$, since $N_G[S_1] \cap S_2 = \emptyset$. Similarly $S_2 \subseteq N_G(J)$, giving the result. $\square$

Corollary 2.4 *Let G be a graph and let $I \subseteq V(G)$ be an independent set of vertices each of which is extendable in G. Then I is an extendable set in G.*

On the other hand, a vertex of an extendable set is not necessarily individually extendable. Indeed, it is possible that no vertex of an extendable set S is extendable even when S is independent (see for example C_4 in Example 1.1).

Lemma 2.5 *Let G be a graph and let $U \subseteq V(G)$ have the property that $U = N_G(J)$ for some independent set J in G. Let S be an extendable set in G such that $S \cap J = \emptyset$. Then $S \cup U$ is an extendable set in G.*

Proof. Let I be a maximal independent set in $G - (U \cup S)$. Then $J \subseteq I$, since J is a set of isolates in $N_G(U \cup S)$. Hence $I \setminus J$ is a maximal independent set in $G - N_G[J]$ and hence $S \setminus U \subseteq N_G(I \setminus U)$. But $U \subseteq N_G(J)$ and so $S \cup U \subseteq N_G(I)$. $\square$

Lemma 2.6 *Let G be a well-covered graph with $g(G) \geq 4$ and let $x, y \in V(G)$ where $xy \in E(G)$. Let $N_G(y) \setminus \{x\}$ be an extendable set in $G - N_G[x]$. Then $N_G(x) \setminus \{y\}$ is an extendable set in G.*

Proof. Note that by the girth restriction, y has no neighbour in $N_G(x)$. Let $S_1 := N_G(y) \setminus \{x\}$, $S_2 := N_G(x) \setminus \{y\}$ and let $H := G - N_G[x]$. By hypothesis, S_1 is an extendable set in H. Let J be a maximal independent set in $G - S_2$. Since x is a leaf in $G - S_2$, exactly one of x and y is in J. Clearly $S_2 \subseteq N_G(J)$ when $x \in J$ and so suppose $y \in J$. Then $J \setminus \{y\}$ is a maximal independent set in

$H - S_1$. But S_1 is an extendable set in H and hence $H - S_1$ is well-covered with $\beta(H - S_1) = \beta(H) = \beta(G) - 1$. But then $|J| = \beta(G)$ so that J is a maximal independent set in G, again giving $S_2 \subseteq N_G(J)$. Hence S_2 is extendable in G. □

The following counter-example shows the condition that G is well-covered in Lemma 2.6 is necessary.

Example 2.1 Let $C := ystu$ be a 4-cycle and construct the graph G by adjoining to C the vertices a, x and the edges ax, xy. Then $\beta(G) = 3$, but G is not well-covered. With the notation of Lemma 2.6, H is the path stu, $S_1 := \{s, u\}$ and $S_2 := \{a\}$. Now choose $J := \{y, t\}$. Then J is maximal independent in $G - S_2$, but $S_2 \nsubseteq N_G(J)$.

Lemma 2.7 *Let G be a well-covered graph with $g(G) \geq 4$ and let $x, y \in V(G)$ be such that $xy \in E(G)$. Then $N_G(y) \setminus \{x\}$ is an extendable set in $G - N_G[x]$ if and only if $N_G(y) \setminus \{x\}$ is an extendable set in G.*

Proof. By the girth restriction, y has no neighbour in $N_G(x)$. Let $S := N_G(y) \setminus \{x\}$ and let $H := G - N_G[x]$. Then H is well-covered with $\beta(H) = \beta(G) - 1$.

Suppose first that S is an extendable set in H. Then $\beta(H - S) = \beta(H) = \beta(G) - 1$. Let J be a maximal independent set in $G - S$. Since y is a leaf in $G - S$, just one of x or y is in J. If $y \in J$, then clearly $S \subseteq N_G(J)$. Otherwise, $x \in J$ and hence $J \setminus \{x\}$ is a maximal independent set in $H - S$ so that again $S \subseteq N_G(J)$. Thus S is an extendable set in G.

The converse follows from Lemma 2.2, with $J := \{x\}$ and $S := N_G(y) \setminus \{x\}$. □

Lemma 2.8 *Let G be a well-covered graph with $g(G) \geq 4$ and let $x, y \in V(G)$ be such that $xy \in E(G)$. Then $N_G(\{x, y\})$ is an extendable set in G if and only if $N_G(x) \setminus \{y\}$ is an extendable set in G.*

Proof. Let $S_1 := N_G(x)\setminus\{y\}$, $S_2 := N_G(y)\setminus\{x\}$. By the girth restriction, $S_1 \cap S_2 = \emptyset$. Then $N_G(\{x, y\}) = S_1 \cup S_2$.

Suppose S_1 is an extendable set in G. Then $H := G - S_1$ is well-covered with $\beta(H) = \beta(G)$. By Lemma 2.6 and Lemma 2.7, S_2 is an extendable set in G. Let J be a maximal independent set in $G - (S_1 \cup S_2)$. Then J contains exactly one of x and y, say $y \in J$. Then J is a maximal independent set in H and hence $|J| = \beta(G)$. Thus $G - (S_1 \cup S_2)$ is well-covered and $S_1 \cup S_2$ is an extendable set in G.

Conversely, suppose that $S_1 \cup S_2$ is an extendable set in G. By Lemma 2.2, $(S_1 \cup S_2) \setminus N_G(y) = S_1$ is extendable in $G - N_G[y]$. But then by Lemma 2.6, S_2 is extendable in G. □

3 Constructions by adding or deleting edges in a well-covered graph

There are a number of instances of a well-covered graph G remaining well-covered with the same independence number when a set of edges is deleted from G or added to G. Proposition 3.1 together with Proposition 3.10 characterize the set of edges that can be respectively added to or removed from a well-covered graph G to obtain a distinct well-covered graph with the same independence number. From these propositions we deduce several constructions, some of which have been used in previous work.

Proposition 3.1 *Let G be a well-covered graph and let $A_1, A_2 \subseteq V(G)$ with $N_G[A_1] \cap A_2 = \emptyset$. Construct a new graph H from G with $V(H) := V(G)$ and $E(H) := E(G) \cup E_H(A_1, A_2)$. Then H is well-covered with $\beta(H) = \beta(G)$ if and only if the edges in $E_H(A_1, A_2)$ are chosen so that the following condition is satisfied.*

(1) *For $i, j \in \{1, 2\}$, whenever $B \subseteq A_i$ is an independent set in G, then $N_H(B) \cap A_j$ $(i \neq j)$ is an extendable set in $G - N_G[B]$.*

Furthermore (i) *any subset $S \subseteq V(G) \setminus N_G[A_1 \cup A_2]$ that is extendable in G is also extendable in H; and* (ii) *any subset $T \subseteq V(G) \setminus (N_G^2[A_1] \cup A_2)$ that is extendable in H is extendable in G.*

Proof. Suppose condition (1) is satisfied. Let J be a maximal independent set in H and let $J_i := J \cap A_i$ and $S_i := N_H(J_i) \cap A_j$, for $i, j \in \{1, 2\}$, $i \neq j$. Set $J_0 := J \setminus (J_1 \cup J_2)$. By condition (1), S_i is extendable in $G - N_G[J_i]$, $i \in \{1, 2\}$. Then by Lemma 2.2, $R_i := S_i \setminus N_G[J_1 \cup J_2]$ is extendable in $G' := G - N_G[J_1 \cup J_2]$, $i \in \{1, 2\}$. However, by Lemma 2.3, $R_1 \cup R_2$ is extendable in G'. But J is a maximal independent set in $G - (S_1 \cup S_2)$. Hence J_0 is a maximal independent set in $G' - (R_1 \cup R_2)$ so that $R_1 \cup R_2 \subseteq N_G(J_0)$. Thus $S_1 \cup S_2 \subseteq N_G(J)$, so that J is a maximal independent set in G and hence $|J| = \beta(G)$. Since J is an arbitrary maximal independent set in H, this implies H is well-covered with $\beta(H) = \beta(G)$.

Conversely, suppose condition (1) does not hold. Then without loss of generality, there exists an independent set $I \subseteq A_1$ such that $S := N_H(I) \cap A_2$ is not extendable in $G - N_G[I]$. This implies there is a maximal independent set J in $G - (N_G[I] \setminus S)$ such that $S \setminus N_G(J) \neq \emptyset$. Hence the independent set $I \cup J$ is not maximal in G, but is maximal in H. We conclude that either H is not well-covered, or H is well-covered with $\beta(H) < \beta(G)$.

To prove (i), suppose that $S \subseteq V(G) \setminus N_G[A_1 \cup A_2]$ is extendable in G and let J be a maximal independent set in $H - S$. Extend J to a maximal independent set K in $G - S$. Then $S \subseteq N_G[K]$, since S is extendable in G. But $(K \setminus J) \subseteq A_1 \cup A_2$ and hence $S \subseteq N_H[J]$ so that S is extendable in H.

To prove (ii), suppose that $T \subseteq V(G) \setminus (N_G^2[A_1] \cup A_2)$ is extendable in H and let I be a maximal independent set in $G - T$. Now $U := A_1 \cap N_H(I \cap A_2)$ is extendable in G, by hypothesis. Thus if we extend $I \setminus U$ to a maximal independent set L in $G - U$, then $U \subseteq N_G(L)$. Hence L is a maximal independent set in G and by construction is also a maximal independent set in H. But $L \setminus I \subseteq N_G(U) \subseteq N_G[A_1]$ and hence $T \cap L = \emptyset$. But since T is extendable in H, $T \subseteq N_H[L]$. But $L \cap (V(G) \setminus N_G[A_1]) = I \cap (V(G) \setminus N_G[A_1])$ and so $T \subseteq N_G[I]$. Hence T is an extendable set in G. □

Corollary 3.2 through Corollary 3.5 are applications of Proposition 3.1.

Corollary 3.2 *Let G be a well-covered graph containing two extendable sets S_1, S_2 with $N_G[S_1] \cap S_2 = \emptyset$. Then the graph $H := G + \{xy : x \in S_1, y \in S_2\}$ is well-covered with $\beta(H) = \beta(G)$. Further, if z is an extendable vertex in G such that $N_G(z) \cap (S_1 \cup S_2) = \emptyset$, then z is also extendable in H.*

The next result was used in [6] and [7] in the construction of families of well-covered graphs.

Corollary 3.3 *Let G be a well-covered graph containing a set S of independent extendable vertices. Let $x, y \in S$. Then $H := G + xy$ is well-covered with $\beta(H) = \beta(G)$. Further, every vertex of S is also extendable in H.*

Note that in general the deletion of an edge joining a pair of extendable vertices in a well-covered graph does not result in a well-covered graph: consider for example C_5.

Corollary 3.4 *Let G be a well-covered graph and let x_1, x_2 be a pair of non-adjacent extendable vertices in G. Suppose further that $S_i \subseteq N_G(x_i)$, for $i = 1, 2$, is an extendable set such that $N_G[S_1] \cap S_2 = \emptyset$. Then $H := G + x_1x_2 + \{uv : u \in S_1, v \in S_2\}$ is well-covered and $\beta(H) = \beta(G)$. Further, if z is an extendable vertex of G that is not adjacent to any vertex of $\{x_1, x_2\} \cup S_1 \cup S_2$, then z is also extendable in H.*

Corollary 3.5 *Let G be a well-covered graph and let x_1, x_2 be a pair of extendable vertices in G such that $d_G(x_1, x_2) \geq 4$. Then $H := G + x_1x_2 + \{uv : u \in N_G(x_1), v \in N_G(x_2)\}$ is well-covered and $\beta(H) = \beta(G)$.*

In the case when the sets A_1, A_2 are in distinct components of G, Proposition 3.1 becomes the following result. Both the construction by S. R. Campbell et al. of their family W in [2] and the O-join operation of A. S. Finbow et al. in [12] can be viewed as applications of this result.

Proposition 3.6 *Let G be a graph with two distinct well-covered components G_1, G_2 and let $A_i \subseteq V(G_i)$, $i = 1, 2$. Construct a new graph H from G with $V(H) := V(G)$ and $E(H) := E(G) \cup E_H(A_1, A_2)$. Then H is well-covered with $\beta(H) = \beta(G)$ if and only if the edges in $E_H(A_1, A_2)$ are chosen to satisfy the following condition.*

(2) *For $i, j \in \{1, 2\}$, whenever $B \subseteq A_i$ is an independent set in G_i, then $N_H(B) \cap A_j$ $(i \neq j)$ is an extendable set in G_j.*

Furthermore (i) *any subset $S \subseteq V(G) \setminus N_G[A_1 \cup A_2]$ that is extendable in G is also extendable in H; and* (ii) *any subset $T \subseteq V(G) \setminus (A_1 \cup A_2)$ that is extendable in H is also extendable in G.*

Definition 3.7 Let H be a connected well-covered graph. We shall say that H is *composite* if it is possible to construct H by adding edges to a well-covered graph G, where G has two components and $\beta(G) = \beta(H)$. Otherwise, we say that H is *prime.* We define the graphs K_1 and K_2 to be prime.

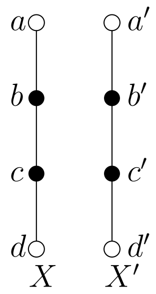


Figure 1

Example 3.1 In Figure 1, the solid vertices are those which are individually extendable in the respective graphs. Let G be the graph with components X, X', where $\beta(X) = \beta(X') = 2$. X contains the following extendable sets: $\{b\}$, $\{c\}$, $\{b, c\}$, $\{a, c\}$, $\{b, d\}$, and similarly for X'. Adding edges between X and X' so that condition (2) of Proposition 3.6 is satisfied and the graph obtained has girth at least 4 will produce a composite graph isomorphic to one of the well-covered graphs H_i, with $\beta(H_i) = 4$, $i = 12, 13, \ldots, 18$, shown in Table 2 in the Appendix.

Corollary 3.8 *Let G be a well-covered graph containing a complete multipartite subgraph G_0 with partition $(X_1, X_2, \ldots, X_r)$ such that X_i is an extendable set in G, for $i \in \{1, 2, \ldots, r\}$. Let $T \cong K_r$, where $V(T) := \{u_1, u_2, \ldots, u_r\}$. Construct a new graph H from the disjoint union of G and T by adding the edge set $F := \{u_1 x : x \in X_1\} \cup \{u_2 x : x \in X_2\} \cup \ldots \cup \{u_r x : x \in X_r\}$. Then H is well-covered with $\beta(H) = \beta(G) + 1$. Further, when $r = 2$, then $g(H) \geq 4$ whenever $g(G) \geq 4$.*

Corollary 3.9 *Let G_1, G_2 be well-covered graphs and let S_i be an extendable set in G_i, for $i = 1, 2$. Construct a new graph H from G_1 and G_2 with $V(H) := V(G_1) \cup V(G_2)$ and $E(H) := E(G_1) \cup E(G_2) \cup F$, where F is the set of edges obtained by joining each vertex of S_1 to every vertex of S_2. Then H is well-covered with $\beta(H) = \beta(G_1) + \beta(G_2)$.*

We now turn our attention to deleting edges from a well-covered graph.

Proposition 3.10 *Let H be a well-covered graph and let $A_1, A_2 \subseteq V(H)$ be such that $A_1 \cap A_2 = \emptyset$. Then $G := H - E(A_1, A_2)$ is well-covered with $\beta(G) = \beta(H)$ if and only if the following conditions are satisfied.*

(1) *For $i, j \in \{1, 2\}$, whenever $B \subseteq A_i$ is an independent set in G, then $N_H(B) \cap A_j$ $(i \neq j)$ is an extendable set in $G - N_G[B]$;*

(3) *$G - N_G[X]$ is well-covered, for any independent set $X \subseteq A_2$.*

Proof. Suppose conditions (1) and (3) hold. Let J be a maximal independent set in G. Let $J_2 := J \cap A_2$ and $S_2 := N_H(J_2) \cap A_1$. Set $M' := J \setminus S_2$ and extend M' to a maximal independent set M in $G - S_2$. Then, since $S_2 \cap J_2 = \emptyset$ we have (*a*) $M \cap J_2 = J \cap J_2 = J_2$.

Note that M is independent in H and that M is a maximal independent subset in $G - (N_G(J_2) \cup S_2)$. Since by condition (1), S_2 is extendable in $G - N_G[J_2]$ it follows that $S_2 \subseteq N_G[M \setminus J_2] \subseteq N_G[M]$. Then $S_2 \subseteq N_H[M]$ and so M is a maximal independent set in H and hence

(*b*) $|M| = \beta(H)$.

Note that both $M \setminus J_2$ and $J \setminus J_2$ are maximal independent sets in $G - N_G[J_2]$. However, $G - N_G[J_2]$ is well-covered by condition (3). Hence

$$|M \setminus J_2| = |J \setminus J_2|.$$

Then equation (a) gives $|J| = |M|$. Thus, since J is an arbitrarily chosen independent set, G is well-covered and $\beta(G) = \beta(H)$, from equation (b).
The converse follows from Proposition 3.1 and Lemma 1.2. □

In the case when the sets A_1, A_2 in the graph G constructed in Proposition 3.10 are sufficiently far apart, as when they are in distinct components of G for example, we can use the following modified version of Proposition 3.10 in which a distance constraint on the sets A_1, A_2 replaces condition (3).

Proposition 3.11 *Let H be a well-covered graph and $A_1, A_2 \subseteq V(H)$ be such that $N^3_G[A_1] \cap A_2 = \emptyset$ in $G := H - E(A_1, A_2)$. Then G is well-covered and $\beta(G) = \beta(H)$ if and only if the following condition is also satisfied.*

(1) *For $i, j \in \{1, 2\}$, whenever $B \subseteq A_i$ is an independent set in G, then $N_H(B) \cap A_j$ $(i \neq j)$ is an extendable set in $G - N_G[B]$.*

Proof. Suppose condition (1) holds. Let J be a maximal independent set in G. For $i \in \{1, 2\}$, set $J_i := J \cap A_i$, $S_i := N_H(J_i) \cap A_j$ and $K_i := S_i \cap J_j$, $i \neq j$. Let $M'_i := J \setminus K_i$ and extend M'_i to a maximal independent set M_i in $G - S_i$ by adding a set U_i of α_i vertices. Then for $i \in \{1, 2\}$, we have

(*a*) $|M_i| = |J| - |K_i| + \alpha_i$.

Note that for $i = 1, 2$, M_i is both an independent set in H and a maximal independent set in $G - (N_G(J_2) \cup S_2)$. Since, by condition (1), S_i is extendable in $G - N_G[J_i]$, it follows that $S_i \subseteq N_G[M_i] \subseteq N_H[M_i]$. Then $S_i \subseteq N_H[M_i]$ and so M_i is a maximal independent set in H and hence $|M_i| = \beta(H)$. Then for $i \in \{1, 2\}$, equation (a) yields

(*b*) $|J| - |K_i| + \alpha_i = \beta(H)$.

Note that all the vertices in U_i are adjacent to vertices in $K_i \subseteq A_i$ and hence since $N^3_G[A_1] \cap A_2 = \emptyset$, we have $U_1 \cap N_G[U_2] = \emptyset$. Hence $M := (J \cup U_1 \cup U_2) \setminus (K_1 \cup K_2)$ is independent in G and in H. We claim that M must also be maximal in H. Indeed if $x \notin N_H[M]$, then by the maximality of M_i, for both $i = 1$ and $i = 2$ it follows that $x \in N_G[K_j]$, $i \neq j$, and so $x \in N_G[K_1] \cap N_G[K_2] \subseteq N_G[A_1] \cap N_G[A_2] = \emptyset$. Thus M is maximal in H. It follows that (*c*) $|M| = |J| - |K_1| - |K_2| + \alpha_1 + \alpha_2 = \beta(H)$.

But (b) and (c) together yield $\alpha_i = |K_i|$, $i \in \{1, 2\}$, and hence $|J| = \beta(H)$. Thus G is well-covered with $\beta(G) = \beta(H)$.
The converse follows from Proposition 3.1. □

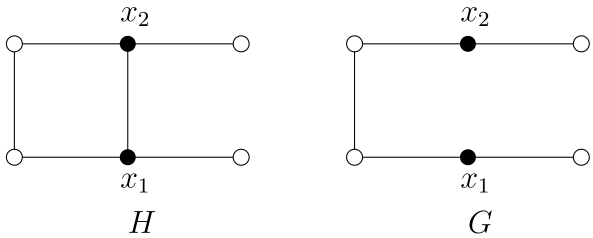


Figure 2

The graphs H and G in Figure 2 illustrate the necessity of the condition in Proposition 3.11 that $N^3_G[A_1] \cap A_2 = \emptyset$ in $G := H - E(A_1, A_2)$. Note that H is well-covered and the sets $\{x_i\}$, $i = 1, 2$, are each extendable in G. However, the graph G is not well-covered.

4 Constructions adding a new neighbourhood set to a well-covered graph

As we have noted (Lemma 1.2), a well-covered graph H with independence number β can always be reduced to a well-covered graph G with independence number $\beta - 1$ by removing the closed neighbourhood set $N_H[v]$ of an arbitrary vertex $v \in V(H)$. In this section, we give in Proposition 4.1 a necessary and sufficient condition for this process to be reversed to produce a well-covered graph H from a well-covered graph G. This result provides in theory a constructive procedure for obtaining all well-covered graphs starting from the empty graph K_0. We also note that many well-covered graphs, including for example K_n for all $n \geq 1$, C_5 and C_7, are prime and can only be constructed using the method of Proposition 4.1.

Proposition 4.1 *Let G be a well-covered graph and let T be a graph with $V(T) := \{v\} \cup U$, where $U := N_T(v) \neq \emptyset$. Let H be a graph constructed from the disjoint union of G and T by adjoining a set of edges between U and G. Then H is well-covered with $\beta(H) = \beta(G) + 1$ if and only if the following condition is satisfied.*

(a) *$H - N_H[u]$ is well-covered with $\beta(H - N_H[u]) = \beta(G)$, for all $u \in U$.*

Proof. Suppose first that condition (a) is satisfied and let J be any maximal independent set in H. When $v \in J$, then $H - N_H[v] = G$, which is well-covered by hypothesis. Otherwise, J contains an element $u \in U$. But by condition (a), $H - N_H[u]$ is well-covered with $\beta(H - N_H[u]) = \beta(G)$, so that in either case, $|J| = \beta(G) + 1$. Since J was chosen arbitrarily, H is a well-covered graph with $\beta(H) = \beta(G) + 1$.

Conversely, suppose that H is well-covered with $\beta(H) = \beta(G) + 1$. Then $H - N_H[u]$ is well-covered with $\beta(H - N_H[u]) = \beta(G)$, by Lemma 1.2, and so condition (a) is satisfied. □

Proposition 4.1 can be considered as a special case of Proposition 3.6 when T is a clique.
The construction described in Proposition 4.1 was used in [6] (where it was called an *expansion*) in order to construct those well-covered graphs of girth at least 5 that contain no extendable vertex.

Corollary 4.2 *If in Proposition 4.1 the following additional conditions are satisfied:*

(b) $T \cong K_{1,r}$, $1 \leq r \leq \beta(G) + 1$;

(c) $N_H(u)$ *is independent, for* $u \in U$.

then H *is well-covered with* $g(H) \geq 4$ *whenever* $g(G) \geq 4$.

In general, it is not easy to choose the sets $N_H(u) \cap V(G)$, $u \in U$, to satisfy condition (a) in Proposition 4.1. However, it is useful to note that for each maximal independent set $I \subseteq U$, $G - N_H(I)$ is well-covered with $\beta(G - N_H(I)) = \beta(H) - |I|$. This method was used in constructing the following example.

Example 4.1 The graph G in Figure 3 below is well-covered with $\beta(G) = 3$. We adjoin $T \cong K_{1,2}$ to G to obtain the well-covered graph H. Note first that G contains a well-covered subgraph $G_0 \cong K_2 + K_2$ with components b_1b_2 and b_3b_4, so that $\beta(G_0) = 2$. Using the notation of Proposition 4.1, we require the remaining vertices of G to be members of $N_H(U)$. Choosing $N_H(u_1) \setminus \{v\} = \{a_1, a_3\}$, $N_H(u_2) \setminus \{v\} = \{a_2, a_3\}$ satisfies the conditions of Proposition 4.1 and gives a well-covered graph H.

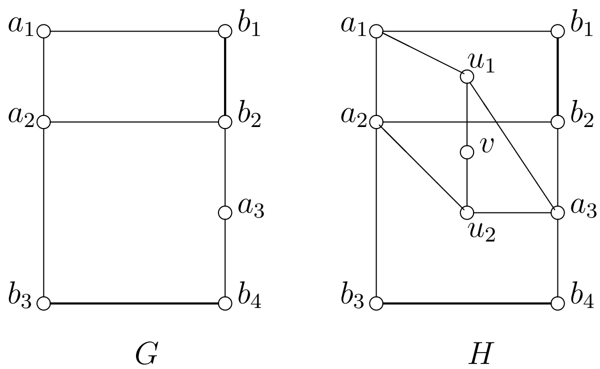


Figure 3

There are well-covered graphs which are prime by Definition 3.7 and can only be constructed from simpler graphs by Proposition 4.1. For example, the primes C_5 and C_7 can each be obtained by adding a new vertex u and its two independent neighbours to P_2 and P_4 respectively.

Appendix

Connected well-covered graphs H with $g(H) \geq 4$, $\beta(H) \leq 4$ and $|V(H)| \leq 9$

Table 1: $\beta(H) \leq 3$

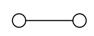
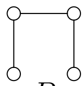
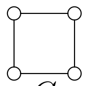
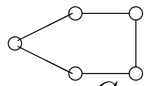
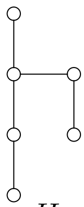
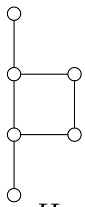
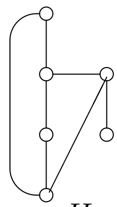
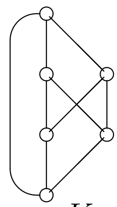
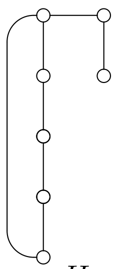
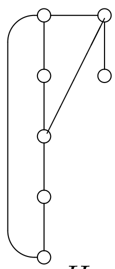
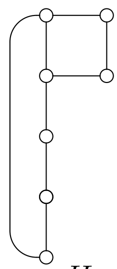
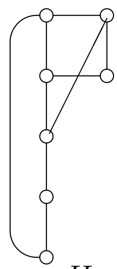
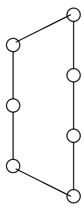
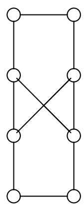
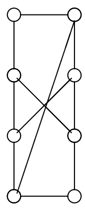
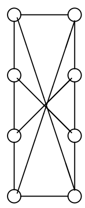
$\beta(H) = 1$ *Prime*	 K_1  K_2		
$\beta(H) = 2$ *Composite*	 P_4  C_4	$\beta(H) = 2$ *Prime*	 C_5
$\beta(H) = 3$ *Composite*	 H_1  H_2  H_3  $K_{3,3}$  H_4  H_5  H_6  H_7		
$\beta(H) = 3$ *Prime*	 C_7  H_8  H_9  H_{10}		

Table 2: $\beta(H) = 4$ **and** $|V(H)| = 8$. All these graphs are composite.

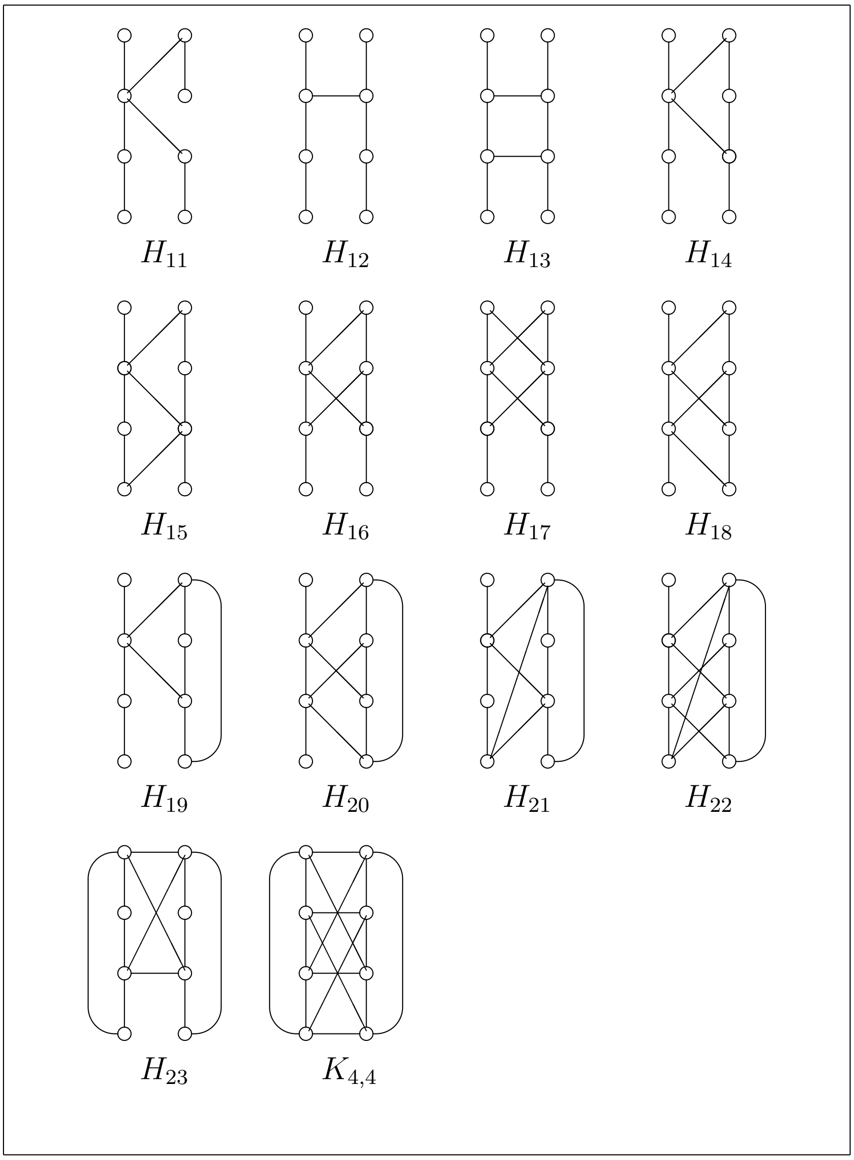
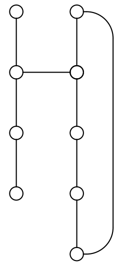
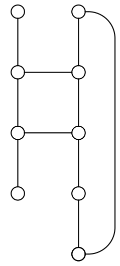


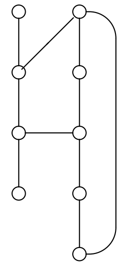
Table 3: $\beta(G) = 4$ **and** $|V(G)| = 9$. All these graphs are composite.



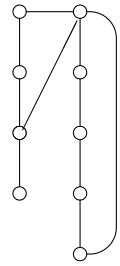
H_{24}



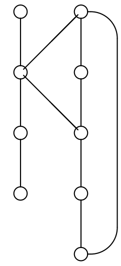
H_{25}



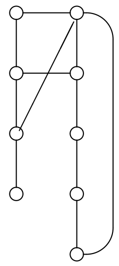
H_{26}



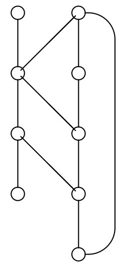
H_{27}



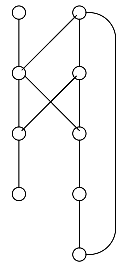
H_{28}



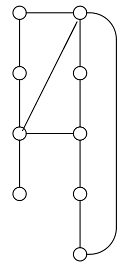
H_{29}



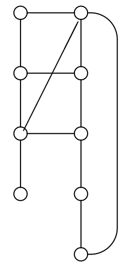
H_{30}



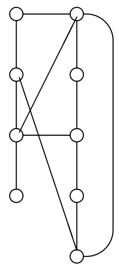
H_{31}



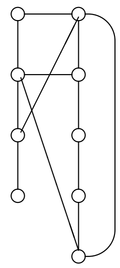
H_{32}



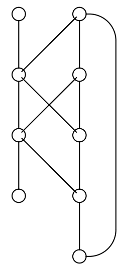
H_{33}



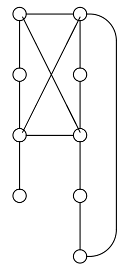
H_{34}



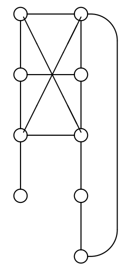
H_{35}



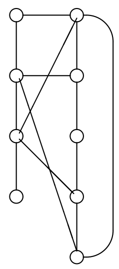
H_{36}



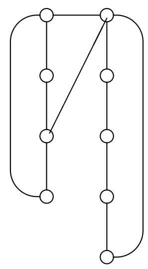
H_{37}



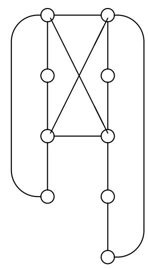
H_{38}



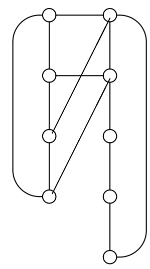
H_{39}



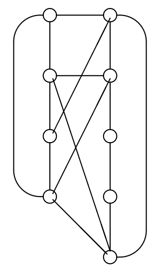
H_{40}



H_{41}

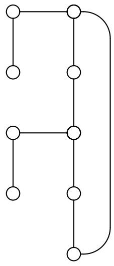


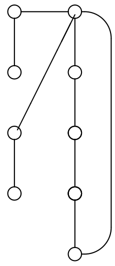
H_{42}

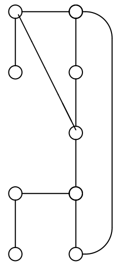


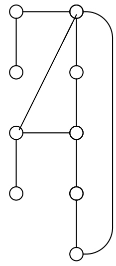
H_{43}

Table 3 (cont)


H_{44}


H_{45}


H_{46}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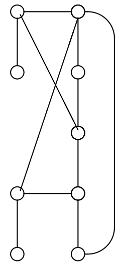
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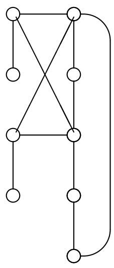
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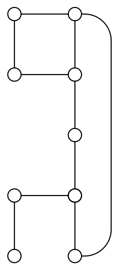
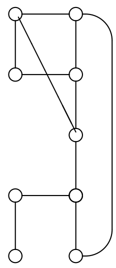
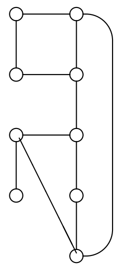
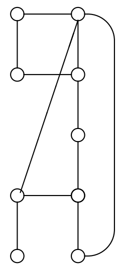
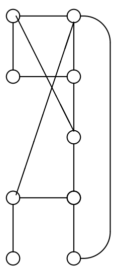
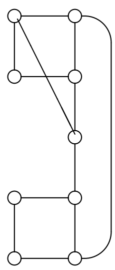

H_{50}

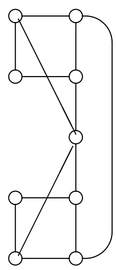

H_{51}

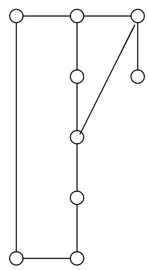

H_{52}

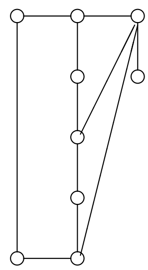

H_{53}


H_{54}


H_{55}


H_{56}


H_{57}


H_{58}

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