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Equating two maximum degrees

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Abstract

Given a graph G and a positive integer k, we would like to find (if it exists) the largest induced subgraph H in which there are at least k vertices realizing the maximum degree of H. This problem was first posed by Caro and Yuster. They proved, for example, that for every graph G on n vertices we can guarantee, for $k = 2$, such an induced subgraph H by deleting at most $2\sqrt{n}$ vertices, but the question of whether $2\sqrt{n}$ is best possible remains open.

Among the results obtained in this paper we prove that:

1. For every graph G on $n \geq 4$ vertices we can delete at most $\lceil \frac{-3+\sqrt{8n-15}}{2} \rceil$ vertices to get an induced subgraph H with at least two vertices realizing $\Delta(H)$, and this bound is sharp, solving the problems left open by Caro and Yuster.
2. For every graph G with maximum degree $\Delta \geq 1$ we can delete at most $\lceil \frac{-3+\sqrt{8\Delta+1}}{2} \rceil$ vertices to get an induced subgraph H with at least two vertices realizing $\Delta(H)$, and this bound is sharp.
3. Every graph G with $\Delta(G) \leq 2$ and at least $2k-1$ vertices (respectively $2k-2$ vertices if k is even) contains an induced subgraph H in which at least k vertices realise $\Delta(H)$, and these bounds are sharp.

1 Introduction

A well-known elementary exercise in graph theory states that every (simple) graph on at least two vertices has two vertices with the same degree. Motivated by this fact, Caro and West [10] formally defined the repetition number of a graph G, $\mathrm{rep}(G)$, to be the maximum multiplicity in the list (degree sequence) of the vertex degrees.

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Research has been done concerning the repetition number or repetitions in the degree sequence. Here we mention some of these directions.

1. The connection between the independence number and K_r-free graphs with given repetition number [5, 7, 13].
2. Hypergraph irregularity—the existence of r-uniform hypergraphs ($r \geq 3$) with no repeated degrees [4, 16].
3. Ramsey type problems with repeated degrees [2, 12].
4. Regular independent sets—vertices of the same degree forming an independent set [1, 3, 8].
5. Forcing k-repetition anywhere in the degree sequence [9].
6. Forcing k-repetition of the maximum degree [11].

In this paper we shall focus on the following problem first stated in [11]. For a graph G and an integer $k \geq 2$, let $f_k(G)$ denote the minimum number of vertices we have to delete from G in order to get an induced subgraph H in which there are at least k vertices that attain the maximum degree $\Delta(H)$, of H, or otherwise $|H| < k$, where as usual, following the notation of [19], $|G| = n$ is the number of vertices of G, $\Delta(G)$ is the maximum degree of G and a vertex of degree t is called a t-vertex. In the case $k = 2$ we use the abbreviation $f(G)$ instead of $f_2(G)$. We define $f(n, k) = \max\{f_k(G) : |G| \leq n\}$ and i $g(\Delta, k) = \max\{f_k(G) : \Delta(G) \leq \Delta\}$.

Clearly there are graphs in which we cannot equate k degrees let alone k maximum degrees. A simple example is the star $K_{1,k-1}$ for $k \geq 3$ having k vertices and by definition $f_j(K_{1,k-1}) = 1$ for $j \geq 2$.

However, it is trivial that in every graph G on at least $R(k, k)$ vertices (where $R(k, k)$ is the diagonal Ramsey number), we can equate k maximum degrees. We call a graph G in which (by deleting vertices) we can equate k maximum degrees a k-*feasible* graph. So, of interest is the following function

$$h(\Delta, k) = \max\{|G| : \Delta(G) \leq \Delta \text{ and } G \text{ is not } k\text{-feasible}\}.$$

Caro and Yuster [11] conjectured that for every $k \geq 2$ there exists a constant $c(k)$ such that $f(n, k) \leq c(k)\sqrt{n}$, and proved the conjecture for $k = 2$ with $c(2) = 2$ and $k = 3$ with $c(3) = 43$. For $k \geq 4$ the conjecture is still open. The question whether $c(2) = 2$ and $c(3) = 43$ are best possible also remains open.

Our main purpose in this paper is to show:

1. $f(G)$ can be computed exactly in polynomial time $O(n^2)$.
2. For $\Delta \geq 1$, $g(\Delta, 2)) \leq \lceil (\frac{-3+\sqrt{8\Delta+1}}{2}\rceil$ and this bound is sharp.

3. For $n \geq 4$, $f(n,2) \leq \lceil (\frac{-3+\sqrt{8n-15}}{2} \rceil$ and this bound is sharp. Hence in particular $c(2) = \sqrt{2}$, solving the problem left open in [11].

4. For a forest F on n vertices, $f_k(F) \leq (2k-1)n^{\frac{1}{3}}$.

5. $g(1,k) = \lfloor \frac{k-1}{2} \rfloor$, $g(2,k) = k-1$, thus determining exactly $g(\Delta,k)$ for $\Delta = 1,2$.

6. $h(0,k) = k-1$, $h(1,k) = \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor$, $h(2,k) = 2k-2$ for odd $k \geq 3$, $h(2,k) = 2k-3$ for even $k \geq 2$.

The paper is organized as follows.

In Section 2 we cover the complexity issue of computing $f(G)$, as well as the sharp upper-bounds for $g(\Delta,2)$ and $f(n,2)$. In Section 3 we consider upper-bounds for $f(F)$ and $f_k(F)$ where F is a forest. In Section 4 we prove exact results about $g(\Delta,k)$ and $h(\Delta,k)$ for $\Delta = 0,1,2$. Finally, in Section 5 we collect open problems and conjectures that deserve further exploration.

2 Determination of exact upper bounds for $f(G)$ in terms of $\Delta(G)$ and $|G| = n$.

We first need a definition and two lemmas.

We call $B \subset V(G)$, a set of vertices in a graph G, a 2-*equating set* if in the induced subgraph H on $V(G)\backslash B$, there are at least two vertices that realise $\Delta(H)$. We say that B is a 2-equating set which realises $f(G)$ if B has the minimum cardinality among all 2-equating sets of G.

Let the degree sequence of the graph G on n vertices be $\Delta = d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n = \delta$ so that Δ is the maximum degree and δ the minimum degree. We define $\mathrm{diff}(G) = d_1 - d_2$.

Lemma 2.1. *Let G be a graph on $n \geq 2$ vertices with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$, with $\deg(v) = d_1$ and $\deg(u) = d_2$. Then $f(G) \leq d_1 - d_2 = \mathrm{diff}(G)$.*

Proof. If $d_1 = d_2$ then clearly $f(G) = 0$. So let $\mathrm{diff}(G) = d_1 - d_2 \geq 1$. But then there is at least one set B of neighbours of v of size $\mathrm{diff}(G)$, none of which is adjacent to u, and clearly $f(G) \leq |B| = \mathrm{diff}(G)$. $\square$

Lemma 2.2. *Let G be a graph on $n \geq 2$ vertices, with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$, with $\deg(v) = d_1$. Then either $f(G) = \mathrm{diff}(G)$, or v must be in every minimal 2-equating set of G.*

Proof. Suppose $f(G) \neq \mathrm{diff}(G)$. Then $f(G) < \mathrm{diff}(G)$, and we may assume that v is the unique vertex in G of degree d_1.

Let $f(G)$ be realised by some induced subgraph H, and $B = V(G) - V(H)$ is a minimum 2-equating set for G. Assume for the contrary that $v \in H$ (then v is not a member of B).

Since $v \in H$ but $f(G)$ is not realised by $\text{diff}(G)$ then either v is not of maximum degree in H in which case at least $\text{diff}(G) + 1$ vertices among the neighbours of v must be deleted contradicting $|B| = f(G) < \text{diff}(G)$, or v is of maximum degree in H and still at least $\text{diff}(G)$ among its neighbours must be deleted and again $f(G) = |B| \geq \text{diff}(G)$ contradicting $f(G) < \text{diff}(G)$. □

Lemma 2.3. *Let G be a graph on at least $n \geq 2$ vertices. Suppose that $f(G) \neq \text{diff}(G)$, then $f(G) = 1 + f(G - v)$, where v is the single vertex of maximum degree in G.*

Proof. Since $f(G) \neq \text{diff}(G)$ it follows that there is a single vertex v of maximum degree in G and also from Lemma 2.2 we infer that v must be in any minimal 2-equating set of G.

Let $G_1 = G\backslash\{v\}$ and let B be a minimal 2-equating set for G_1, namely $f(G_1) = |B|$. Then clearly $B \cup \{v\}$ is a 2-equating set for G hence $f(G) \leq 1 + f(G_1)$.

On the other hand let B be a minimum 2-equating set for G. Then by assumption and Lemma 2.2 $v \in B$. Set $B_1 = B\backslash\{v\}$. Clearly B_1 is a 2-equating set of G_1 hence $f(G_1) \leq |B_1| = |B| - 1 = f(G) - 1$ which gives $f(G_1) + 1 \leq f(G)$.

Hence combining both inequalities we get $f(G) = f(G\backslash\{v\}) + 1$. □

Theorem 2.4. *Let G be a graph on $n \geq 2$ vertices, then*

$$f(G) = \min\{\text{diff}(G_j) + j : j = 0 \ldots n - 2\},$$

where G_{j+1} is obtained from G_j by deleting a vertex $v_{1,j}$ of the maximum degree $d_{1,j}$ from G_j (where G_0 is taken to be G), and $d_{2,j}$ is the second largest degree in G_j.

Moreover $f(G)$ can be determined in time $O(n^2)$.

Proof. By Lemma 2.2, either $f(G) = \text{diff}(G)$ or $v_{1,0}$ must be deleted to obtain G_1 and in this case by Lemma 2.3, $f(G) = f(G_1) + 1$.

Hence $f(G) = \min\{\text{diff}(G), f(G_1) + 1\}$. Now either $f(G_1) = \text{diff}(G_1)$, or by Lemma 2.2 and Lemma 2.3 the maximum degree in G_1 must be deleted to obtain G_2 and then $f(G_1) = f(G_2) + 1$.

Hence $f(G) = \min\{\text{diff}(G), \text{diff}(G_1) + 1, f(G_2) + 2\}$.

We continue this process until for some first j, $\text{diff}(G_j) = 0$ and there we stop, having two vertices realizing the maximum degree of G_j (the later steps will always give a larger value then $\text{diff}(G_j) + j = j$).

Each step is forced by Lemma 2.2 and Lemma 2.3 , hence

$$f(G) = \min\{\text{diff}(G_j) + j : j = 0 \ldots n - 2\}.$$

Now in each iteration we have to construct G_j from G_{j-1} by deleting the maximum degree $v_{1,j-1}$ from G_{j-1} and compute $d_{1,j}$ and $d_{2,j}$ which can be done in $O(n)$ time running over the new degree sequence of G_j that can be computed from the degree sequence of G_{j-1} by updating $d_{1,j}$ values in it .

So the total running time for the algorithm is $O(n^2 + e(G)) = O(n^2)$. □

Theorem 2.5. *Let G be a graph on $n \geq 2$ vertices with maximum degree Δ, and $t \geq 1$ be an integer.*

(i) If $0 \leq \Delta \leq 1$, then $f(G) = 0$.

(ii) If $\binom{t+1}{2} + 1 \leq \Delta \leq \binom{t+2}{2}$, then $f(G) \leq t$, and this bound is sharp for every Δ in the range.

(iii) For $\Delta \geq 1$, $f(G) \leq \left\lceil \frac{-3+\sqrt{8\Delta+1}}{2} \right\rceil$.

Proof. Clearly, if $0 \leq \Delta \leq 1$ and $n \geq 2$ then $f(G) = 0$.

For (ii), we use induction on t. For $t = 1$, $2 \leq \Delta \leq 3$. If there is one vertex v of degree $\Delta = 2$ then removing v clearly leaves at least two vertices of maximum degree equal to one or zero, and hence $f(G) = 1$. If there is a vertex v of degree $\Delta = 3$ and a vertex u of degree 2, then by Lemma 2.1, $f(G) \leq 3 - 2 = 1$. Otherwise, all other vertices have degree 0 or 1 and deleting v leaves at least two vertices of maximum degree equal to one or zero, and $f(G) = 1$.

So assume statement is true for $t - 1$ and we shall prove it is true for t.

By assumption, $\binom{t+1}{2} + 1 \leq \Delta \leq \binom{t+2}{2}$. Let v be a vertex of maximum degree and u a vertex with the second largest degree; clearly

$$\deg(u) \leq \deg(v) \leq \binom{t+2}{2}.$$

Now consider $\deg(u)$.

1. If $\deg(u) \leq \binom{t+1}{2}$ we drop v to get $G\backslash\{v\} = H$ where $\Delta(H) \leq \binom{t+1}{2}$, and by induction $f(G) \leq f(H) + 1 \leq t - 1 + 1 = t$ and we are done.

2. If $\deg(u) \geq \binom{t+1}{2} + 1$ then clearly

$$\mathrm{diff}(G) = \deg(v) - \deg(u) \leq \binom{t+2}{2} - \binom{t+1}{2} - 1 = t,$$

hence by Theorem 2.4 $f(G) \leq \mathrm{diff}(G) \leq t$ and we are done.

Sharpness: consider the sequence $a_j = \binom{j+1}{2} + 1$ i.e. $a_1 = 2$, $a_2 = 4$, $a_3 = 7$ etc. and let $\Delta = \binom{t+1}{2} + j$, $j = 1, \ldots, t+1$. For example, if $t = 4$, $\Delta = 11, 12, 13, 14, 15$.

Consider the graph G_Δ consisting of the stars K_{1,a_j} for $j = 1, \ldots, t-1$ and a "big star" $K_{1,\Delta}$.

Suppose for example $\Delta = 13$, which is the case $t = 4$, since $\binom{t+1}{2} = \binom{5}{2} < 13 < \binom{6}{2} = \binom{t+2}{2}$. The sequence of stars we choose involves a_1, a_2, a_3 and Δ, that is $K_{1,2} \cup K_{1,4} \cup K_{1,7} \cup K_{1,13}$, and this realises $f(G) = 4$ as required. The validity of this construction is a simple application of Theorem 2.4.

So this construction shows the bound is sharp for every $\Delta \geq 1$.

For (iii), from part (ii) above ($t \geq 1$ and $\Delta \geq 2$), we get $t^2 + 3t + 2 - 2\Delta \leq 0$. Solving the quadratic and rounding up, since t must be an integer, we get

$$f(G) \leq t = \left\lceil \frac{-3 + \sqrt{1 + 8\Delta}}{2} \right\rceil,$$

which holds true also for the case $\Delta = 1$. □

Theorem 2.6. *Let G be a graph on $n \geq 4$, and $t \geq 1$ an integer such that*

$$\binom{t+1}{2} + 3 \leq n \leq \binom{t+2}{2} + 2.$$

Then $f(G) \leq t$, and this is sharp for all values of n in the range. Also, for $n \geq 4$,

$$f(G) \leq \left\lceil \frac{-3 + \sqrt{8n - 15}}{2} \right\rceil.$$

Proof. Observe that for $\binom{t+1}{2} + 3 \leq n \leq \binom{t+2}{2} + 1$ it follows that if $|G| = n$ then $\Delta(G) \leq n - 1$ hence $\Delta(G) \leq \binom{t+2}{2}$ and $f(G) \leq t$ by Theorem 2.5.

We now construct for every n, such that $\binom{t+1}{2} + 3 \leq n \leq \binom{t+2}{2} + 1$, a graph $G_n = G$ with $f(G) = t$ proving sharpness.

Let $n = \binom{t+1}{2} + j : j = 3, \ldots, t + 2$.

Let $A = \{v_1, v_2, \ldots, v_t\}$ and $B = \{u_1, \ldots, u_{n-t}\}$.

Vertex v_t is adjacent to all other vertices so that $\deg(v_t) = n - 1 = \binom{t+1}{2} + j - 1$. Vertex v_q, for $q = t - 1, \ldots, 1$ has degree $\deg(v_q) = \frac{q^2+q+2}{2} + 1 = a_q + 1$, where v_q is adjacent to v_t and to $u_1, \ldots, u_{a_q}$. Figure 1 shows the case when $t = 3$ and $j = 3$ i.e. $n = \binom{3+1}{2} + 3 = 9$.

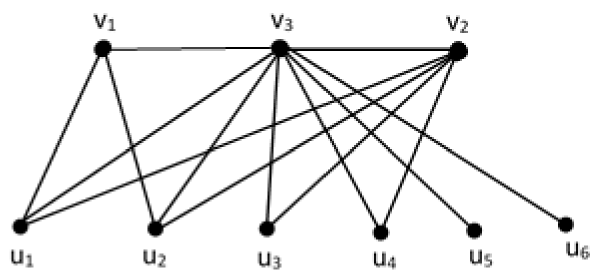

Figure 1: The graph G_n for $t = 3$ and $j = 3$

We now apply Theorem 2.4 to G. Then

$$\operatorname{diff}(G) = \binom{t+1}{2} + j - 1 - \frac{(t-1)^2 + t - 1 + 2}{2} + 1 = t + j - 1 \geq t.$$

Hence, we apply Theorem 2.4 by deleting v_t to give a new graph G_1 on $n-1$ vertices in which $\deg(v_i)$, $i = 1 \ldots t - 1$, as well as all the degrees of vertices in B adjacent to v_t are reduced by 1, and hence $\text{diff}(G_1) = \deg(v_{t-1}) - \deg(v_{t-2}) = t-1$ Therefore $f(G) \geq 1+\text{diff}(G_1) \geq t$ and we again apply Theorem 2.4 to delete v_{t-1}. The degrees of $v_{t-2} \dots v_1$ now remain unchanged, and for $i = 2 \dots t-1$, $\deg(v_{t-i}) - \deg(v_{t-i-1}) = t-i$, and the vertices are not adjacent to each other. Hence it follows that, at each step, $\text{diff}(G_i) = t-i$, which, by Theorem 2.4, implies that $f(G) = \min\{\text{diff}(G_j) + j : j = 0, \dots, n-2\} = t$.

Let us now look at the case when $|G| = n = \binom{t+2}{2} + 2$.

Let $\Delta = \Delta(G)$ — then if $\Delta(G) \leq \binom{t+2}{2}$ then by Theorem 2.5, $f(G) \leq t$ and we are done. So $\Delta = \binom{t+2}{2} + 1$. Let v_1 and v_2 be such that $\deg(v_1) = \Delta$ and v_2 has the second largest degree. Observe that v_1 is adjacent to all vertices of G. Now if $\deg(v_2) \geq \binom{t+1}{2} + 2$, then $\text{diff}(G) \leq t$ and again we are done by Theorem 2.4.

So $\deg(v_2) \leq \binom{t+1}{2} + 1$. We delete v_1 to get the graph G_1. Clearly $\Delta(G_1) = \deg(v_2) - 1 \leq \binom{t+1}{2}$ and by the Theorem 2.5, $f(G_1) \leq t-1$ hence $f(G) \leq t$.

For, sharpness we can take the graph G constructed above on n vertices for $n = \binom{t+2}{2} + 1$ and add an isolated vertex.

Now for a graph G on n vertices with $4 \leq n \leq \binom{t+2}{2} + 2$, we know $f(G) \leq t$, hence we get $t^2 + 3t - 2n + 6 \geq 0$, and solving the quadratic gives

$$f(G) = t \leq \left\lceil \frac{-3 + \sqrt{8n-15}}{2} \right\rceil .$$

□

3 Trees and Forests

We have determined the maximum possible value for $f(G)$ with respect to $\Delta(G)$ (Theorem 2.5) and with respect to $|G| = n$ (Theorem 2.6).

In the submitted version of this paper we proposed the problem of finding $\max\{f(G) : G \text{ is a forest on } n \text{ vertices}\}$ and we made the following conjecture:

Conjecture 3.1. *If F is a forest on n vertices, where $n \leq \frac{t^3+6t^2+17t+12}{6}$ then $f(F) \leq t$ and this is sharp.*

Since then this conjecture has been proved in [14] so we limit ourselves to giving a construction which achieves this upper bound.

Consider the sequence $a_j = \binom{j+1}{2} + 1$. For $t \geq 0$ we define a tree T_t as follows: Let P_{2t+3} be a path on $2t+3$ vertices $\{v_1, \dots, v_{2t+3}\}$. Now to the vertex v_{2j} for $j = 1, \dots, t+1$ of the path we add exactly $x_j = a_j - 2 = \binom{j+1}{2} - 1$ leaves, so that $x_1 = 0$, $x_2 = 2$ and so on. Let b_t be the number of vertices of T_t.

Clearly $\deg(v_1) = \deg(v_{2t+3}) = 1$, and for $t \geq 1$, $\deg(v_{2j+1}) = 2$ for $j = 1, \dots, t$ while $\deg(v_{2j}) = a_j$ for $j = 1, \dots, t+1$.

Now for $t = 0$ we get $T_0 = K_{1,2}$ with $b_t = 3$, for $t = 1$ we get $b_t = 7$ and for $t = 2$, $b_t = 14$, as shown in Figure 2.

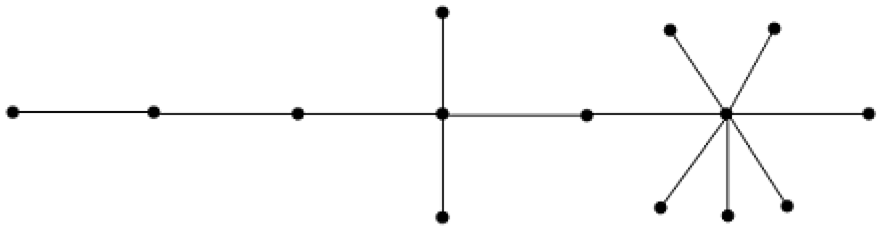

Figure 2: The tree T_2 on 14 vertices

The number of vertices in T_t is $b_t = \frac{t^3+6t^2+17t+18}{6}$. We prove this by induction on t. For $t = 0$, $b_0 = 3$ and for $t = 1$, $b_1 = 7$ as required. So let us assume it is true for b_{k-1}. Then

$$\begin{aligned} b_k &= b_{k-1} + \frac{k^2+3k+4}{2} \\ &= \frac{(k-1)^3+6(k-1)^2+17(k-1)+18}{6} + \frac{k^2+3k+4}{2} \\ &= \frac{k^3+6k^2+17k+18}{6} \end{aligned}$$

as required. So $\mathrm{diff}(T_t) = \frac{t^2+3t+4}{2} - \frac{(t-1)^2+3(t-1)+4}{2} = t+1$. Again by induction on t, we can show that $f(T_t) = t+1$, for $t \geq 0$. Clearly $f(T_0) = 1$. Suppose that $f(T_t) \leq t < \mathrm{diff}(T_t)$. Then we should remove the vertex of degree Δ in order to obtain a minimal 2-equating set. But this leaves isolated vertices and the tree T_{t-1}. But $f(T_{t-1}) = t$, by induction, and hence $f(T_t) = t+1$, a contradiction.

We now prove the following result:

Theorem 3.2. *Let F be a forest on n vertices and $k \geq 2$ an integer. Suppose $n^{\frac{1}{3}} \geq 2k-1$. Then*

$$f_k(F) \leq (2k-1)\lfloor n^{\frac{1}{3}} \rfloor.$$

We first prove the following lemmas:

Lemma 3.3. *For every $k \geq 2$ and every graph G with maximum degree Δ, $f_k(G) \leq (k-1)\Delta$.*

Proof. By induction on Δ. If $\Delta = 0$, then the result is trivial since either $|G| < k$ or there are k vertices of degree 0.

So suppose the result holds for $\Delta \leq r$ and let G have $\Delta = r+1$. If there are k vertices of maximum degree $r+1$ we are done. Otherwise, remove all the vertices of maximum degree Δ in G —there are at most $k-1$ such vertices. The resulting graph H has maximum degree at most r and hence by the induction hypothesis, $f_k(H) \leq (k-1)r$. Hence

$$f_k(G) \leq (k-1)r + k - 1 = (k-1)(r+1) = (k-1)\Delta(G)$$

as required. □

Lemma 3.4. *Let G be a forest and let A be any subset of $k \geq 2$ vertices of G. Define $M(A)$ to be the set of vertices of $V(G)\backslash A$ each having at least two neighbours in A. Then $|M(A)| < |A| = k$.*

Proof. Suppose $|M(A)| \geq |A| = k$. Let B be any subset of $M(A)$ of cardinality k and let H be the bipartite graph with vertices $A \cup B$ and only those edges connecting vertices in A to vertices in B. Since each vertex of B has degree at least 2 in H, $|E(H)| \geq 2k$. But $|V(H)| = 2k$, therefore H has a cycle contradicting the fact that G is a forest. □

We now prove Theorem 3.2.

Proof. The maximum degree of F can range from 0 to $n-1$. Let us divide the interval $[0, n)$ into subintervals

$$S_j = [jn^{\frac{1}{3}}, (j+1)n^{\frac{1}{3}}) \text{ for } j = 0, \ldots, \lfloor n^{\frac{1}{3}} \rfloor - 1$$

with the last two intervals being $S_{\lfloor n^{\frac{1}{3}} \rfloor} = [\lfloor n^{\frac{1}{3}} \rfloor n^{\frac{1}{3}}, n^{\frac{2}{3}})$ and $S_L = [\lceil n^{\frac{2}{3}} \rceil, n)$. Let us denote by A_j or A_L the set of vertices of F whose degrees fall in the intervals S_j or S_L respectively.

We first claim that A_L contains at most $\lfloor n^{\frac{1}{3}} \rfloor$ vertices. Suppose not and let x_j be the number of vertices of F having degree j. Consider the forest F^* on $n^* \leq n$ vertices obtained by deleting all isolated vertices in F. Then clearly the number of vertices x_j of degree $j \geq 1$ in F^* in the same as in F, and we have $x_1 + \ldots + x_{n-1} = n^*$ and $x_1 + 2x_2 + \ldots + (n-1)x_{n-1} \leq 2n^* - 2$. Multiplying the first equation by 2 and subtracting the second gives:

$$x_1 - \sum_{j=3}^{n-1} (j-2)x_j \geq 2.$$

Hence

$$x_1 \geq 2 + \sum_{j \geq 3} (j-2)x_j.$$

In particular

$$\begin{aligned} x_1 &\geq 2 + (\lfloor n^{\frac{2}{3}} \rfloor - 2)|A_L| \\ &\geq 2 + (\lfloor n^{\frac{2}{3}} \rfloor - 2)(\lfloor n^{\frac{1}{3}} \rfloor + 1) = n + \lfloor n^{\frac{2}{3}} \rfloor - 2\lfloor n^{\frac{1}{3}} \rfloor. \end{aligned}$$

But

$$n^* \geq x_1 + |A_L| \geq n + n^{\frac{2}{3}} - 2n^{\frac{1}{3}} + n^{\frac{1}{3}} + 1 = n + n^{\frac{2}{3}} - n^{\frac{1}{3}} + 1 > n^*$$

a contradiction.

We now proceed as follows. We remove from F the vertices in A_L and redistribute the resulting degrees among the intervals S_j, $j = 0, \ldots, \lfloor n^{\frac{1}{3}} \rfloor$, recalculating A_j for $j = 0, \ldots, \lfloor n^{\frac{1}{3}} \rfloor$. If there are at least k vertices with degrees in the last interval we stop. Otherwise we remove these vertices and again we redistribute the degrees among the intervals S_j, $j = 0, \ldots, \lfloor n^{\frac{1}{3}} \rfloor - 1$, recalculating A_j for $j = 0, \ldots, \lfloor n^{\frac{1}{3}} \rfloor - 1$. This process continues until we reach one of the following possibilites:

Case 1: We have deleted all vertices and are left with only those vertices in A_0.

Case 2: For some $j \geq 1$, A_j contains at least k vertices.

We consider these cases separately:

Case 1: In this case we have deleted $\lfloor n^{\frac{1}{3}} \rfloor$ vertices from A_L and at most $(k-1)\lfloor n^{\frac{1}{3}} \rfloor$ further vertices by deleting at most $(k-1)$ vertices that were in the respective sets $A_1, \ldots, A_{\lfloor n^{\frac{1}{3}} \rfloor}$ at each stage. So altogether $k\lfloor n^{\frac{1}{3}} \rfloor$ vertices have been deleted.

But now the resulting graph has maximum degree at most $\lfloor n^{\frac{1}{3}} \rfloor$ and therefore, by Lemma 3.3, by deleting at most a further $(k-1)\lfloor n^{\frac{1}{3}} \rfloor$ vertices we arrive at a graph with k vertices of maximum degree (or at most $k-1$ vertices at all). To do this we have altogether deleted at most $(2k-1)\lfloor n^{\frac{1}{3}} \rfloor$ vertices, as required.

Case 2: We have stopped the deletion process when A_j, $j \geq 1$, contains at least k vertices, A_j being the set of vertices of the reduced forest having degrees in $S_j = [jn^{\frac{1}{3}}, (j+1)n^{\frac{1}{3}})$.

Let $v_1, v_2, \ldots, v_k$ be the k vertices in A_j of largest degrees, say $d_1 \geq d_2 \geq \ldots \geq d_k$. Let us call this set of vertices A. By Lemma 3.4, $|M(A)| < k$ where we recall that $M(A)$ is the set of vertices adjacent to at least two vertices of A. Since a vertex $v \in A$ can be adjacent to at most $k-1$ other vertices in A and $k-1$ vertices in $M(A)$, there are at least $\deg(v) - 2k + 2$ vertices that are neighbours of v but which are not in $A \cup M(A)$. Since such vertices are adjacent to at most one vertex from A, these $\deg(v) - 2k + 2$ vertices are only adjacent to $v \in A$ and not to any other vertex in A. Let $B(v)$ be the set of these neighbours of v.

Now consider any vertex $v_i \in A$, $i = 1 \ldots k$. Suppose $\deg(v_i) = \deg(v_k) + t_i$. Then

$$|B(v_i)| \geq \deg(v_i) - 2k + 2 \geq \deg(v_k) + t_i - 2k + 2 \geq n^{\frac{1}{3}} + t_i - 2k + 2 > t_i$$

since $n^{\frac{1}{3}} \geq 2k - 1 > 2k - 2$.

We therefore need to remove t_i vertices of $B(v_i)$ (and this will not change the degree of any other vertex in A) in order to equate $\deg(v_i)$ and $\deg(v_k)$. However, since $|B(v_i)| \geq t_i$, we can do this.

Hence, equating all the degrees of the vertices $v_1, \ldots, v_{k-1}$ to $\deg(v_k)$ can be done at the cost of deleting at most a further $(k-1)\lfloor n^{\frac{1}{3}} \rfloor$ vertices. This means that we have deleted altogether at most $(2k-1)\lfloor n^{\frac{1}{3}} \rfloor$ vertices, so we are done. □

Remark: The above proof also works for the more general class of graphs without even cycles. Lemma 3.4 remains unchanged since the graph H used in the proof is bipartite by construction. A graph G on n vertices and without even cycles contains at most $\frac{3n}{2}$ edges (Theorem 5.1 on page 31 in [6]). Therefore, in the proof of the

Theorem, instead of the computation involving x_1 we compute an upper bound on the number of vertices in A_L, by noting that if this number is at least $3n^{\frac{1}{3}}+1$ then

$$3n \geq 2|E(G)| \geq \sum_{v\in A_L} \geq (3n^{\frac{1}{3}}+1)(n^{\frac{2}{3}}) = 3n + n^{\frac{2}{3}} > 3n,$$

a contradiction.

We then remove the vertices of A_L and redistribute the resulting degrees, sacrificing at most $3n^{\frac{1}{3}}$ vertices, and continue as in the proof. This gives

$$f_k(G) \leq (2k+1)n^{\frac{1}{3}},$$

giving a weaker bound for a more general class of graphs.

4 The functions $g(\Delta, k)$ and $h(\Delta, k)$

Lemma 3.3, which states that $g(\Delta, k) \leq (k-1)\Delta$, plays a crucial rule in the proof of Theorem 3.2. Another motivation to study $g(\Delta, k) = \max\{f_k(G) : \Delta(G) \leq \Delta\}$ comes from the Proposition 4.1 below, which gives a weak support for the conjecture $f_k(G) \leq c(k)\sqrt{|G|}$ mentioned in the introduction, and also demonstrates that for graphs with $e(G) = o(n^2)$, $f_k(G) = o(n)$ (where $e(G)$ is the number of edges of G) .

Observe that it has not yet been proved in general that for fixed k and G a graph on n vertices, $f_k(G) = o(n)$.

Proposition 4.1. *Suppose G is a graph on n vertices and $e(G) \leq cn^{1+\beta}$ where $0 \leq \beta < 1$ and let $\alpha = \frac{1+\beta}{2}$. Then $f_k(G) \leq (k-1+2c)n^\alpha$, and in particular for $\beta = 0$, $f_k(G) \leq (k-1+2c)\sqrt{n}$.*

Proof. Define $V_\alpha = \{v : \deg(v) \geq n^\alpha\}$ and suppose $|V_\alpha| > 2cn^\alpha$.

Then

$$2e(G) = \sum_{v\in V(G)} \deg(v) \geq \sum_{v\in V_\alpha} \deg(v) > n^\alpha 2cn^\alpha = 2cn^{2\alpha} = 2cn^{1+\beta} \geq 2e(G),$$

a contradiction.

Hence $|V_\alpha| \leq 2cn^\alpha$. Deleting V_α we get a graph H with $\Delta(H) \leq n^\alpha$. Hence applying Lemma 3.3 we get

$$f_k(G) \leq |V_\alpha| + (k-1)\Delta(H) \leq 2cn^\alpha + (k-1)n^\alpha = (k-1+2c)n^\alpha.$$

□

So a better knowledge of the behavior of $g(\Delta, k)$ will help to obtain better bounds on $f_k(G)$ as well as $f(n, k) = \max\{f_k(G) : |G| = n\}$.

Proposition 4.2. *For every $\Delta \geq 0$ and $k \geq 2$,* *1.* $g(0, k) = 0$.

2. $g(1, k) = \lfloor \frac{k-1}{2} \rfloor$.

3. *For* $\Delta \geq 1$, $g(\Delta, 2) = \left\lceil \frac{-3+\sqrt{8\Delta+1}}{2} \right\rceil$.

Proof. 1. Clearly if G is a graph with maximum degree $\Delta = 0$ then either $|G| \geq k$ and the result holds or else $|G| \leq k - 1$ and the result holds by the definition of $f_k(G)$, hence $g(0, k) = 0$.

2. Consider G with maximum degree $\Delta(G) = 1$. If there are already k vertices of degree 1 we are done. So assume there are at most $k - 1$ vertices of degree 1. By parity these vertices induce at most $\lfloor \frac{k-1}{2} \rfloor$ isolated edges containing at most $2\lfloor \frac{k-1}{2} \rfloor$ vertices of degree 1. We delete from each isolated edge one vertex of degree 1 to get an induced subgraph H with $\Delta(H) = 0$. It follows, since $g(0, k) = 0$, that $f_k(G) \leq \lfloor \frac{k-1}{2} \rfloor$. This bound is sharp as demonstrated by the graph $tK_2 \cup mK_1$ where $m \geq 0$ and $t = \lfloor \frac{k-1}{2} \rfloor$, $k \geq 3$.

3. This is a restatement of Theorem 2.5.

□

Determining $g(2, k)$ requires more efforts, in particular we will use Ore's observation [18] that if G is a graph on n vertices without isolated vertices, then the domination number of G, denoted $\gamma(G)$, satisfies $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 4.3. *For* $k \geq 2$, $g(2, k) = k - 1$.

Proof. The graph G made up of $k - 1$ vertex-disjoint copies of the star $K_{1,2}$ has $f_k(G) = k - 1$ as is easily checked. So $g(2, k) \geq k - 1$.

Let us prove the reverse inequality.

Consider a graph G with $\Delta(G) = 2$, otherwise by Proposition 4.2 (part 2) we are done.

Let $n_2 = |\{v : \deg(v) = 2\}|$. Clearly if $n_2 \geq k$ we are done so we may assume $1 \leq n_2 \leq k - 1$.

We collect the (possible) components of G into three subgraphs: $A = \{$ all isolated vertices and isolated edges$\}$, $B = \{$all copies of $K_{1,2}\}$, $C = \{$all other components$\}$.

We denote by t the number of copies of $K_{1,2}$ in B and we observe that $t \leq n_2$ and that in each component in C the vertices of degree 2 induce either a path (including a single edge) or a cycle.

We claim that if $t > \lfloor \frac{k-1}{2} \rfloor$ we are done by deleting all $n_2 - t$ vertices of degree 2 in C and from each copy of $K_{1,2}$ in B we delete a leaf to get from G an induced subgraph H with $\Delta(H) = 1$ and with at least $2(\lfloor \frac{k-1}{2} \rfloor + 1) \geq k$ vertices of degree 1, and we have deleted altogether $n_2 =< k - 1$ vertices. So we shall assume $t \leq \lfloor \frac{k-1}{2} \rfloor$.

Consider the subgraph F induced by the vertices of degree 2 in C.

Case 1: $|F| = 0$.

If $|F| = 0$ (namely C is empty) then $n_2 = t \leq \lfloor \frac{k-1}{2} \rfloor$. Delete a leaf from each copy of $K_{1,2}$ in B. We get from G a graph H with $\Delta(H) = 1$ (as in A all components have maximum degree at most 1).

If in H there are already k vertices of degree 1, we are done and we have deleted $t \leq \lfloor \frac{k-1}{2} \rfloor$ vertices. Otherwise by Proposition 4.2 (part 2), $f_k(H) \leq \lfloor \frac{k-1}{2} \rfloor$ and hence $f_k(G) \leq 2\lfloor \frac{k-1}{2} \rfloor \leq k-1$.

Case 2: $|F| > 0$.

Then as we have noted before, due to the components of C, there are no isolated vertices in F, and by Ore's result $\gamma(F) \leq \lfloor \frac{n_2-t}{2} \rfloor \leq \lfloor \frac{k-1-t}{2} \rfloor$.

Let D be a dominating set for F that realises $\gamma(F)$, hence $|D| \leq \frac{n_2-t}{2}$.

Delete D and consider the induced subgraph H on $A \cup C$. Clearly $\Delta(H) \leq 1$ and denote by n_1 the number of vertices of degree 1 in H.

Now we look again at B.

If $t = 0$ then B is empty, and either $n_1 \geq k$ and we are done as we have deleted $|D| \leq \lfloor \frac{n_2}{2} \rfloor \leq \lfloor \frac{k-1}{2} \rfloor$ vertices or by Proposition 4.2 (part 2),

$$f_k(G) \leq f_k(H) + |D| \leq 2\lfloor \frac{k-1}{2} \rfloor \leq k-1.$$

On the other hand, if $1 \leq t \leq \lfloor \frac{k-1}{2} \rfloor$, we consider two possibilities:

1. if $n_1 \geq k - 2t$ then deleting a leaf from every copy of $K_{1,2}$ in B, we get an induced graph H^* on $A \cup B \cup C$ (extending H to the leftover vertices of B) with $\Delta(H^*) = 1$ and at least $k - 2t + 2t = k$ vertices of degree 1 and we are done as we have deleted altogether

$$|D| + t \leq \frac{n_2 - t}{2} + t = \frac{n_2 + t}{2} \leq n_2 \leq k-1$$

vertices.

2. If $n_1 \leq k - 1 - 2t$ (recall n_1 is the number of vertices of degree 1 in H formed from $A \cup \{C \backslash D\}$), then we delete $\frac{n_1}{2}$ independent vertices of degree 1 in H, and t vertices of degree 2 in B to get an induced subgraph H^* with $\Delta(H^*) = 0$.

   But $g(0, k) = 0$ hence $f_k(H^*) = 0$ and

$$f_k(G) \leq \frac{n_1}{2} + |D| + t \leq \frac{k-1-2t}{2} + \frac{k-1-t}{2} + t = \frac{2k-2-t}{2} \leq k-1$$

   and the proof is complete.

□

The following construction supplies a lower bound for $g(\Delta, k)$ in terms of $g(\Delta, 2)$.

Proposition 4.4. *For even $k \geq 2$, $g(\Delta, k) \geq g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$.*

Proof. Recall the sequence $a_t = \binom{t+1}{2}+1$, which for $t \geq 1$ gives the smallest maximum degree for which there is a graph G with $f(G) = t$. The graph which we constructed to prove sharpness in Theorem 2.5 is made up of $\bigcup K_{1,a_j}$ for $j = 1, \ldots, t$ and in case we have $a_t \leq \Delta < a_{t+1}$, $G = K_{1,\Delta} \cup K_{1,a_j} : j = 1, \ldots, t-1$.

Now we construct G_k as follows: take $k-1$ copies of K_{1,a_t} and $\frac{k}{2}$ copies of $K_{1,a_j} : j = 1, \ldots, t-1$. In case $a_t \leq \Delta < a_{t+1}$ we take $k-1$ copies of $K_{1,\Delta}$ and $\frac{k}{2}$ copies of $K_{1,a_j} : j = 1, \ldots, t-1$.

Note that for $k = 2$ this is exactly the construction that realises Theorem 2.5.

Observe now that, in G_k we cannot equate to degree Δ as there are just $k-1$ such degrees. So we can equate to the second largest degree a_{t-1} by deleting exactly $\Delta - a_{t-1}$ leaves from $\frac{k}{2}$ vertices of the maximum degree and $k - 1 - \frac{k}{2}$ other centres. Altogether we deleted

$$\frac{(\Delta - a_t)k}{2} + \frac{k}{2} - 1 \geq \frac{g(\Delta, 2)k}{2} + \frac{k}{2} - 1.$$

In case $\Delta = a_t$ we have deleted exactly $\frac{g(\Delta,2)k}{2} + \frac{k}{2} - 1$.

We can now equate to some value x such that $a_{t-1} > x \geq a_{t-2}+j$, $j \geq 1$. However clearly this requires the deletion of more vertices then just to equate to a_{t-1} and in particular the deletion of at least $g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$ vertices.

Now we can try to equate to a_{t-2}.

The cheapest way is to delete the $k-1$ vertices of degree Δ and $a_{t-1} - a_{t-2}$ leaves from each of the $\frac{k}{2}$ vertices of degree a_{t-1}. So altogether we deleted

$$k - 1 + (a_{t-1} - a_{t-2})\frac{k}{2} = k - 1 + (g(\Delta, 2) - 1)\frac{k}{2} = g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$$

vertices.

Again we can now try to equate to some value x such that $a_{t-2} > x \geq a_{t-3} + j$, $j \geq 1$. Clearly this requires the deletion of more vertices than just to equate to a_{t-2} and in particular the deletion of at least $g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$ vertices.

So this deletion process continues and we are always forced to delete at least $g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$ vertices, even if we delete all the centres of the stars to get an induced subgraph with all degrees equal 0.

Hence for even $k \geq 2$ we get $g(\Delta, k) \geq g(\Delta, 2)\frac{k}{2} + \frac{k}{2} - 1$ (which is sharp for $k = 2$).

While slight improvements on this lower bound are possible for odd $k \geq 3$, our goal in this construction is only to demonstrate a linear lower bound on $g(\Delta, k)$ in terms of $g(\Delta, 2)$ and k for which the construction suffices.

□

We now turn our attention to $h(\Delta, k)$. Recall that for $k \geq 2$, a graph G is k-feasible if it contains an induced subgraph H (possibly also $H = G$) such that in H there are at least k vertices that realise $\Delta(H)$, and we define

$$h(\Delta, k) = \max\{|G| : \Delta(G) \leq \Delta \text{ and } G \text{ is not } k\text{-feasible}\}.$$

Theorem 4.5. *For every $\Delta \geq 0$ and $k \geq 2$,*

1. $h(\Delta, k) \leq R(k, k) - 1$.
2. $h(0, k) = k - 1$.
3. $h(1, k) = \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor$.
4. *For odd $k \geq 3$, $h(2, k) = 2k - 2$, and for even $k \geq 2$, $h(2, k) = 2k - 3$.*
5. $h(\Delta, k) \leq g(\Delta, k) + k - 1$.

Proof.

1. Clearly if $|G| \geq R(k, k)$ then G has a vertex-set A, $|A| \geq k$ such that the induced subgraph on A is either a clique or an independent set .

   Hence deleting $V - A$ we are left with a regular graph on at least k vertices hence G is k-feasible and $h(\Delta, k) \leq R(k, k) - 1$.

2. $h(0, k) = k - 1$ is trivially realised by $(k - 1)K_1$ i.e. $k - 1$ isolated vertices.

3. A lower bound for $h(1, k)$ is $h(1, k) \geq \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor$ realised by the graph $G = \lfloor \frac{k}{2} \rfloor K_1 \cup \lfloor \frac{k-1}{2} \rfloor K_2$ which is trivially seen to be non-k-feasible.

   Next suppose G is a graph having $\lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor + j$ vertices, $j \geq 1$.

   Write $\lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor + j = x + 2y$ where x denotes the number of 0-vertices and 2y the number of 1-vertices in G.

   Now if $y > \lfloor \frac{k-1}{2} \rfloor$ we have at least k vertices of degree 1 and we are done. If $0 \leq y \leq \lfloor \frac{k-1}{2} \rfloor$ then delete y 1-vertices, one of each copy of K_2, and we are left with at least $\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + j \geq \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor + 1 = k$ vertices of degree 0.

   Hence G is k-feasible, and $h(1, k) = \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor$.

4. Clearly $h(2, k) \leq 2k - 2$ since if G has $\Delta = 2$ and at least $2k - 1$ vertices then by deleting at most $k - 1 = g(2, k)$ vertices, there must be induced H with at least k vertices realizing the maximum degree.

   Suppose k **is odd** and $k \geq 3$. Consider the graph $G = \frac{k-1}{2} P_4$ ($\frac{k-1}{2}$ copies of the path on four vertices P_4). Clearly $|G| = 2k - 2$ having exactly $k - 1$ 2-vertices and $k - 1$ 1-vertices.

   Observe that if G is k-feasible then in at least one of the P_4 we should be able to delete just one vertex to get the remaining three vertices of the same degree, otherwise if in each copy of P_4 (or what remains of it after deleting some vertices) we will have at most two vertices of the same degree then over all G we will have at most $k-1$ vertices of the same degree, meaning G is not k-feasible.

However it is impossible to delete one vertex from P_4 to get all the remaining three vertices of the same degree hence G is not k-feasible proving $h(2,k) = 2k-2$ for odd $k \geq 3$.

Suppose k **is even**, $k \geq 2$.

The case $k = 2$ is trivial hence we assume $k \geq 4$.

Consider the graph $G = \frac{k-1}{2}P_4 \cup K_1$. Clearly $|G| = 4\frac{k-2}{2} + 1 = 2k-3$ having exactly $k-2$ 1-vertices, $k-2$ 2-vertices and one 0-vertex. If G was k-feasible then by deleting the 0-vertex v, $H = G - v$ would be at least $k-1$-feasible with odd $t = k-1 \geq 3$.

But H is exactly the graph which was proved above to be non t-feasible for odd $t \geq 3$, so G is not k-feasible, proving $h(2,k) \geq 2k-3$ for even $k \geq 4$.

We have to show that if $|G| = 2k-2$ and $\Delta(G) = 2$, then for even $k \geq 4$, G is k-feasible, this will complete the proof that for even $k \geq 2$, $h(2,k) = 2k-3$.

Suppose on the contrary that $|G| = 2k-2$ and $\Delta(G) = 2$ but G is not k-feasible. Let n_j, $j = 0,1,2$ be the number of vertices of degree $j = 0,1,2$ respectively in G.

Since G is non-k-feasible and by the value of $h(1,k)$ we may assume $1 \leq n_2 \leq k-1$. However $2k-2 > h(2,k-1) = 2(k-1)-2 = 2k-4$. Hence G is $k-1$-feasible.

So either $n_2 = k-1$ or else, by removing at most $k-2$ vertices, we get an induced subgraph H, $|H| \geq k$ with at least $k-1$ vertices realising the maximum degree of H.

If $\Delta(H) = 1$ then, since $k-1$ is odd, it forces that there are at least k 1-vertices but then G is k-feasible. Otherwise $\Delta(H) = 0$ but $|H| \geq k$ and again G is k-feasible.

So only the case $n_2 = k-1$ is left. Since $n = 2k-2$ and $n_2 = k-1$ then by parity $n_1 \leq k-2$ and $n_0 \geq 1$.

We collect the (possible) components of G into three subgraphs: $A = \{$all isolated vertices and isolated edges$\}$, $B = \{$all copies of $K_{1,2}\}$, $C = \{$all other components$\}$.

We denote by t the number of copies of $K_{1,2}$ in B and also observe that $t < n_2 = k-1$ since otherwise $|G| = 3k-3 > 2k-2 = |G|$ a contradiction since $k \geq 2$.

Also observe that in each component in C the vertices of degree 2 induced either on a path (including a single edge) or a cycle.

Claim: If $t > \lfloor \frac{k-1}{2} \rfloor$ we are done.

This is because F is not empty since $t < n_2$, and by the observation above $\delta(F) \geq 1$ hence by Ore's result the domination number of F satisfies $\gamma(F) \leq \lfloor \frac{n_2-t}{2} \rfloor \leq \lfloor \frac{k-1-t}{2} \rfloor$.

Let D be a minimum dominating set for F. Deleting D from C and from each copy of $K_{1,2}$ in B we delete a leaf to get an induced subgraph H with $\Delta(H) = 1$ and with at least $2(\lfloor \frac{k-1}{2} \rfloor + 1) \geq k$ vertices of degree 1, meaning G is k-feasible. Observe we have deleted at most

$$t + \lfloor \frac{n_2 - t}{2} \rfloor \leq \frac{n_2 + t}{2} \leq \lfloor \frac{2n_2 - 1}{2} \rfloor = \lfloor \frac{2k-3}{2} \rfloor = k - 2$$

vertices, proving the claim.

Consider the subgraph F induced by the vertices of degree 2 in C and recall $|F| > 0$, hence $|F| \geq 2$.

Then as we have noted before, due to the components of C, there are no isolated vertices in F, and by Ore's result $\gamma(F) \leq \lfloor \frac{n_2-t}{2} \rfloor \leq \lfloor \frac{k-1-t}{2} \rfloor$.

Let D be a dominating set for D that realises $\gamma(F)$, hence $|D| \leq \frac{n_2-t}{2}$.

Delete D and consider the induced subgraph H on $A \cup C$. Clearly $\Delta(H) \leq 1$ and denote by x_1 the number of vertices of degree 1 in H.

Now let us look again at B.

Case 1: $t = 0$.

Since $t = 0$, B is empty, and we have deleted $|D| \leq \lfloor \frac{n_2}{2} \rfloor \leq \lfloor \frac{k-1}{2} \rfloor = \frac{k-2}{2}$ vertices since k is even. So the number of vertices remains is at least $2k-2-\frac{k-2}{2} = \frac{3k-2}{2}$.

But as k is even, $h(1,k) = \lfloor \frac{k}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor = \frac{k}{2} + \frac{2(k-2)}{2} = \frac{3k-4}{2} < \frac{3k-2}{2}$ hence H is k-feasible and so G is k-feasible.

Case 2: $1 \leq t \leq \lfloor \frac{k-1}{2} \rfloor$.

We consider two cases:

(a) if $x_1 \geq k - 2t$ then deleting a leaf from every copy of $K_{1,2}$ in B we get an induced graph H^* on $A \cup B \cup C$ (extending H to the leftover of B) with $\Delta(H^*) = 1$ and at least $k - 2t + 2t = k$ vertices of degree 1 and we are done as we have deleted altogether $|D| + t \leq \frac{n_2-t}{2} + t = \frac{n_2+t}{2} \leq k-2$ (as before). Hence G is k-feasible.

(b) if $x_1 \leq k - 1 - 2t$ (recall x_1 is the number of vertices of degree 1 in H formed from $A \cup \{C \backslash D\}$), then by the even parity of x_1 and as k is even we must have $x_1 \leq k - 2 - 2t$.

Now delete $\frac{x_1}{2}$ independent vertices of degree 1 in H, and t vertices of degree 2 in B to get an induced subgraph H^* with $\Delta(H^*) = 0$.

We have removed

$$\frac{x_1}{2} + |D| + t \leq \frac{k-2-2t}{2} + \frac{k-1-t}{2} + t = \frac{2k-3-t}{2} \leq \frac{2k-4}{2} = k-2$$

vertices (since $t \geq 1$), hence $|H^*| \geq k$ and we have k vertices of degree 0 realizing $\Delta(H^*)$. Hence H^* is k-feasible and so does G, completing the proof.

5. Suppose $|G| = g(\Delta, k) + k$ and $\Delta(G) = \Delta$. Then by the definition of $g(\Delta, k)$, by deleting at most $g(\Delta, k)$ vertices we either get below k vertices or have an induced subgraph H with at least k vertices realizing the maximum degree of H.

   But deleting $g(\Delta, k)$ vertices from G will leave us with a graph on at least k vertices hence the second possibility above holds and G is k-feasible, and we conclude that $h(\Delta, k) \leq g(\Delta, k) + k - 1$.

□

5 Open Problems

We conclude by proposing the following open problems:

1. Certainly the most intriguing problem is to solve the Caro-Yuster conjecture that $f(n, k) \leq f(k)\sqrt{n}$. As mentioned we proved that $f(2) = \sqrt{2}$ is sharp and best possible, and it is known that $f(3) \leq 43$. For $k \geq 4$ the conjecture remains open. Even a proof that $f(n, k) = o(n)$ is of interest.

2. Theorem 2.4 supplies an $O(n^2)$ algorithm to compute $f(G) = f_2(G)$. Can $f_3(G)$ be computed in polynomial time?

3. We have calculated, in section 4 , the exact values of $g(\Delta, k)$ for $\Delta = 0, 1, 2$, and we have given a general constructive lower bound for $g(\Delta, k)$. Determining $g(3, k)$ seems a considerably more involved task, as well as proving a conjecture inspired by the Caro-Yuster conjecture namely:

   Conjecture 5.1. *For $k \geq 2$ there is a constant $g(k)$ such that $g(\Delta, k) \leq g(k)\sqrt{\Delta}$.*

   This conjecture, if true, implies the Caro-Yuster conjecture.

4. We introduced the notion of a k-feasible graph and the corresponding function $h(\Delta, k)$ discussed in Section 4. We have determined the exact values of $h(\Delta, k)$ for $\Delta = 0, 1, 2$. We pose the problem to determine more exact values of $h(\Delta, k)$ in particular for $\Delta = 3$ as well as to determine $h(k) = \max\{h(\Delta, k) : \Delta \geq 0\}$. Clearly as already proved in section 4, $h(k) \leq R(k, k) - 1$.

5. Lastly we mention again the conjecture about forests:

   Conjecture 5.2. *If F is a forest on n vertices, where $n \leq \frac{t^3+6t^2+17t+12}{6}$ then $f(F) \leq t$ and this bound is sharp.*

Edit in proof: Since this paper was submitted, Conjecture 3.1 has been proved in [14], and this topic has been extended in [15, 17].

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