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Hadamard Matrices, Orthogonal Designs and Clifford-Gastineau-Hills Algebras

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Dedicated with great respect to Anne Penfold Street

Abstract

Research into the construction of Hadamard matrices and orthogonal designs has led to deeper algebraic and combinatorial concepts. This paper surveys the place of amicability, repeat designs and the Clifford and Clifford-Gastineau-Hills algebras in laying the foundations for a *Theory of Orthogonal Designs*.

Research into the existence question for Hadamard matrices has been crucial in forcing the study of related theoretical results. The pioneering work by Kathy Horadam in her studies on the five-fold path [12], and her ground-breaking efforts with Warwick de Launey on cocyclic Hadamard matrices [4] are examples, and the foundational efforts by Warwick de Launey and Dane Flannery [3] on algebraic design theory yet another. Paul Leopardi has explored their relationship to amicability/anti-amicability graphs [16]. Other authors have concentrated further on their applications and structure in multidimensional space.

To construct Hadamard matrices, Geramita and Seberry [8] used orthogonal designs. This survey discusses the path from Hadamard matrices to orthogonal designs, amicable Hadamard matrices and anti-amicable Hadamard matrices to amicable orthogonal designs, and then to constructs called product designs and repeat designs. In each case the number of variables possible has been solved by converting the question into algebra. The study of the role of algebras in orthogonal design constructions leads us to see that product designs are subsets of repeat designs. The algebras of orthogonal designs are Clifford algebras and the algebras of repeat designs are Clifford-Gastineau-Hills algebras. The study of the algebras allows us to obtain exactly the maximum possible number of variables in each of the designs studied.

We consider Clifford algebras in a more complex context, over fields of characteristic 2: we observe that in fact characteristic $\neq 2$ is easier to deal with, and characteristic 2 is a special case. We do not treat these, but refer the reader to Lam [15], O'Meara [17], Kawada and Iwahori [14] and Artin [1]. The more modern view has been that Clifford algebras arise naturally from quadratic forms. In fact the class of all Clifford algebras corresponding to non-singular quadratic forms over a field F of characteristic not 2 coincides with the class of all F-algebras, C, on a finite number of generators $\{\alpha_i\}$ with defining equations of the form

$$\begin{array}{ll} \alpha_i^2 = k_i & (\text{some } k_i \in \mathcal{F} = F \setminus \{0\}) \\ \alpha_j\alpha_i = -\alpha_i\alpha_j & (i \neq j). \end{array} \tag{1}$$

We identify k_i in F with $k_i 1_C$ in C.

This leads to questions about how this knowledge, when applied to Hadamard matrices of orders which are powers of two, may be able to have embedded substructures to hide messages and/or improve some error correction capabilities. Conceivably such deeper knowledge may have applications in other areas such as spectrometry, sound enhancement or compression and other signal processing.

1 Introduction

Eddington, in 1932, in his studies of relativity, raised the combinatorial question "What is the largest number of matrices of a given order which can anti-commute and square to $-I$, I the identity matrix?" (see [5,6]). Strongly related to this is the work of Radon and Hurwitz [7,13,18] on orthogonal matrices and the composition of quadratic forms which we also use. We will see that a set of p $n \times n$ matrices E_i which satisfy the algebraic conditions

$$\begin{array}{ll} E_i^2 = -I & (1 \leq i \leq p) \\ E_jE_i = -E_iE_j & (1 \leq i < j \leq p)\,, \end{array} \tag{2}$$

is *necessary* for the existence of an orthogonal design of order n on $p+1$ variables.

That it is *sufficient* is not immediately clear, since the E_i must satisfy other combinatorial conditions, namely

$$\text{each } E_i \text{ is a } \{0, \pm 1\} \text{ matrix and } E_j * E_i = 0 \ (i \neq j)\,, \tag{3}$$

where "*" is the *Hadamard product* defined as

$$(a_{ij}) * (b_{ij}) = (a_{ij}b_{ij})$$

the component-wise multiplication.

An algebra, which is associative with a "1", on p generators, $\alpha_1, \ldots, \alpha_p$ say, with defining equations

$$\begin{aligned} \alpha_i^2 &= -1 && (1 \le i \le p) \\ \alpha_j\alpha_i &= -\alpha_i\alpha_j && (1 \le i < j \le p)\,. \end{aligned} \tag{4}$$

is an example of the well-known *Clifford Algebras*.

2 Orthogonal Designs

While orthogonal designs are known (see [23]) with complex and quaternion elements, we shall only consider cases with real entries.

Definition 1. An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted

$$OD(n; s_1, s_2, \ldots, s_u)$$

on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with real entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words it has the *additive property*,

$$AA^\top = \left(s_1x_1^2 + \ldots + s_ux_u^2\right) I_n \tag{5}$$

where I_n is the identity matrix.

We use the notation '$-$' for -1. Later we use the following notation:

$OD(n; s_1, \ldots, s_m)$ orthogonal design or OD

$$XX^\top = (\sum_{i=1}^{m} s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. That is, $\pm x_i$ occurs s_i times in each row (and column) of X. See [23, p1].

$AOD(n : (u_1, \ldots, u_s); (v_1, \ldots v_t))$ amicable orthogonal designs or AOD when (i) X is an $OD(n; u_1, \ldots, u_s)$, (ii) Y is an $OD(n; v_1, \ldots, u_s)$, and (iii) $XY^\top = YX^\top$. See [23, p157].

$POD(n : a_1, a_2 \ldots, a_r; b_1, \ldots b_s; c_1, \ldots, c_t)$ product (orthogonal) design or POD when (i) M_1 is an $OD(n; a_1, \ldots, a_r)$, (ii) M_2 is an $OD(n; b_1, \ldots, b_s)$, (iii) N is an $OD(n; c_1, \ldots, c_t)$ and (I) $M_1 * N = M_2 * N = 0$ (* the Hadamard product), (II) $M_1 + N$ is an $OD(n; a_1, \ldots, a_r, c_1, \ldots, c_t)$, (III) $M_2 + N$ is an $OD(n; b_1, \ldots, b_s, c_1, \ldots, c_t)$ and (IV) M_1 and M_2 are AOD$(n : (a_1, \ldots, a_r); (b_1, \ldots, b_s))$. See [23, p215].

$ROD(n : (r_1, r_2 \ldots); (p_{11}, p_{12} \ldots; p_{21}, p_{22}, \ldots; \ldots; \ldots); (h_1, h_2 \ldots))$ repeat (orthogonal) design or ROD when
(i) X is an $OD(n; r_1, r_2, \ldots)$,
(ii) Y_i is an $OD(n; p_{i1}, p_{i2}, \ldots)$,
(iii) Z is an $OD(n; h_1, h_2, \ldots)$ and (I) $X * Y_i = 0$ (* the Hadamard product),
(II) $X + Y_i$ are $OD(n; r_1, r_2, \ldots, p_{i1}, p_{i2}, \ldots,)$,
(III) $X + Y_i$ and Z are $AOD(n; (r_1, r_2, \ldots, p_{i1}, p_{i2}, \ldots); (h_1, h_2, \ldots))$ and
(IV) Y_i and Y_j are AOD$(n : (p_{i1}, p_{i2}, \ldots); (p_{j1}, p_{j2}, \ldots))$, $i \ne j$. See [23, pp.221-222].

Example 1. We observe the $OD(4;1,1,1,1)$, D, and note it can be written, up to equivalence, as either

$$D = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \tag{6}$$

or as the sum

$$D = aE_1 + bE_2 + cE_3 + dE_4\,,$$

where a, b, c and d are commuting variables (they do not need to be real) and

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix}.$$

Since D is an orthogonal design,

$$DD^\top = \left(a^2 + b^2 + c^2 + d^2\right) I_4\,.$$

The algebraic conditions which make this an orthogonal design are

$$\begin{array}{ll} E_1^2 = I\,,\ E_i^2 = -I & (2 \le i \le 4) \\ E_jE_i = -E_iE_j & (1 \le i < j \le 4)\,, \end{array} \tag{7}$$

and the combinatorial conditions which make this an orthogonal design are

$$\text{each } \ E_i \text{ is a } \{0, \pm 1\} \text{ matrix and } \ E_j * E_i = 0 \ (i \ne j)\,. \tag{8}$$

Thus we have linked the orthogonal design, the quadratic form $a^2+b^2+c^2+d^2$ and the Clifford-type algebras together. The orthogonal design has the extra properties that $E_1^2 = I$ and disjointness of matrices in the combinatorial conditions.

The fact that the structure and representation theory of the Clifford algebra (4) are known means that Eddington's problem can be solved (see Kawada and Iwahori, [14]). Moreover this representation theory is known to give a complete solution to the problem of determining the possible orders of orthogonal designs on any number of variables. As noted above, the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, and $a = 4c + d, 0 \le d < 4$, we have $\rho(n) = 8c + 2^d$ [8].

We note the similarity of equations (2) with those of (7) and (1).

3 Amicable Orthogonal Designs

In the paper, Geramita-Geramita-Wallis [10], the following remarkable pairs of matrices are given:

$$X_2 = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}; \qquad Y_2 = \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix}; \tag{9}$$

and

$$X_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_1 & -x_2 \\ x_3 & -x_3 & x_2 & -x_1 \end{bmatrix}; \qquad Y_4 = \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_2 & y_1 \\ -y_3 & y_3 & y_1 & -y_2 \end{bmatrix}. \tag{10}$$

They are remarkable in that they satisfy $X_iY_i^\top = Y_iX_i^\top$ and are called amicable orthogonal designs $AOD(2 : (1,1;1,1))$ and $AOD(4 : (1,1,2;1,1,2))$ respectively. The first pair satisfy the following equations

$$\begin{aligned} XX^\top &= \left(x_1^2 + x_2^2\right) I_2 \\ YY^\top &= \left(y_1^2 + y_2^2\right) I_2 \\ XY^\top &= \begin{bmatrix} x_1y_1 + x_2y_2 & -x_1y_2 + x_2y_1 \\ x_2y_1 - x_1y_2 & -x_1y_1 - x_2y_2 \end{bmatrix} = YX^\top. \end{aligned} \tag{11}$$

Thus the quadratic form has the unique property that it factors into $(x_1^2+x_2^2)(y_1^2+y_2^2)$.

$$\left[XY^\top\right]\left[XY^\top\right]^\top = XY^\top YX^\top = \left(x_1^2 + x_2^2\right)\left(y_1^2 + y_2^2\right) I_2\,. \tag{12}$$

The second pair satisfies the equations

$$\begin{aligned} XX^\top &= \left(x_1^2 + x_2^2 + 2x_3^2\right) I_4\,, \\ YY^\top &= \left(y_1^2 + y_2^2 + 2y_3^2\right) I_4\,, \\ XY^\top &= YX^\top\,, \end{aligned} \tag{13}$$

and

$$\left[XY^\top\right]\left[XY^\top\right]^\top = XY^\top YX^\top = \left(x_1^2 + x_2^2 + 2x_3^2\right)\left(y_1^2 + y_2^2 + 2y_3^2\right) I_4\,.$$

We then ask do any more such amicable pairs of matrices exist? If so, then how many variables can occur in each of any such pair of orthogonal designs, called *amicable orthogonal designs*, for a given order. The question of the maximum number of variables has been solved completely by Shapiro in his PhD thesis and published in [24]. Orders 2, 4 and 8 are constructed in the PhD Thesis of D. J. Street [25] and Y. Zhao [28]. Seberry [23, Sections 5.5, 5.9] discusses orders 2, 4, and 8 but other orders remain, as yet, unconstructed. The next problem is to determine whether orthogonal designs actually exist satisfying these necessary conditions for the existence of amicable orthogonal designs. Fortunately the papers and PhD Thesis of P. Robinson give many possibilities [19, 21, 23].

In constructing Hadamard matrices *amicability* and *anti-amicability* proved a useful tool. Its extension to orthogonal designs proved decisive in the equating and killing theorem of Geramita and Seberry [8]. Indeed it is crucial to Craigen's [2] extension to the previously known asymptotic existence results [26].

So let us be more precise and investigate further.

Definition 2. Two orthogonal designs X and Y are said to be *amicable* if $XY^\top = YX^\top$ and to be *anti-amicable* if $XY^\top = -YX^\top$. An *amicable k-set* will be used to describe a set of k matrices $X_1, \dots, X_k$ which pairwise satisfy $X_iX_j^\top = X_jX_i^\top$ for all $1 \le i,\ j \le k$ and an *anti-amicable k-set* if $X_1, \dots, X_k$ pairwise satisfy $X_iX_j^\top = -X_jX_i^\top$ for all $1 \le i,\ j \le k$, $i \ne j$.

Remark 1. We note that the definitions of *amicable k-set* and *anti-amicable k-set* are mentioned here for purely historical reasons. It was Wolfe's [27] inspiration in considering amicable pairs and amicable triples that led to the insight of the importance of Clifford algebras (4) in solving the question of the number of variables possible in an orthogonal design. However, as we will see, amicable k-sets or k-tuples are a special case of repeat designs.

Example 2. We now use part of a proof of Lemma 5.142 of Geramita and Seberry [8]. Where the matrices below have blank space, they should be considered to be filled with zeros. We give two examples of amicable triples (three matrices which are pairwise amicable) to show their existence. A. Neeman found the following $(1, 7, 1)$ which has 1, 7, and 1 non-zero entries in each row and column respectively:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ - & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & - & 0 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & - & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & - & 0 \end{array}\right], \quad \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & - & - & 1 & 1 & 1 \\ - & 0 & 1 & 1 & - & - & - & 1 \\ - & - & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & - & 0 & - & 1 & - & 1 \\ \hline 1 & 1 & - & 1 & 0 & - & 1 & 1 \\ - & 1 & - & - & 1 & 0 & - & 1 \\ - & 1 & - & 1 & - & 1 & 0 & - \\ - & - & - & - & - & - & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 1 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 0 \end{array}\right].$$

The following three matrices give a $(2,7,1)$, illustrating that the concept of amicable triples might lead to different kinds of construction:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 0 & & & & \\ - & 0 & 0 & - & & & & \\ - & 0 & 0 & 1 & & & & \\ 0 & 1 & - & 0 & & & & \\ \hline & & & & 0 & 0 & 1 & 1 \\ & & & & 0 & 0 & 1 & - \\ & & & & - & - & 0 & 0 \\ & & & & - & 1 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & - & 1 & - & - & 1 \\ - & 0 & 1 & - & - & 1 & 1 & 1 \\ - & - & 0 & 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & 0 & 1 & 1 & 1 & 1 \\ \hline - & 1 & - & - & 0 & 1 & - & - \\ 1 & - & - & - & - & 0 & - & 1 \\ 1 & - & 1 & - & 1 & 1 & 0 & - \\ - & - & - & - & 1 & - & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ \hline & & & & 0 & - & 0 & 0 \\ & & & & - & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & - \end{array}\right].$$

□

We note that the pair of matrices, X_4 and Y_4, given by equation (10) may be written as

$$X_4 = \sum_{i=1}^{3} x_i A_i\,, \qquad Y_4 = \sum_{i=1}^{3} y_i B_i\,, \tag{14}$$

$$(A_i, B_j\ \{0, \pm 1, \pm 2\} \text{ matrices where } A_i * A_j = 0, B_i * B_j = 0, \text{ for } i \neq j)\,. \tag{15}$$

Substituting (14) into (13) and comparing like terms gives:

$$\begin{cases} A_i A_i^\top = u_i I, \qquad B_j B_j^\top = v_j I, \\ A_i A_j^\top + A_j A_i^\top = 0\ \ (i \neq j), \quad B_i B_j^\top + B_j B_i^\top = 0\ \ (i \neq j), \\ A_i B_j^\top = B_j A_i^\top \text{ (for all } i, j), \end{cases}$$

and similar equations with products reversed.

Set

$$E_i = \frac{1}{\sqrt{u_i u_0}} A_i A_0^\top\,, \qquad F_j = \frac{1}{\sqrt{v_j u_0}} B_j A_0^\top.$$

It is easily verified that $E_0 = I$ and E_1, E_2, F_1, F_2, F_3, satisfy

$$\begin{cases} E_i^2 = -I\,, \qquad F_j^2 = -I\,, \\ E_j E_i = -E_i E_j\ \ (i \neq j) \qquad F_i F_j = -F_j F_i\ \ (i \neq j)\,, \\ E_i F_j = F_j E_i \text{ (for all } i, j)\,. \end{cases}$$

The equations can be considered as a “Clifford-like algebra" with generators α_1, α_2, β_1, β_2 and β_3,

$$\begin{array}{lll} \alpha_i^2 = -1 & \beta_i^2 = -1 & (-1 \in \mathcal{F} = F/\{0\}) \\ \alpha_j\alpha_i = -\alpha_i\alpha_j & \beta_j\beta_i = -\beta_i\beta_j & (i \neq j)\,, \\ \alpha_i\beta_j = \beta_j\alpha_i & & \end{array}$$

4 Foundational Motivating Constructions for Orthogonal Designs

Geramita and Seberry [8] gave a number of constructions; these were first named product designs and repeat designs in Robinson's PhD Thesis [19]. The next construction for orthogonal designs appears in a slightly different form in [8].

Construction 1. *Let x_1, x_2 be commuting variables and W, Y_1 and Y_2 be matrices of order n described by*

1. $W * Y_i = 0$, *for* $i = 1,\ 2$;
2. $Y_1 Y_2^\top = Y_2 Y_1^\top$ *more precisely* $AOD(n : (u_1, u_2, \ldots; v_1, v_2; \ldots; w))$;
3. W *is an* $OD(n; w)$; *and*
4. $Y_i W^\top = -W Y_i^\top$ *for* i = 1, 2.

Then the following matrix is an $OD(2n; w, w, u_1, u_2, \ldots, v_1, v_2, \ldots)$*:*

$$\begin{bmatrix} Y_1 + x_1 W & Y_2 + x_2 W \\ Y_2 - x_2 W & -Y_1 + x_1 W \end{bmatrix}.$$

Construction 2 (Geramita-(Seberry) Wallis [11])**.** *Let Y_1, Y_2, Y_3 be skew-symmetric orthogonal designs of types $(p_{i1}, p_{i2},$
ldots), $i = 1, 2, 3$ in order n, and Z a symmetric $OD(n; h_1, h_2, \ldots)$. Further, suppose $Y_i Y_j^\top = Y_j Y_i^\top$ and $Y_k Z^\top = Z Y_k^\top$. Then*

$$\begin{bmatrix} x_1 I_n + Y_1 & x_2 I_n + Y_2 & x_3 I_n + Y_3 & Z \\ -x_2 I_n + Y_2 & x_1 I_n - Y_1 & Z & -x_3 I_n - Y_3 \\ -x_3 I_n + Y_3 & -Z & x_1 I_n - Y_1 & x_2 I_n + Y_2 \\ -Z & x_3 I_n - Y_3 & -x_2 I_n + Y_2 & x_1 I_n + Y_1 \end{bmatrix}$$

is an $OD(4n; 1, p_{11}, p_{12}, \ldots, 1,\ p_{21}, p_{22}, \ldots, 1,\ p_{31}, p_{32}, \ldots, h_1, h_2, \ldots$.

Proof. Both proofs are by straightforward verification. □

Closer study of these two constructions shows that if we replace W by the identity matrix and Z by the zero matrix O the matrices satisfy the same equations. The first was previously used as an illustration of a product design and the second given as an illustration of a repeat design. We now proceed to study the more general concept of repeat designs.

5 Repeat Orthogonal Designs

Robinson and Seberry [22] defined a *repeat design*, but we prefer to give the formal definition in an alternative form given by Gastineau-Hills [6, pp.29-30]:

Definition 3. Suppose $X, Y_1, \ldots, Y_k, Z$ are orthogonal designs of order n, types $(u_1, \ldots, u_p)$, $(v_{11}, \ldots, v_{1q_1}), \ldots, (v_{k1}, \ldots, v_{kq_k})$, and $(w_1, \ldots, w_r)$ on the variables $(x_1, \ldots, x_p)$, $(y_{11}, \ldots, y_{1q_1}), \ldots, (y_{k1}, \ldots, v_{kp_k})$, and $(z_1, \ldots, z_r)$ respectively, and that

(i) $X * Y_i = 0$ (for all i)
(ii) $Y_i X^\top = -XY_i^\top$
(iii) $Y_j Y_i^\top = Y_i Y_j^\top, \qquad ZX^\top = XZ^\top, \qquad ZY_i^\top = Y_i Z^\top$ (all i, j).

Then we call the $(k+2)$-set $(X, Y_1, \ldots, Y_k, Z)$ a *repeat design* of order n,

$$ROD(n : u_1, \ldots, u_p; v_{11}, \ldots, v_{1q_1}; \ldots; v_{k1}, \ldots, v_{kq_k}; w_1, \ldots, w_r)$$

on the variables $(x_1, \ldots, x_p;\ y_{11}, \ldots, y_{1q_1}; \ldots; y_{k1}, \ldots, v_{kp_k}; z_1, \ldots, z_r)$.

$X, Y_1, \ldots, Y_k, Z$ in Definition 3 correspond to $R, P_1, \ldots, P_k, H$ respectively in [8]. Otherwise, apart from the fact that we have allowed X in Definition 3 to be on more than one variable, the conditions here and in [8] are equivalent.

Product designs [9] may be regarded as particular cases of repeat designs, given by $k = 2$, $r = 0$ and $Z = 0$ (zero matrix, which may be regarded as an orthogonal design on no variables).

A theory of repeat designs should yield a theory of amicable k-sets, if we can allow $X = Z = 0$. In the immediate following we assume that X has at least one variable (while allowing $Y_1, \ldots, Y_k, Z$ to have as few as no variables each), but it will be found that this restriction may be removed painlessly.

Remark 2. The existence problem for triples (R, S, H) which are repeat designs $(I; (R; S); H)$ is very difficult and far from resolved.

The following is given in Gastineau-Hills [6, pp.31-32]

$$\begin{array}{l} XX^\top = \left(\sum_0^p u_j x_j^2\right) I\,, \quad Y_i Y_i^\top = \left(\sum_1^{q_i} v_{ij} y_{ij}^2\right) I\,, \quad ZZ^\top = \left(\sum_1^r w_j z_j^2\right) I \\ Y_i X^\top = -XY_i^\top\,. \end{array} \tag{16}$$

with similar equations for $X^\top X$, etc.,

Write

$$X = \sum_0^p x_j A_j\,, \qquad Y_i = \sum_1^{q_i} y_{ij} B_{ij}\,, \qquad Z = \sum_1^r z_j C_j \tag{17}$$

$$(A_j, B_{ij}, C_j\,, \{0 \pm 1\} \text{ matrices}) \tag{18}$$

Substituting into (16) and comparing like terms gives:

$$\begin{cases} A_jA_j^\top = u_jI, \qquad B_{ij}B_{ij}^\top = v_{ij}I, \qquad C_jC_j^\top = w_jI, \\ A_iA_j^\top + A_jA_i^\top = 0 \ \ (i \neq j), \qquad B_{ij}B_{ik}^\top + B_{ik}B_{ij}^\top = 0 \ \ (j \neq k), \\ C_iC_j^\top + C_jC_i^\top = 0 \ \ (i \neq j), \\ B_{jk}A_i^\top = -A_iB_{jk}^\top\,, \\ B_{k\ell}B_{ij}^\top = B_{ij}B_{k\ell}^\top \ \ (i \neq k)\,, \quad C_kB_{ij}^\top = B_{ij}C_k^\top\,, \quad C_jA_i^\top = A_iC_j^\top\,, \end{cases}$$

and similar equations with products reversed.

Set

$$E_i = \frac{1}{\sqrt{u_iu_0}}A_iA_0^\top\,, \quad F_{ij} = \frac{1}{\sqrt{v_{ij}u_0}}B_{ij}A_0^\top\,, \quad G_i = \frac{1}{\sqrt{w_iu_0}}C_iA_0^\top\,.$$

It is easy to verify $E_0 = I$ and $E_1, \ldots, E_p$, $F_{11}, \ldots, F_{1p_1}$, $F_{k1}, \ldots, F_{kp_k}$, $G_1, \ldots, G_r$ satisfy

$$\begin{cases} E_i^2 = -I\,, \qquad F_{ij}^2 = -I\,, \qquad G_i^2 = I \\ E_jE_i = -E_iE_j \ \ (i \neq j) \qquad F_{ik}F_{ij} = -F_{ik}F_{ij} \ \ (j \neq k)\,, \\ G_jG_i = -G_jG_i \ \ (i \neq j) \\ F_{jk}E_i = -E_iF_{jk}\,, \qquad G_jE_i = -E_iG_j\,, \qquad G_kF_{ij} = -F_{ij}G_k \\ F_{k\ell}F_{ij} = F_{ij}F_{k\ell} \ \ (i \neq k)\,, \end{cases}$$

Thus we have arrived at degree n representation of a real algebra which is "Clifford-like", with the one "non-Clifford" property that some pairs of distinct generators commute.

6 Quasi-Clifford Algebras and Clifford-Gastineau-Hills Algebras

We write CGH-algebra for a *Clifford-Gastineau-Hills algebra.*

Definition 4. A *Clifford-Gastineau-Hills algebra* is a real algebra on $p+q_1+\cdots q_k+r$ generators $\alpha_1, \ldots, \alpha_p$, $\beta_{11}, \ldots, \beta_{1q_1}, \ldots, \beta_{k1}, \ldots, \beta_{kq_k}$, $\gamma_1, \ldots, \gamma_r$, with defining equations:

$$\begin{cases} \alpha_i^2 = -1\,, \quad \beta_{ij}^2 = -1\,, \quad \gamma_i^2 = 1\,, \\ \alpha_j\alpha_i = -\alpha_i\alpha_j \ \ (i \neq j)\,, \quad \beta_{ik}\beta_{ij} = -\beta_{ij}\beta_{ik} \ \ (j \neq k)\,, \\ \gamma_j\gamma_i = -\gamma_i\gamma_j \ \ (i \neq j)\,, \\ \beta_{jk}\alpha_i = -\alpha_i\beta_{jk}\,, \quad \gamma_j\alpha_i = -\alpha_i\gamma_j\,, \quad \gamma_k\beta_{ij} = -\beta_{ij}\gamma_k\,, \\ \beta_{k\ell}\beta_{ij} = \beta_{ij}\beta_{k\ell} \ \ (i \neq k)\,. \end{cases} \tag{19}$$

For a repeat design of order n on $p+1, q_1, \ldots, q_k, r$ variables to exist it is *necessary* for a real degree n faithful representation of this algebra to exist.

Gastineau-Hills [6] answers completely the questions of just what are the possible orders of representations of (19), and whether the existence of a degree n representation of (19) is sufficient for the existence of repeat design (16).

Observe that the case of product designs is included in what we have just done — we simply take $k = 2$ and $r = 0$.

If we also rewrite $q_1, q_2, \beta_{1j}, \beta_{2j}$ as q, r, β_j, γ_j respectively we find that the existence of an order n product design on $(p + 1, q, r)$ variables implies the existence of a degree n representation of the real algebra on $p + q + r$ generators $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_r$ with defining equations

$$\begin{cases} \alpha_i^2 = \beta_j^2 = \gamma_k^2 = -1 \\ \alpha_j\alpha_i = -\alpha_i\alpha_j\,, \quad \beta_j\beta_i = -\beta_i\beta_j\,, \quad \gamma_j\gamma_i = -\gamma_i\gamma_j \;\; (i \neq j) \\ \beta_j\alpha_i = -\alpha_i\beta_j\,, \quad \gamma_j\alpha_i = -\alpha_i\gamma_j\,, \quad \gamma_j\beta_i = \beta_i\gamma_j\,, \end{cases} \tag{20}$$

again a "not-quite-Clifford" algebra in that the defining equations differ slightly from those of a Clifford algebra.

Note that (20) is not quite the same as equation (3.10) in [6, p.20], so that a theory of amicable triples need not necessarily by itself yield a theory of product designs.

In fact not even equation (3.8) in [6, p.18] (the algebra corresponding to more general amicable k-sets), seems to contain (20) as a particular case.

Then we have

Theorem 1. *Let $(L; M_1 + M_2 + \cdots + M_s; N)$ be product designs $POD(n : a_1, \ldots, a_p; b_{11}, \ldots, b_{1_{q_1}}, b_{21}, \ldots, b_{2_{q_2}}, \ldots, b_{s1}, \ldots, b_{sq_s}; c_1, \ldots, c_t)$, where M_i is of type $(b_{i1}, \ldots, b_{i_{q_i}})$.*

Further, let $(X; (Y_1; Y_2; \ldots; Y_u); Z)$ be repeat orthogonal designs, $ROD(m : (r_1, \ldots, r_w); (p_{11}, \ldots, p_{1_{v_1}}; p_{21}, \ldots, p_{2_{v_2}}; p_{u1}, \ldots, p_{uv_u});$ $h_1, \ldots, h_x)$. Then, with $\times$ the Kronecker product

$$L \times X + M_1 \times Y_{j1} + \cdots + M_k \times P_{jk} + N \times Z$$

is an orthogonal design of order mn and type 1 with one of the following four sets of parameters

(i) $OD(mn; a_1r, \ldots, a_pr,\, b_1p_{11}, \ldots,\, b_1p_{1_{v_1}}, \ldots,\, b_sp_{s1}, \ldots,\, b_sp_{sq_s},\, ch_1, \ldots, ch_x)$,

(ii) $OD(mn; a_1r, \ldots, a_pr,\, b_1p_{11}, \ldots,\, b_1p_{1_{v_1}}, \ldots,\, b_sp_{s1}, \ldots,\, b_sp_{sq_s},\, c_1h, \ldots, c_th)$,

(iii) $OD(mn; ar_1, \ldots, ar_w,\, b_1p_{11}, \ldots,\, b_1p_{1_{v_1}}, \ldots,\, b_sp_{s1}, \ldots,\, b_sp_{sq_s},\, ch_1, \ldots, ch_x)$,

(iv) $OD(mn; ar_1, \ldots, ar_w,\, b_1p_{11}, \ldots,\, b_1p_{1_{v_1}}, \ldots,\, b_sp_{s1}, \ldots,\, b_sp_{sq_s},\, c_1h, \ldots, c_th)$.

where a, c, r, h *are the sum of some or all of the* a_i, c_i, r_i, h_i, *respectively, and* $b_i = b_{i1} + \cdots + b_{iq_i}$.

Proof. We use different combinations of the parameters by equating as necessary. □

This construction is at first sight quite formidable, but we see it does lead to new orthogonal designs [23].

Geramita and Seberry [8], using many theorems by P.J. Robinson [19], give many results on the usefulness of the previously mentioned product designs. However, we need to give some repeat designs as our argument is that product designs are a subset of repeat designs. First we see that repeat designs do lead to new designs:

Example 3. The list below of repeat designs are examples of creating new designs: for example to choose the the X, Y_1, Y_2 Z of the $ROD(4 : (1; (1; 3); 1, 3))$ we use $X = I$, $Y_1 = T_1$, $Y_2 = T_4$ and $Z = T_0$.

ROD	Design
$ROD(4 : (1; (1; 3); 1, 3))$	$(I; (T_1; T_4); T_0)$
$ROD(4 : (1; (2; 3); 1, 3))$	$(I; (T_3; T_4); T_0)$
$ROD(4 : (1; (1; 2); 1, 1, 2))$	$(I; (T_1; T_3); T_3)$
$ROD(4 : (1; (2; 1, 2); 1, 2))$	$(I; (T_2; T_6); T_7)$

where

$$T_0 = \begin{bmatrix} x & y & y & y \\ y & -x & y & y \\ y & -y & y & -x \\ y & y & -x & -y \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & - \\ - & - & 0 & 0 \\ - & + & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & - & + \\ - & + & 0 & 0 \\ - & - & 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} u & v & w & w \\ v & -u & -w & w \\ w & -w & v & -u \\ w & w & -u & -v \end{bmatrix},$$

$$T_6 = \begin{bmatrix} 0 & a & b & b \\ -a & 0 & -b & b \\ -b & b & 0 & -a \\ -b & -b & a & 0 \end{bmatrix}, \quad T_7 = \begin{bmatrix} u & 0 & w & w \\ 0 & -u & -w & w \\ w & -w & 0 & -u \\ w & w & -u & 0 \end{bmatrix}.$$

These repeat designs can be constructed using Theorem 1.

$ROD(4 : (1;(1, 1; 1, 1); 1))$ $ROD(4 : (1;(1, 1; 1, 2); 2))$
$ROD(4 : (1; (1, 1; 2); 1, 2))$ $ROD(4 : (1; (1; 1, 2); 2, 2))$
$ROD(4 : (1; (1, 2; 1, 2); 4))$

Example 4. There are product designs $POD(8:1,1,2,3;1,3,3;1)$, $POD(8:2,2;1,1,1,1;4)$ and $POD(8:1,1,1;1,1,1;5)$. Then using the repeat design $ROD(4:1;(2;3);1,3)$ with the matrix of weight 2 used once only, we have $OD(32;(1,1,2,3,2,9,9,1,3))$, $OD(32;(2,2,2,3,3,3,4,12))$ and $OD(32;(1,1,1,2,3,3,5,15))$.

Since all of these have weight 31, and have 8 variables, we use the Geramita-Verner Theorem 2.5 in [23] to increase the number of variables to 9 and obtain the following orthogonal designs: $OD(32;1,1,1,1,2,2,3,3,9,9)$, $OD(32;1,2,2,2,3,3,3,4,12)$ and $OD(32;1,1,1,1,2,3,3,5,15)$. These last two designs are exciting.

The product designs $POD(4:1,1,1;1,1,1;1)$ can be used with the repeat designs of types $ROD(4:1;(p;3);1,3)$, $p=1,2$, to obtain $OD(16;1,1,1,1,p,p,3,3)$, $p=1,2$. These were first given in Geramita and Seberry [8].

Remark 3. In the preceding example we have concentrated on constructing orthogonal designs with no zero. There is considerable scope to exploit these constructions to look for other orthogonal designs in order 32 and higher powers of 2.

We can collect the results from Example 3 in the following statement:

Proposition 1. *In order 4 there exist repeat designs of types $(1;(r;s);h)$ for $0\le r$, $s\le 3$, $0\le h\le 4$.*

Noting that the repeat designs $(R;(P);H)$ are just amicable orthogonal designs $R+P$ and H, we see that:

Corollary 1. *There exist $AOD(4;(1,r),(h))$ for $0\le r\le 3$, $0\le h\le 4$.*

Remark 4. The non-existence of $AOD(8;(1,7),(5))$ and $AOD(16;(1,15),(1))$ means there are no repeat designs of types $(1;(r;7);5)$ in order 8 and $(1;(r;15);1)$ in order 16 (see Robinson [20]).

6.1 Construction and Replication of Repeat Designs

We now show that many repeat designs can be constructed.

Lemma 1. *Suppose $AOD(n_1:(a);(b_1,b_2)$ and $AOD(n_2:(c);(d_1,d_2)$ are amicable orthogonal designs. Then there is a repeat design in order n_1n_2 of type $ROD(n_1n_2:(b_1d_1;(ad_2,b_2d_1;b_2c,b_1d_2);ac)$.*

Proof. Let $A, x_1B_1+x_2B_2$ and $C, y_1D_1+y_2D_2$ be the amicable orthogonal designs. Then $(B_1\times D_1;(xA\times D_2+yB_2\times D_1;uB_2\times C+wB_1\times D_2);A\times C)$ is the required repeat design. □

Example 5. Let $A=C=\left[\begin{smallmatrix}1&1\\1&-\end{smallmatrix}\right]$, $B_1=D_1=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, and $B_2=D_2=\left[\begin{smallmatrix}0&1\\-&0\end{smallmatrix}\right]$. Then the repeat design in order 4 and type $(1;(1,2;1,2);4)$ is

$$\left(I_4\,;\left(\left[\begin{array}{cc|cc}0&y&x&x\\\bar y&0&x&\bar x\\\hline \bar x&\bar x&0&y\\\bar x&x&\bar y&0\end{array}\right];\left[\begin{array}{cc|cc}0&u&w&u\\\bar u&0&\bar u&w\\\hline \bar w&u&0&\bar u\\\bar u&\bar w&u&0\end{array}\right]\right);\quad z\left[\begin{array}{cc|cc}1&1&1&1\\1&-&1&-\\\hline 1&1&-&-\\1&-&-&1\end{array}\right]\right).$$

We refer the reader to Geramita and Seberry [8] for many replication results, and we also note from [8] the following powerful result given there as Corollary 5.129.

Corollary 2. *There are repeat designs of type*

$$ROD(2^t : 1; (1, 2, \ldots, 2^{t-1}; 1, 2, \ldots, 2^{t-1}); 2^t).$$

The construction and replication lemmas given later allow us to say:

Comment 1. In order 8 there exist, in fact, repeat designs $(1; (r); h)$ for all $0 \le r \le 7$ and $0 \le h \le 8$, except $r = 7$, $h = 5$ (which cannot exist).

In order 16 there exist repeat designs $(1; (r); h)$ for all $r = 1, 2, 3, \ldots, 15$, $h = 1, 2, \ldots, 16$, except possibly the following pairs (r, h): $(13, 1)$, $(13, 5)$, $(13, 9)$, $(15, 7)$, $(15, 9)$, $(15, 15)$ which are undecided, and $(15, 1)$ which does not exist.

6.2 Construction of Orthogonal Designs

The use of repeat designs is so powerful a source of orthogonal designs that it is quite impossible to indicate all the designs constructed here. We use Robinson's Ph.D. Thesis [19] and Seberry [23] as a source for product designs.

The constructions using these methods [8] allow us to say:

Theorem 2. *All orthogonal designs of type* $(2^t; a, b, c, 2^t - a - b - c)$ *and of type* (a, b, c), $0 \le a + b + c \le 2^t$, *exist for* $t = 2, 3, 4, 5, 6, 7, 8, 9$.

Remark 5. We believe these results do, in fact, allow the construction of all full orthogonal designs (that is, with no zero) with four variables in every power of 2, but we have not been able to prove this result.

Example 6. There is a product design of type $POD(2^t : 1, 1, 1, 1, 2, 4, \ldots, 2^{t-4}; 2, 2^{t-3};\ 2, 4, \ldots, 2^{t-4}, 2^{t-3}, 2^{t-3})$ in order 2^t. So using an amicable pair of weights (a, b) in order n gives an $OD(2^t n; 1, 1, 1, 1, 2, 4, \ldots, 2^{t-4},\ 2a, 2^{t-3}a,\ 2b, 4b, \ldots,\ 2^{t-4}b, 2^{t-3}b, 2^{t-3}b)$.

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