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A note on a finite version of Euler's partition identity

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Abstract

Recently, George Andrews has given a Glaisher style proof of a finite version of Euler's partition identity. We generalise this result by giving a finite version of Glaisher's partition identity. Both the generating function and bijective proofs are presented.

1 Introduction

A partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, 2, \ldots, \ell - 1$ and $\sum_{i=1}^{\ell} \lambda_i = n$. In a partition, the multiplicity of a part is defined to be the number of times that part occurs. An alternative notation for a partition λ is $(\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_\ell^{f_\ell})$ where $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell$ and f_i is the multiplicity of λ_i. If we restrict the multiplicities of parts to be 1, we are said to have a partition into distinct parts. For instance, partitions of 5 into distinct parts are: $(4, 1)$, $(3, 2)$ and (5). Surprisingly, such a number is related to partitions into odd parts, as in the following theorem due to Euler.

Theorem 1.1 (Euler [1]). *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

J. W. L. Glaisher gave a bijective proof of the identity (see [4]). This easily stated theorem was generalised as follows.

Theorem 1.2 (Glaisher [4]). *The number of partitions of n into parts not divisible by s is equal to the number of partitions of n into parts not repeated more than $s - 1$ times.*

We now describe Glaisher's bijection for this theorem.

Consider a partition $\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_\ell^{f_\ell})$ of n into parts not divisible by s. Take the s-ary expansion of the multiplicity f_i of the part λ_i, i.e.

$$f_i = \sum_{j=0}^{m_i} a_{i,j} s^j,$$

where $a_{i,j} \in \{0, 1, \ldots, s-1\}$. The map is then defined as

$$\lambda \mapsto \cup_{i=1}^{\ell} \cup_{j=0}^{m_i} (\lambda_i s^j)^{a_{i,j}}$$

where union is the multi-set union operation, and the parts are $\lambda_i s^j$ with multiplicities $a_{i,j}$. We give an example for this.

Let $n = 6$ and $s = 3$. Partitions of 6 whose parts are not divisible by 3 are:

$$(5, 1), (4, 2), (4, 1^2), (2^3), (2^2, 1^2), (2, 1^4), (1^6).$$

Applying the map, we observe that

$$(5, 1) \mapsto (5, 1), (4, 2) \mapsto (4, 2), (4, 1^2) \mapsto (4, 1^2)$$
$$(2^3) \mapsto (6), (2^2, 1^2) \mapsto (2^2, 1^2), (2, 1^4) \mapsto (3, 2, 1)$$
$$(1^6) \mapsto (3^2).$$

The image partitions are all partitions of 6 whose parts are not repeated more than twice. The Glaisher map is clearly reversible.

Theorem 1.2 has been made finite. Its finite version was given together with bijective proofs (see [2, 3]).We recall this version below.

Theorem 1.3 (Euler's theorem—finite version)**.** *The number of partitions of n into odd parts, each at most $2N$, is equal to the number of partitions of n into parts, each at most $2N$, and every part that is at most N is distinct.*

However, the bijections for Theorem 1.3 given in [2] are complicated, and motivated by their complexity, George Andrews gave a simpler proof that is Glaisher style (see [1]).

It is clear that Euler's partition identity (see Theorem 1.1) is a specific case of Glaisher's partition identity (see Theorem 1.2) when $s = 2$.

We are then naturally led to ask whether a finite version of Glaisher's partition identity that generalises Theorem 1.3 is possible. If so, can we find a bijective proof thereof reminiscent of Andrews' Glaisher style proof?

The goal of this paper is to fully address the questions above. Our main result is as follows:

Theorem 1.4. *Let s be a positive integer. The number of partitions of n into parts not divisible by s, each at most sN, is equal to the number of partitions of n into parts, each at most sN, and each part at most N appears not more than $s-1$ times.*

In the subsequent section we give a generating function proof, and in the section thereafter, a bijective proof that is Glaisher style.

2 First Proof of Theorem 1.4

Let $\mathcal{O}_{s,N}(n)$ denote the number of partitions of n into parts not divisible by s, each at most sN. On the other hand, let $\mathcal{D}_{s,N}(n)$ denote the number of partitions of n into parts, each at most sN, and each part at most N appears not more than $s-1$ times. Thus

$$\sum_{n=0}^{\infty} \mathcal{O}_{s,N}(n)q^n = \prod_{n=1}^{N} \frac{1}{(1-q^{sn-1})(1-q^{sn-2})\dots(1-q^{sn-s+1})}$$

and

$$\sum_{n=0}^{\infty} \mathcal{D}_{s,N}(n)q^n = \frac{\prod_{n=1}^{N}(1+q^n+q^{2n}+\dots+q^{(s-1)n})}{\prod_{n=1}^{(s-1)N}(1-q^{n+N})}.$$

Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{D}_{s,N}(n)q^n &= \frac{\prod_{n=1}^{N}(1+q^n+q^{2n}+\dots+q^{(s-1)n})}{\prod_{n=1}^{(s-1)N}(1-q^{n+N})} \\ &= \frac{\prod_{n=1}^{N}(1-q^n)(1+q^n+q^{2n}+\dots+q^{(s-1)n})}{\prod_{n=1}^{N}(1-q^n)\prod_{n=1}^{(s-1)N}(1-q^{n+N})} \\ &= \frac{\prod_{n=1}^{N}(1-q^{sn})}{\prod_{n=1}^{sN}(1-q^n)} \\ &= \prod_{n=1}^{N} \frac{1}{(1-q^{sn-1})(1-q^{sn-2})\dots(1-q^{sn-s+1})} \\ &= \sum_{n=0}^{\infty} \mathcal{O}_{s,N}(n)q^n.\end{aligned}$$

3 Second Proof of Theorem 1.4

We give a simple Glaisher style extension of the bijection given by George Andrews [1].

The bijection:

Consider a partition $\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \lambda_3^{f_3}, \dots, \lambda_\ell^{f_\ell})$ enumerated by $\mathcal{O}_{s,N}(n)$. Perform the following steps:

For each λ_i, find a unique α_i such that $N < \lambda_i s^{\alpha_i} \leq sN$. Then compute $\beta_i = \lfloor \frac{f_i}{s^{\alpha_i}} \rfloor$. Finally, take the s-ary expansion of $f_i - \beta_i s^{\alpha_i}$, i.e.

$$f_i - \beta_i s^{\alpha_i} = \sum_{j=0}^{m_i} a_{i,j} s^j.$$

Then the bijection is given by

$$\lambda \mapsto \bigcup_{i=1}^{\ell} \bigcup_{j=0}^{m_i} \left((\lambda_i s^{\alpha_i})^{\beta_i}, (\lambda_i s^j)^{a_{i,j}}\right)$$

where the union in the image is the multi-set union and β_i's and $a_{i,j}$'s are the multiplicities of the parts $\lambda_i s^{\alpha_i}$ and $\lambda_i s^j$, respectively.

It is not difficult to see that the image partition is enumerated by $\mathcal{D}_{s,N}(n)$.

Example 3.1. *Let* $s = 3, N = 4$ *and* $\lambda = (11^6, 7^5, 5^7, 4^5, 1^{17})$.

In this case, $\lambda_1 = 11$, $f_1 = 6$, $\lambda_2 = 7$, $f_2 = 5, \ldots, \lambda_5 = 1$, $f_5 = 17$. Following the steps, we have $\alpha_1 = 0$, $\beta_1 = 6$, $a_{1,j} = 0$ for all $j \geq 0$, and the reader can verify the rest of the computations and observe that

$$(11^6, 7^5, 5^7, 4^5, 1^{17}) \mapsto (12, 11^6, 9, 7^5, 5^7, 4^2, 3^2, 1^2).$$

The inverse:

Let μ be a partition enumerated by $\mathcal{D}_{s,N}(n)$. Separate the parts divisible by s from μ. Write each of the multiples of s as vs^j for some $j \geq 0$ and $s \nmid v$. Then the multiplicity of v in the resulting partition enumerated by $\mathcal{O}_{s,N}(n)$ is

$$\sum_{\mu} s^j$$

where the sum is over μ and each part in μ is written as vs^j for some $j \geq 0$.

Since each part v in the resulting partition comes from the representation of parts as vs^j with $s \nmid v$, it is clear that the new partition has all its parts not divisible by s. Furthermore, since each part in μ is at most sN, it is also clear that $v \leq sN$. Hence, each part in the resulting partition is at most sN.

Example 3.2. *Consider* $\mu = (12, 11^6, 9, 7^5, 5^7, 4^2, 3^2, 1^2)$, $s = 3$, $N = 4$.

Those parts divisible by 3 are $12, 9, 3, 3$. Note that $12 = 4 \cdot 3^1, 9 = 1 \cdot 3^2$, $3 = 1 \cdot 3^1$ (repeated). From these parts, we find that $v = 4$ and $v = 1$. The multiplicity of $v = 4$ is

$$\sum_{\mu} s^j = 3^1 + 3^0 + 3^0 = 5.$$

The multiplicity of $v = 1$ is

$$\sum_{\mu} s^j = 3^2 + 3^1 + 3^1 + 3^0 + 3^0 = 17.$$

For the rest of the parts (non-multiples of 3 and excluding $v = 1$ and $v = 4$), we have $v = 11$, which has multiplicity

$$\sum_{\mu} s^j = 3^0 + 3^0 + 3^0 + 3^0 + 3^0 + 3^0 = 6;$$

$v = 7$ has multiplicity

$$\sum_{\mu} s^j = 3^0 + 3^0 + 3^0 + 3^0 + 3^0 = 5;$$

$v = 5$ has multiplicity

$$\sum_{\mu} s^j = 3^0 + 3^0 + 3^0 + 3^0 + 3^0 + 3^0 + 3^0 = 7.$$

Hence the resulting partition is

$$(11^6, 7^5, 5^7, 4^5, 1^{17}).$$

Thus

$$(12, 11^6, 9, 7^5, 5^7, 4^2, 3^2, 1^2) \mapsto (11^6, 7^5, 5^7, 4^5, 1^{17}).$$

Indeed, $(11^6, 7^5, 5^7, 4^5, 1^{17})$ is the partition we started with in Example 3.1.

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