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Well primitive digraphs

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Abstract

A primitive digraph D is said to be well primitive if the local exponents of D are all equal. In this paper we consider well primitive digraphs of two special types: digraphs that contain loops, and symmetric digraphs with shortest odd cycle of length r. We show that the upper bound of the exponent of the well primitive digraph is $n-1$ in both these classes of digraphs, and we characterize the extremal digraphs.

1 Introduction

We consider digraphs in which loops are permitted but no multiple arcs. The set of all loop vertices in a digraph D is denoted by $V^0(D)$. Unless otherwise stated we consider a digraph D on the vertex set $\{1, 2, \ldots, n\}$. For simplicity we write the interval of integers from i to j as $[i, j]$. An arc (i, j) will be written as $\overrightarrow{i,j}$. A sequence i_0,a_1,i_1,a_2,...,i_{p-1},a_p,i_p, where $p \geq 1$ and $a_k = \overrightarrow{i_{k-1}, i_k} \in A(D)$ for $k = 1, \ldots, p$, is called a walk from i_0 to i_p in D (or an $i_0 \to i_p$ walk), and p is called its length. A set of the vertices that belong to a walk W is denoted by V_W. If $i \neq j$, then $\mathrm{d}_D(i, j)$ is defined to be the length of the shortest $i \to j$ walk in D and $\mathrm{d}_D(i, i)$ is defined to equal 0. The maximum $\mathrm{d}_D(i, j)$ taken over all $j \in V(D)$ is denoted by $\mathrm{e}^+_D(i)$ and the maximum $\mathrm{e}^+_D(i)$ taken over all $i \in V(D)$ is called the diameter of D, denoted $\mathrm{diam}(D)$. If $\mathrm{e}^+_D(i) = \mathrm{diam}(D)$, then i is called a peripheral vertex in D. A cycle is a walk which begins and ends in the same vertex. A walk is said to be simple if no vertex appears at least twice in it, and a cycle is said to be simple if each vertex except the last appears at most once in it.

Let $N^+_D(i) = \{j \in V(D) : \overrightarrow{i,j} \in A(D)\}$ and $N^-_D(i) = \{j \in V(D) : \overrightarrow{j,i} \in A(D)\}$. The cardinalities of the sets $N^+_D(i)$ and $N^-_D(i)$ are called the out-degree and the in-degree of a vertex i in D, respectively. The minimum out-degree in D is denoted by $\delta^+(D)$.

A digraph D is strongly connected if there is an $i \to j$ walk in D for all $i, j \in V(D)$. A digraph D is called primitive if for some positive integer t there is an $i \to j$

walk of length t in D for all $i, j \in V(D)$. The smallest such t is denoted by $\exp(D)$ and it is called the exponent of D. It is well known that a strongly connected digraph D is primitive if and only if the lengths of cycles in D are relatively prime.

The primitivity of a digraph can be also defined by its adjacency matrix. Namely, if $M(D)$ is the adjacency matrix of the digraph D, then D is primitive if and only if there exists an integer k such that all entries of a matrix $[M(D)]^k$ are positive. The k-th power of a digraph D, denoted by D^k, is a digraph on the same set of vertices as D in which $\overrightarrow{i,j} \in A(D)$ if there is an $i \to j$ walk of length k in D. It is well known that $[M(D)]^k = M(D^k)$

Let D be a primitive digraph and $i, j \in V(D)$. The least integer p such that for each $t \geq p$ there is an $i \to j$ walk of length t in D is denoted by $\exp_D(i,j)$ and $\max_{j \in V(D)} \exp_D(i,j)$ is called the exponent of a vertex i, denoted by $\exp_D(i)$. The latter is also called the local exponent at vertex i. Obviously, the maximum of the exponents of vertices equals $\exp(D)$. A primitive digraph D is said to be well primitive if each of the local exponents of D equals $\exp(D)$.

The concept of local exponents can be found in [3]; Brualdi and Liu in [1] considered them as the special case of generalization of the exponent of the primitive digraph (for details see [4]). The motivation for introducing generalized exponents stemmed from its interpretation in an application model of a memoryless network communication. Such a network can be represented by a directed graph in natural way. Suppose that at time $t = 0$, each vertex holds one bit of information with no two of the information bits the same. At time $t = 1$ each vertex passes its information to each of its out-neighbours (so each vertex also receives bits of information from its in-neighbours). After sending a bit of information a vertex forgets it. The system continues in this way. If a digraph is primitive, then from certain moment all the vertices receive each bit of information simultaneously.

With respect to local exponents, Brualdi and Liu [1] noted that how much time it takes to a given bit of information was delivered simultaneously to all the vertices of D, depends on its location in D. Thus one can ask in which networks there is no need to care for arranging bits of information. It is therefore of interest to consider well primitive digraphs. In natural way the exponent of a well primitive digraph is bounded by its first local exponent. The first local exponent set is completely described in [6]. However, the complete characterization of well primitive digraphs and the exponent set of these cannot be directly inferred from the results obtained by Neufeld and Shen. In particular, the problem of determining the strict upper bound of the exponent of a well primitive digraph remains open.

In this paper we focus on two special classes of primitive digraphs. The class of all the primitive digraphs on n vertices is denoted by $\mathcal{DP}(n)$. We denote by $\mathcal{SDP}(n)$ the class of all primitive symmetric digraphs on n vertices. For $m \in [1, n]$ we use the symbol $\mathcal{DP}^0(n,m)$ to denote the class of all strongly connected digraphs on n vertices in each of which there are exactly m loop vertices. In Section 2 we study well primitive digraphs in $\mathcal{SDP}(n)$ and $\mathcal{DP}^0(n,m)$. It follows from the results known in the literature that the upper bound of the exponent of well primitive digraphs is

$n-1$ in both of these classes. For the symmetric well primitive digraph we extend this result, showing how the exponent of a well primitive symmetric digraph depends on the length of the shortest odd cycle and on existence of a pair of disjoin odd cycles in it. We also characterize the extremal digraphs. The main results concerning well primitive symmetric digraphs are formulated and proved in Theorems 2.1, 2.3 and 2.4. Theorem 3.2 indicates all non-isomorphic extremal well primitive digraphs in $\mathcal{DP}^0(n,m)$.

2 Well primitive symmetric digraphs

A symmetric digraph can be viewed as a pseudograph (with no multiple edges), for an edge (i,j) can be replaced by a pair of arcs $\overrightarrow{i,j}$ and $\overrightarrow{j,i}$. We adopt the conventional notations for undirected graphs as P_n and C_n, $n \geq 2$, thinking of them as of directed graphs. In additional the symbol C_1 denotes the digraph consisting of one vertex and one loop. It is well known that symmetric digraph is primitive if and only if it is connected and there is at least one odd cycle in it. The local indices of symmetric digraphs were thoroughly studied in [2].

The upper bound of the exponent of well primitive symmetric digraphs and the complete characterization of extremal digraphs are simple consequences of the results proved in [8]. Let $n \geq 2$ and let P_n^* and P_n^{**} be the digraphs obtained from P_n by adding a loop in one of the endvertices and by adding a loops in both endvertices, respectively. Shao, Wang and Li [8] showed that if $D \in \mathcal{SDP}(n)$, then $\min_{i\in V(D)}\{\exp_D(i)\} \leq n-1$, with equality if and only if D is isomorphic to one of the following digraphs: C_n for odd n, P_n^* or P_n^{**}. It is well known that $\exp(P_n^*) = 2n-2 > n-1$ ([7]), so P_n^* is not well primitive. Since C_n is vertex transitive, it is in particular well primitive whenever n is odd. Next, it can easily be observed that $\exp_{P_n^{**}}(i) = n-1$ for $i = 1, 2, \dots, n$. Therefore if $D \in \mathcal{SDP}(n)$ is well primitive, then $\exp(D) \leq n-1$, with equality if and only if $D \cong C_n$ and n is odd or $D \cong P_n^{**}$.

Let $\mathcal{SDP}(n,r)$ be the class of symmetric primitive digraphs on n vertices in each of which the length of the shortest odd cycle is r. The characterization of the extremal well primitive digraphs in $\mathcal{SDP}(n,r)$ will be broken down into two cases, depending on whether there exist disjoint odd cycles in D. Let us first mention some known results which be used to prove our results.

Lemma 2.1 ([7]) *Let D be a symmetric primitive digraph. Then*

$$\exp_D(i,j) = \max\{a(i,j), b(i,j)\} - 1, \tag{1}$$

where $a(i,j)$ and $b(i,j)$ are the lengths of the shortest odd and even $i \to j$ walk in D, respectively.

Lemma 2.1 indicates a close relationship between $\exp_D(i,j)$ and $\mathrm{d}_D(i,j)$ in symmetric digraphs.

Corollary 2.1 *If D is a symmetric primitive digraph, $i, j \in V(D)$ and $\mu_{i,j}$ is the length of the shortest odd cycle containing i and j, then*

$$\exp_D(i,j) = \mu_{i,j} - 1 - \mathrm{d}_D(i,j). \tag{2}$$

Lemma 2.2 ([9]) *Let $D \in \mathcal{SDP}(n,r)$ and let μ_i be the length of the shortest odd cycle containing $i \in V(D)$. Then $\exp(D) \leq \max\{\mu - 1, n - r\}$, where $\mu = \max_{i \in V(D)}\{\mu_i\}$.*

It is convenient for our further investigations to pick out a result which is, in fact, a part of the proof of Lemma 2.2.

Lemma 2.3 ([9]) *Under the hypotheses of Lemma 2.2, if the shortest odd cycles containing vertices i and j have at least one vertex in common, then $\exp_D(i,j) \leq \max\{\mu_i, \mu_j\} - 1$.*

Let $\mathcal{K} \subset \mathcal{SDP}(n,r)$ be the class of all the digraphs that contain a pair of disjoint odd cycles. Obviously, $\mathcal{K}$ is nonempty while $n \geq 2r$. Theorem 2.1 gives a complete characterization of well primitive symmetric digraphs not in $\mathcal{K}$.

Theorem 2.1 *If $r \geq 3$ and $D \in \mathcal{SDP}(n,r) \backslash \mathcal{K}$, then D is well primitive if and only if $\exp(D) = r - 1$.*

Proof. Since $D \in \mathcal{SDP}(n,r)\backslash\mathcal{K}$, Lemma 2.3 implies that $\exp(D) \leq \mu - 1$. On the other hand, $\exp(D) \geq \exp_D(i,i)$ for each $i \in V(D)$, so $\exp(D) \geq \mu - 1$ by (1). Thus $\exp(D) = \mu - 1$ and now our purpose is to show that $\mu = r$. Since $D \in \mathcal{SDP}(n,r)$, there is a subgraph of D isomorphic to C_r. Let $D_0 \cong C_r$ and $i \in V(D_0)$. Since D is well primitive, there exists $j \in V(D)$ such that $\exp_D(i,j) = \mu - 1$. It is obvious for $i = j$, so let us suppose that $i \neq j$. From Lemma 2.3 it follows that $\mu_j = \mu$. From (2) we deduce that if there exists a cycle of length μ containing both of the distinct vertices i and j, then $\exp_D(i,j) \leq \mu - 2$. Therefore if $i \neq j$, then i does not belong to any cycle of length μ that contains j. Let W_j be a cycle of length μ that contains j. Then there exists $i' \in V(D_0) \cap V_{W_j}$, and $\exp_D(i',j) \leq \mu - 1 - \mathrm{d}_D(i',j)$ by (2). We have $\exp_D(i,j) \leq \mathrm{d}_D(i,i') + \exp_D(i',j)$, so $\mu - 1 \leq \mathrm{d}_D(i,i') + \mu - 1 - \mathrm{d}_D(i',j)$, and hence $\mathrm{d}_D(i',j) \leq \mathrm{d}_D(i,i')$. On the other hand, $\exp_D(i,j) \leq \exp_D(i,i') + \mathrm{d}_D(i',j)$, which gives

$$\mu - 1 = \exp_D(i,j) \leq r - 1 - \mathrm{d}_D(i,i') + \mathrm{d}_D(i',j) \leq r - 1, \text{ so } \mu = r.$$

Conversely, assume that $\exp(D) = r - 1$. Suppose that there exists $i \in V(D)$ such that $\exp_D(i) \leq r - 2$. Then in particular $\exp_D(i,i) \leq r - 2$. Since $\exp_D(i,i)$ is an even number, vertex i belongs to some cycle of length $r - 2$, but this contradicts the choice of D. Therefore, $\exp_D(i) = r - 1$ for each $i \in V(D)$. □

Corollary 2.2 *If $D \in \mathcal{SDP}(n,r)$, $r \geq 3$ and $n \leq 2r - 1$, then D is well primitive if and only if $\exp(D) = r - 1$.*

Now we proceed to characterize the extremal well primitive digraphs in $\mathcal{K}$. Let $p, q \geq 1$ and $m \geq 2$. We denote by $P_m(C_p, C_q)$ a graph (or a pseudograph if $p = 1$) on the vertex set $[1, p + q + m - 2]$ with the edge set

$$\{(i, i+1) : i \in [1, p+q+m-3]\} \cup \{(1, p), (p+m-1, p+q+m-2)\}.$$

A graph $P_m(C_p, C_q)$ can be viewed as a graph obtained by joining graphs C_p and C_q by a graph P_m. In particular, $P_n(C_1, C_1) = P_n^{**}$.

Lemma 2.4 *Let p and q be odd integers, $1 \leq p \leq q$. A digraph $P_m(C_p, C_q)$ is well primitive if and only if $q \leq p + 2m - 2$. If $P_m(C_p, C_q)$ is well primitive, then*

$$\exp(P_m(C_p, C_q)) = \mathrm{diam}(P_m(C_p, C_q)) = (p+q)/2 + m - 2.$$

Proof. For simplicity of notation D stands for $P_m(C_p, C_q)$. It can easily be notified that $\mathrm{diam}(D) = (p+q)/2 + m - 2$. First assume that $q \geq p + 2m$. Let V_1 be the set of peripheral vertices in $P_m(C_p, C_q)$. Then, by (1), $\exp_D(i) = \mathrm{diam}(D) \leq q - 2$ for $i \in V(C_p) \cap V_1$. On the other hand, $\exp_D(j) \geq \exp_D(j, j) = q - 1$ for $j \in V(C_q) \cap V_1$. Therefore, D is not well primitive whenever $q \geq p + 2m$.

Next assume that $q \leq p + 2m - 2$. Then $\mathrm{diam}(D) \geq q - 1$. If $\mathrm{diam}(D)$ is even, then $c = (3p + q + 2m - 2)/4$ is the unique central vertex in D. More precisely, $c \notin V(C_p) \cup V(C_q)$ unless $\mathrm{diam}(D) = q - 1$. Next, $\mu_c = \mathrm{diam}(D) + 1$ and $\mu_{i,c} = \mu_c$ for each $i \in V(D)$, so $\exp_D(c) = \mathrm{diam}(D)$ by (2). Let $i \neq c$. We now show that $\exp_D(i) = \mathrm{diam}(D)$. If $j = c$ or j is 'by the same side' of c as i, we obtain $\mu_{i,j} \leq \mu_c$, and hence, by (2), $\exp_D(i, j) \leq \mathrm{diam}(D)$. Now choose j such that $\min\{i, j\} < c < \max\{i, j\}$. Since $\mathrm{d}_D(i, c) + \mathrm{d}_D(c, j)$ and $|\mathrm{d}_D(i, c) - \mathrm{d}_D(c, j)|$ have the same parity, we obtain

$$\{a(i, j), b(i, j)\} = \{\mathrm{d}_D(i, c) + \mathrm{d}_D(c, j), \mu_c - |\mathrm{d}_D(i, c) - \mathrm{d}_D(c, j)|\}.$$

Thus $\exp_D(i, j) = \mathrm{diam}(D)$ while $\mathrm{d}_D(i, c) = \mathrm{d}_D(j, c)$; otherwise $\exp_D(i, j) < \mathrm{diam}(D)$. Combining all these cases we obtain $\exp_D(i) = \mathrm{diam}(D)$ for each vertex $i \in V(D)$.

If $\mathrm{diam}(D)$ is odd, then there are exactly two central vertices in D and they are adjacent. Analogously, it can easily be verified that if $i \leq c$, $c + 1 \leq j$ and $\mathrm{d}_D(i, c) = \mathrm{d}_D(j, c+1)$, then $\exp_D(i, j) = \mathrm{diam}(D)$ and $\exp_D(i, j) \leq \mathrm{diam}(D)$ for all remaining pairs (i, j). □

Theorem 2.2 ([3]) *If $D \in \mathcal{DP}(n)$ and t is a positive integer, then $\exp_{D^t}(i) = \lceil(\exp_D(i))/t\rceil$ for each $i \in [1, n]$.*

Remark 2.1 If D is well primitive and $V^0(D) \neq \emptyset$, then $\exp(D) = \mathrm{diam}(D)$.

Theorem 2.3 *If $D \in \mathcal{K}$ is well primitive, then $\exp(D) \leq n - r$.*

Proof. Let $D \in \mathcal{K}$ be a well primitive digraph and let p and q be the lengths of odd disjoint cycles in D, where $p \leq q$. Let $q = p+2t$, $t \geq 0$. A digraph D is well primitive, so is D^2 and $\exp(D) \leq 2\exp(D^2)$, by Theorem 2.2. Since D is symmetric, there is a loop in each vertex of D, so $\exp(D^2) = \mathrm{diam}(D^2)$ by Remark 2.1. Moreover, each vertex of D^2 is both peripheral and central, and C_q, C_p are subgraphs of D^2, so

$$\mathrm{diam}(D^2) \leq \lfloor (n-(q-1)/2-(p-1)/2+1)/2 \rfloor = \lfloor (n-p-t+1)/2 \rfloor .$$

If $p \geq r+2$ or $t \geq 1$, then $\mathrm{diam}(D^2) \leq (n-r)/2$, and hence $\exp(D) \leq n-r$.

Now let $p = q = r$. If n is odd, then $\lfloor (n-r+1)/2 \rfloor = (n-r)/2$, so $\exp(D) \leq 2\lfloor n-r+1/2 \rfloor = n-r$. If n is even, then $2\lfloor (n-r+1)/2 \rfloor = n-r+1$. Suppose $\exp(D) = n-r+1$. Then $n-r+1 \leq \max\{\mu-1, n-r\}$ by Lemma 2.2, so $\mu \geq n-r+2$. Next, there exist disjoint subgraphs D_1 and D_2 of D such that they are both isomorphic to C_r. Choose $v \in V(D)$ so that $\mu = \mu_v$, and for $i \in \{1,2\}$ let W_i be the shortest walk joining v with D_i, and let d_i be the length of W_i. Note that, since D is symmetric, we can, for convenience, ignore directions of walks. A walk $W = W_1 + W_2$ joins D_1 with D_2. From $\mu \leq r + 2\min\{d_1, d_2\}$ we obtain $\min\{d_1, d_2\} \geq n/2 - r + 1$, so the length of W is at least $d_1 + d_2 \geq n - 2r + 2$. On the other hand, $\mathrm{d}_D(V(D_1), V(D_2)) \leq n-2r+1$, thus, there is at least one vertex which appears in W twice. Let v' be the first such vertex and let $d_i' = \mathrm{d}_D(v', V(D_i))$ for $i \in \{1,2\}$. Next, denote by W_i' the shortest walk joining $V(D_i)$ with v', contained in W_i, for $i \in \{1,2\}$. Then $W' = W_1' + W_2'$ is a simple walk. We show that $\exp_D(v') \leq \mu - 2 = \exp(D) - 1$. Let D_0 be the subgraph induced by the set $V(D_1) \cup V(D_2) \cup V_{W'}$ and $u \in V(D_0)$. Then $D_0 \cong P_{1+d_1'+d_2'}(C_r, C_r)$ and $v' \in V(D_0)$, so $\exp_D(v', u) \leq \exp_{D_0}(v', u) \leq r - 1 + d_1' + d_2'$ by Lemma 2.4, and hence $\exp_D(v', u) \leq n - r \leq \mu - 2$. For $u \in V(D) \setminus V(D_0)$ we have $\mathrm{d}_D(u, v') \leq n - (d_1' + d_2' + 2r - 1)$. Next, since $\mu_{v'} \leq 2\min\{d_1', d_2'\} + r \leq d_1' + d_2' + r$, by (1), $\exp_D(v', u) \leq \mathrm{d}_D(u, v') + \mu_{v'} - 1$, and applying two previous inequalities, we obtain $\exp_D(v', u) \leq n - r \leq \mu - 2$. Therefore $\exp_D(v') \leq \mu - 2 < \exp(D)$, which contradicts the assumption that D is well primitive. Hence, $\exp(D) \leq n - r + 1$ and $\exp(D) \neq n - r + 1$, so $\exp(D) \leq n - r$. □

Remark 2.2 If $D \in \mathcal{DP}(n)$ is well primitive, then $\delta^+(D) \geq 2$.

Theorem 2.4 *If $r \geq 3$ and $n \geq 2r$ and $D \in \mathcal{SDP}(n,r)$ is well primitive, then $\exp(D) = n - r$ if and only if $D \cong P_{n-2r+2}(C_r, C_r)$.*

Proof. Obviously, by Lemma 2.4, $\exp(P_{n-2r+2}(C_r, C_r)) = n - r$. Now assume that $D \in \mathcal{SDP}(n,r)$ is well primitive and $\exp(D) = n - r$. Then $n \geq 2r$ implies $\exp(D) \geq r$, so, by Theorem 2.1, $D \in \mathcal{K}$. Let p and q be the lengths of disjoint odd cycles. There is no loss of generality in assuming that these cycles are simple. Since D is strongly connected, we can choose a subgraph D_0 of D that is isomorphic to $P_m(C_p, C_q)$ with $2 \leq m \leq n - p - q + 2$.

First, suppose that $|V(D_0)| = n - a$ for some $a \geq 1$. Since D is well primitive, for each $i \in V(D_0)$ there exists $j' \in V(D)$ such that $\exp_D(i, j') = n - r$. Next, by Lemma

2.4, $\exp_D(i,j) \le \exp_{D_0}(i,j) \le n-a-r$ for all $i,j \in V(D_0)$. Thus $j' \notin V(D_0)$. On the other hand,

$$\exp_D(i,j') \le \min_{j\in V(D_0)} \{\exp_{D_0}(i,j) + \mathrm{d}_D(j,j')\},$$

so $\mathrm{d}_D(j, V(D_0)) = a$. Thus if there exists $j' \in V(D)\backslash V(D_0)$ such that $\mathrm{d}_D(j', V(D_0)) = a$, then, by Remark 2.2, j must be a loop vertex, contrary to $D \in \mathcal{SDP}(n,r)$. Therefore, $V(D_0) = V(D)$, so $n-r \le n-(p+q)/2$ and hence $p=q=r$, i.e. $P_{n-2r+2}(C_r, C_r)$ is the subgraph of D. To complete the proof it remains to verify that if $P_{n-2r+2}(C_r, C_r)$ is the proper subgraph of D, then $\exp_D(1) < n-r$ or $D \notin \mathcal{SDP}(n,r)$. This simple verification is left to the reader. $\square$

Remark 2.3 If $n \ge 2r$, then Theorem 2.4 asserts that the extremal well primitive digraph in $\mathcal{DP}(n,r)$ is unique. However, if the assumption on the length of the shortest odd cycle is dropped, then using $P_{n-2r+2}(C_r, C_r)$ we can indicate the family of symmetric digraphs on n vertices, each of them being well primitive, containing $P_{n-2r+2}(C_r, C_r)$ as a subgraph and having the equal exponent. Let us consider, as the example, the case $r=9, n=20$ (Fig. 1).

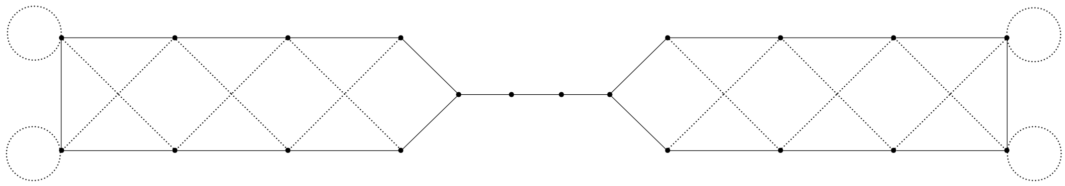

Figure 1: $P_4(C_9, C_9)$ as a subgraph

One can easily verify that adding any set of the dotted pairs of arcs (represented in Fig. 1 as dotted edges) we obtain a symmetric digraph which is well primitive, and its exponent equals $\exp(P_4(C_9, C_9))$.

3 Well primitive digraphs with loops

By Remark 2.1 the upper bound of the exponent of a well primitive digraph $D \in \mathcal{DP}^0(n,m)$ is $n-1$. Moreover, Remark 2.1 implies that there is a loop at each of peripheral vertices.

Lemma 3.1 *Let D be a strongly connected digraph on n vertices such that* $\mathrm{diam}(D) = n-1$. *If D is non-Hamiltonian, then there are at most two peripheral vertices in D.*

Proof. We may assume D to be loopless. A digraph D is strongly connected and $\mathrm{diam}(D) = n-1$, so the vertices of D can be relabelled from 1 up to n so that

$\mathrm{d}_D(1,i) = i-1$ for $i = 2,\dots,n$. Then all the arcs in D are of the form $\overrightarrow{i,i+1}$ for $i = 1,\dots,n-1$ or $\overrightarrow{i,j}$, where $1 \le j < i \le n$, except the case where $i = n$ and $j = 1$. Now assume that there exists $i \ne 1$ such that $\mathrm{e}^+_D(i) = n-1$, and let j be a vertex satisfying $\mathrm{d}_D(i,j) = n-1$. Then $j < i$ and no simple cycle in D contains both i and j, for D is non-Hamiltonian. Next, observe that a vertex i belongs to a simple cycle which also contains a vertex n, since otherwise the shortest $i \to j$ walk does not contain a vertex n. Let $b = \min\{j : \overrightarrow{n,j} \in A(D)\}$ and $c = \max\{l : \overrightarrow{l,m} \in A(D) \wedge m < b \le l\}$. Obviously, $b \ge 2$, $i \ge b$ and $c \le n-1$. Now our goal is to see that $i = c+1$. First suppose that $i \le c$. Since $j < i$ implies $\mathrm{d}_D(i,j) \le c-1 < n-1$, we obtain a contradiction. Thus $i \ge c+1$, and, in particular, $c = n-1$ implies $i = n = c+1$. Next, if $c \le n-2$, then we claim that the assumption $i \ge c+2$ leads to a contradiction. Indeed, for each $i' \ge c+2$ and $j' \in V(D)$ no simple $i' \to j'$ walk contains both $c+1$ and $b-1$, which implies $\mathrm{e}^+_D(i') \le n-2$. Therefore $i = c+1$, and we conclude that there exist at most two peripheral vertices in D. $\square$

Theorem 3.1 states the necessary condition for a well primitive digraph $D \in \mathcal{PD}^0(n,m)$ to attain the upper bound.

Lemma 3.2 ([5]) *If $D \in \mathcal{PD}(n)$, $i \in V(D)$ and $g_D(i)$ is the length of the shortest cycle in D containing the vertex i, then* $\exp_D(i) \le g_D(i)(n - \deg^+_D(i)) + 1$.

Denote by $\overrightarrow{C_n}$ a digraph with the arc set $\{\overrightarrow{1,2}, \overrightarrow{2,3}, \dots, \overrightarrow{n-1,n}, \overrightarrow{n,1}\}$.

Theorem 3.1 *If $D \in \mathcal{PD}^0(n,m)$ is well primitive and* $\exp(D) = n-1$, *then $m = 2$ or $m = n$.*

Proof. First note that Lemma 3.2 and Remark 2.2 provide $\deg^+_D(v) = 2$ for each $v \in V^0(D)$. Suppose that there exists a well primitive digraph in $D \in \mathcal{PD}^0(n,1)$ such that $\exp(D) = n-1$, and let $V^0(D) = \{v\}$. Let $N^+_D(v) = \{u,v\}$. Then $\overrightarrow{uv} \in A(D)$, since otherwise $\mathrm{e}^+_D(u) \le n-3$, and hence $\exp_D(v) = e^+_D(v) \le n-2$, a contradiction. Next, by Remark 2.1, there exists $w \in V(D)$ such that $\mathrm{d}_D(v,w) = n-1$. Let W_0 be a $u \to v$ walk in D of length $n-1$. The length of the shortest $u \to w$ walk containing v is at least n, so $v \notin W_0$. On the other hand, since $\mathrm{d}_D(v,w) = n-1$ and each $v \to w$ walk contains a vertex u, the length of each $u \to w$ walk is of the form $n-2+a_1p_1+a_2p_2+\ldots+a_tp_t$, where $a_i \ge 0$ and p_i are the lengths of simple cycles in D for $1 \le i \le t$. We may assume that $1 \le p_1 < p_2 < \ldots < p_{t-1} < p_t$. Then the equality $n-2+a_1p_1+a_2p_2+\ldots+a_tp_t = n-1$ implies $a_1 = p_1 = 1$ and $a_2 = \ldots = a_t = 0$, which means that there must be another loop vertex in D, contrary to $m = 1$. Therefore there is no well primitive $D \in \mathcal{PD}^0(n,1)$ with $\exp(D) = n-1$.

Now let $m \in [3, n-1]$ and $n \ge 4$. Suppose that there exists a well primitive digraph $D \in \mathcal{PD}^0(n,m)$ with $\exp(D) = n-1$. Remark 2.1 yields that there exist at least m peripheral vertices in D. Since $m \ge 3$, Lemma 3.1 provides the existence of the Hamilton cycle in D, so the vertices of D can be relabelled as $1,2,\dots,n$ so that $A(\overrightarrow{C_n}) \subset A(D)$. Next, $m \leq n-1$, so there exist $v \in V^0(D)$ and $v' \notin V^0(D)$ such that $\overrightarrow{vv'} \in A(D)$. Without loss of generality put $v = n \in V^0(D)$ and $v' = 1 \in V(D)\backslash V^0(D)$. We show that $\exp_D(1) \leq n-2$. Let $i = \min\{t : t \in V^0(D)\}$. The assumption $m \geq 3$ implies $i \leq n-2$. Since $\exp_D(1,j) = \mathrm{d}_D(1,j)$ whenever $j \in [i, n-1]$, we have $\exp_D(1,j) = j-1 \leq n-2$. Next, $\exp_D(1,1) = 2$, so

$$\exp_D(1,j) \leq 2 + \mathrm{d}_D(1,j) = j+1 \leq i \leq n-2 \text{ for } j \leq i-1.$$

Therefore $\exp_D(1) < \exp(D)$, which contradicts the assumption that D is well primitive. □

Theorem 3.2 gives the complete characterization of all (non-isomorphic) well primitive digraphs on n vertices for which the upper bound is attained. Let $\mathcal{DP}^0(n) = \bigcup_{1 \leq m \leq n} \mathcal{DP}^0(n,m)$. Let $\overrightarrow{C_n^*}$ be a digraph with the arc set $A(\overrightarrow{C_n^*}) = A(\overrightarrow{C_n}) \cup \{\overrightarrow{i,i} : i \in [1,n]\}$ and D_n^* (Fig. 2) be a digraph with $A(D_n^*) = A(\overrightarrow{C_n}) \cup \{\overrightarrow{1,1}, \overrightarrow{2,2}\} \cup \{\overrightarrow{i,2} : 3 \leq i \leq n\}$.

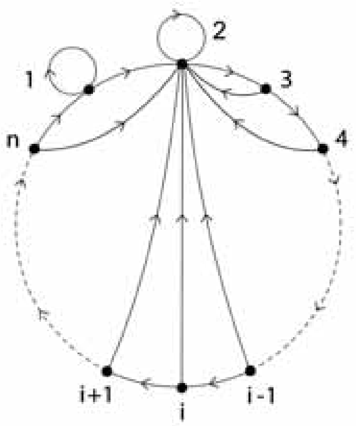

Figure 2: A digraph D_n^*

Theorem 3.2 *If $n \geq 4$ and $D \in \mathcal{PD}^0(n)$ is well primitive, then* $\exp(D) = n-1$ *if and only if D is isomorphic to one of the following digraphs: P_n^{**}, $\overrightarrow{C_n^*}$ or D_n^*.*

Proof. The necessity part of the proof can easily be verified so it is omitted. Let $D \in \mathcal{PD}^0(n)$ be a well primitive digraph satisfying $\exp(D) = n-1$. Then, by Theorem 3.1, $m = n$ or $m = 2$. If $m = n$, then Lemma 3.2 implies that D is isomorphic to $\overrightarrow{C_n^*}$. Let $m = 2$. By Remark 2.1, there exists a simple walk in D of length $n-1$ that begins at a loop vertex. Moreover, if $V^0(D) = \{i,j\}$, then $i \notin N_D^-(j)$ or $j \notin N_D^-(i)$. There is no loss of generality in assuming that $1 \in V^0(D)$ and $N_D^-(1)$ does not contain the other loop vertex. Let the vertices of D be renumbered so that $\mathrm{d}_D(1,i) = i-1$ for $i = 2, \ldots, n$. Then, as in the proof of Lemma 3.1, we see that all the arcs not of the form $\overrightarrow{i,i+1}$ are 'backward' or loops. In particular,

$N_D^+(2) \subseteq \{1, 2, 3\}$. However, if $\{\overrightarrow{2, 1}, \overrightarrow{2, 2}\} \subset A(D)$, then $\exp_D(2) \leq n - 2$. On the other hand, $\deg_D^+(2) \geq 2$ by Remark 2.2, so $N_D^+(2) = \{1, 3\}$ or $N_D^+(2) = \{2, 3\}$.

First assume that $N_D^+(2) = \{1, 3\}$. Then $\exp_D(2, i) \leq n - 2$ for $i \in [1, n - 2]$, so $\exp_D(2) = n - 1$ implies $\exp_D(2, n - 1) = n - 1$ or $\exp_D(2, n) = n - 1$. However, the latter is impossible, because if it was true, that would imply, that there was no $2 \to n$ walk of length $n-2$, contrary to $\mathrm{d}_D(2, n) = n-2$. Therefore, $\exp_D(2, n-1) = n-1$, so none of the vertices $3, \dots, n - 1$ is a loop vertex, which means that the second loop in D is at the vertex n. Applying again Remark 2.1, we can see that the vertex n is peripheral, so $\mathrm{d}_D(n, 1) = n - 1$ or $\mathrm{d}_D(n, n - 1) = n - 1$. Since $N_D^-(1)$ does not contain the loop vertex, we have $\mathrm{d}_D(n, 1) = n - 1$, and hence $D \cong P_n^{**}$.

Now assume that $N_D^+(2) = \{2, 3\}$, that is, a loop is at the vertex 2. Then $\exp_D(2, i) \leq n - 2$ for $i \in [2, n]$, so the assumption $\exp_D(2) = n - 1$ implies that $\exp_D(2, 1) = n - 1$, and hence $A(\overrightarrow{C_n}) \subset A(D)$, $\overrightarrow{i, 1} \notin A(D)$ for $i \in [2, n - 1]$ and $\exp_D(i, 1) = n - i + 1 \leq n - 2$ for $i \in [3, n]$.

Our aim is to show that $N_D^+(i) = \{2, i + 1\}$ for $i \in [3, n - 1]$ and $N_D^+(n) = \{1, 2\}$. It is trivial for $i = 3$, because in this case the only possibility is $N_D^+(3) = \{2, 4\}$. Now $\exp_D(3, 3) = 2$ implies $\exp_D(3, i) \leq n - 2$ for $i \leq n - 1$ and $\exp_D(3, n) = n - 1$. If $n \geq 5$, then $N_D^+(4) \subseteq \{2, 3, 5\}$. We claim that $N_D^+(4) = \{2, 5\}$. Suppose not. Then $N_D^+(4) = \{2, 3, 5\}$ or $N_D^+(4) = \{3, 5\}$. But in the former case we have $\exp_D(4, 4) \leq 2$, and hence $\exp_D(4) \leq n - 2$, a contradiction. Next, suppose that $N_D^+(4) = \{3, 5\}$. If $\overrightarrow{4, 3} \in A(D)$, then $\exp_D(4, 4) \leq 4$, and hence $\exp_D(4, i) \leq n - 2$ for $i \in [5, n - 2]$. Thus, if $\overrightarrow{4, 3} \in A(D)$, then $\exp_D(4) = n - 1$ implies that $\exp_D(4, n - 1) = n - 1$ or $\exp_D(4, n) = n-1$. However, since $\mathrm{d}_D(4, n) = n-4$ and $\exp_D(4) = n-1$, there exists a $4 \to n$ walk in D, say W, which contains cycles of lengths adding up to 3. In fact, a walk W must contain a cycle of length 3, for no vertex among $4, \dots, n$ is a loop vertex. This means, in particular, that there is a $4 \to n - 1$ walk in D which consists of a cycle of length 3 and the shortest $4 \to n - 1$ walk, that is, there is a $4 \to n - 1$ walk of length $n - 2$. Likewise, since $\overrightarrow{4, 3} \in A(D)$, there is a $4 \to n$ walk of length $n - 2$ (it consists of a cycle of length 2 and the shortest $4 \to n$ walk). Therefore, if $\overrightarrow{4, 3} \in A(D)$, then $\exp_D(4) \leq n - 2$, a contradiction. Since both $N_D^+(4) = \{2, 3, 5\}$ and $N_D^+(4) = \{3, 5\}$ lead to a contradiction, our claim holds.

Moreover, $\overrightarrow{i, i - 1} \notin A(D)$ for $i \geq 4$, for if not, there is a cycle of length 2 and we obtain $\exp_D(4) = n - 2$ again. Next, if $n \geq 6$, then $N_D^+(5) \subseteq \{2, 3, 6\}$, since $\overrightarrow{5, 4} \notin A(D)$.

Now we prove that $\overrightarrow{5, 3} \notin A(D)$. We obtain this in a manner similar to that used for the case $\exp_D(4)$, thereby showing that $\overrightarrow{i, i - 2} \notin A(D)$ for $i \geq 5$.

Continuing in this way for the consecutive vertices, we see that for $n \geq 6$ it follows that each vertex i does not belong to any cycle of length $3, \dots, i - 2$ for $i \geq 5$, and hence $N_D^+(i) = \{2, i + 1\}$ for $i \in [5, n - 1]$ and $N_D^+(n) = \{1, 2\}$. Finally, $D = D_n^*$, which completes the proof. □

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