AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(3)** (2018), Pages 349–361

Rainbow domination in the Cartesian product of directed paths

Guoliang Hao*

College of Science, East China University of Technology
Nanchang 330013, Jiangxi, P.R. China
guoliang-hao@163.com

Doost Ali Mojdeh

Department of Mathematics
University of Mazandaran Babolsar, Iran
damojdeh@umz.ac.ir

Shouliu Wei

Department of Mathematics, Minjiang University
Fuzhou 350121, Fujian, P.R. China
wslwillow@126.com

Zhihong Xie

College of Science, East China University of Technology
Nanchang 330013, Jiangxi, P.R. China
xiezh168@163.com

Abstract

For a positive integer k, a k-rainbow dominating function (kRDF) of a digraph D is a function f from the vertex set $V(D)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex v with $f(v) = \emptyset$, $\bigcup_{u \in N^-(v)} f(u) = \{1, 2, \ldots, k\}$, where $N^-(v)$ is the set of in-neighbors of v. The weight of a kRDF f of D is the value $\sum_{v \in V(D)} |f(v)|$. The k-rainbow domination number of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. Let $P_m \Box P_n$ denote the Cartesian product of P_m and P_n, where P_m and P_n denote the directed paths of order m and n, respectively. In this paper, we determine the exact values of $\gamma_{rk}(P_m \Box P_n)$ for any positive integers $k \geq 2$, m and n.

* Corresponding author.

1 Introduction and notation

The concept of domination in graphs, with its many variations, has been extensively studied (see, for example, [3, 4, 6, 7, 8, 9]). One of the variations on the domination theme is rainbow domination. There are many results on rainbow domination in undirected graphs; for example, see [2, 5, 10, 11]. However, there exists a smaller number of results on rainbow domination in digraphs. Our aim in this paper is to study the rainbow domination in digraphs.

Throughout this paper, $D = (V(D), A(D))$ denotes a digraph with vertex set $V(D)$ and arc set $A(D)$. For two vertices $u, v \in V(D)$, we use (u, v) to denote the arc with direction from u to v, and we say that u is an *in-neighbor* of v. For $v \in V(D)$ we denote the set of in-neighbors of v by $N^-(v)$, and we define the *in-degree* of v by $d^-(v) = |N^-(v)|$.

For a positive integer k, we use $\mathcal{P}(\{1, 2, \ldots, k\})$ to denote the set of all subsets of the set $\{1, 2, \ldots, k\}$. A *k-rainbow dominating function* (kRDF) of a digraph D is a function $f : V(D) \to \mathcal{P}(\{1, 2, \ldots, k\})$ such that for any vertex v with $f(v) = \emptyset$, $\bigcup_{u\in N^-(v)} f(u) = \{1, 2, \ldots, k\}$. The *weight* of a kRDF f of D is the value $\omega(f) = \sum_{v\in V(D)} |f(v)|$. The *k-rainbow domination number* of a digraph D, denoted by $\gamma_{rk}(D)$, is the minimum weight of a kRDF of D. A kRDF f of D with $\omega(f) = \gamma_{rk}(D)$ is called a *$\gamma_{rk}(D)$-function*.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs with disjoint vertex sets V_1 and V_2 and disjoint arc sets A_1 and A_2, respectively. The *Cartesian product* $D_1\Box D_2$ is the digraph with vertex set $V_1 \times V_2$ and for $(x_1, y_1), (x_2, y_2) \in V(D_1\Box D_2)$, $((x_1, y_1), (x_2, y_2)) \in A(D_1\Box D_2)$ if and only if either $(x_1, x_2) \in A_1$ and $y_1 = y_2$, or $x_1 = x_2$ and $(y_1, y_2) \in A_2$. For any $y \in V_2$, we denote by D_1^y the subdigraph of $D_1\Box D_2$ induced by the vertex set $\{(x, y) : x \in V_1\}$. Then it is easy to see that D_1^y is isomorphic to D_1.

Let P_n denote the directed path of order n. We emphasize that $V(P_n) = \{0, 1 \ldots, n-1\}$ and $A(P_n) = \{(i, i+1) : i = 0, 1, \ldots, n-2\}$, throughout this paper. As defined earlier, for each $j \in \{0, 1, \ldots, n-1\}$, we denote by P_m^j the subdigraph of $P_m\Box P_n$ induced by the vertex set $\{(i, j) : i = 0, 1, \ldots, m-1\}$.

In 2013, rainbow domination in digraphs was introduced by Amjadi et al. [1]. However, to date no research has been done for the Cartesian product of two directed paths. In this paper, we give the exact values of $\gamma_{rk}(P_m\Box P_n)$ for any positive integers $k \geq 2$, m and n.

2 The 2-rainbow domination number

In this section, we will determine the exact values of the 2-rainbow domination number in Cartesian products of directed paths.

Theorem 2.1. *For any positive integer n,*

$$\gamma_{r2}(P_1\Box P_n) = \gamma_{r2}(P_n) = n.$$

Proof. Let f be a $\gamma_{r2}(P_n)$-function. Since $d^-(0) = 0$, it follows from the definition of $\gamma_{r2}(P_n)$-function that $f(0) \neq \emptyset$. Moreover, if there exists some $i \in \{1, 2, \ldots, n-1\}$ such that $f(i) = \emptyset$, then clearly $f(i-1) = \{1, 2\}$. This implies that $\gamma_{r2}(P_n) = \omega(f) = \sum_{i=0}^{n-1} |f(i)| \geq n$. On the other hand, it is easy to see that $\gamma_{r2}(P_n) \leq n$. Therefore, $\gamma_{r2}(P_1 \Box P_n) = \gamma_{r2}(P_n) = n$. □

Lemma 2.2. *Let $n \geq 2$ be any integer and let f be a $\gamma_{r2}(P_2 \Box P_n)$-function such that $|\{(i, j) \in V(P_2 \Box P_n) : f((i, j)) = \emptyset\}|$ is minimum. Then for each $j \in \{0, 1, \ldots, n-1\}$,*

$$|f((0, j))| + |f((1, j))| \geq 1.$$

Proof. If $f((1, 0)) \neq \emptyset$, then clearly $|f((0, 0))|+|f((1, 0))| \geq 1$. Otherwise, $f((1, 0)) = \emptyset$. Then by the definition of $\gamma_{r2}(P_2 \Box P_n)$-function, we have $f((0, 0)) = \{1, 2\}$ and hence $|f((0, 0))| + |f((1, 0))| = 2 \geq 1$.

We now claim that for each $j \in \{1, 2, \ldots, n-1\}$, $|f((0, j))| + |f((1, j))| \geq 1$. Suppose, to the contrary, that there exists some $j \in \{1, 2, \ldots, n-1\}$ such that $|f((0, j))| + |f((1, j))| = 0$. This implies that $f((0, j)) = f((1, j)) = \emptyset$. Then by the definition of $\gamma_{r2}(P_2 \Box P_n)$-function, we have $f((1, j-1)) = \{1, 2\}$. Define a function $g : V(P_2 \Box P_n) \to \mathcal{P}(\{1, 2\})$ by

$$g(v) = \begin{cases} \{1\}, & \text{if } v = (1, j-1), (1, j), \\ f(v), & \text{otherwise.} \end{cases}$$

Then it is easy to see that g is a 2RDF of $P_2 \Box P_n$ with weight $\omega(g) = \omega(f) = \gamma_{r2}(P_2 \Box P_n)$, implying that g is also a $\gamma_{r2}(P_2 \Box P_n)$-function. Moreover, it is easy to see that $|\{(i, j) \in V(P_2 \Box P_n) : f((i, j)) = \emptyset\}| - |\{(i, j) \in V(P_2 \Box P_n) : g((i, j)) = \emptyset\}| = 1$, contradicting the choice of f. This completes the proof. □

Theorem 2.3. *For any integer $n \geq 2$,*

$$\gamma_{r2}(P_2 \Box P_n) = \left\lceil \frac{4n}{3} \right\rceil.$$

Proof. Let f be a $\gamma_{r2}(P_2 \Box P_n)$-function such that $|\{(i, j) \in V(P_m \Box P_n) : f((i, j)) = \emptyset\}|$ is minimum. For each $j \in \{0, 1, \ldots, n-1\}$, let $a_j = |f((0, j))| + |f((1, j))|$ and for each $j \in \{2, 3, \ldots, n-1\}$, let $b_j = a_{j-2} + a_{j-1} + a_j$. Then by Lemma 2.2, we have that for each $j \in \{1, 2, \ldots, n-1\}$, $a_j \geq 1$. Moreover, by the definition of $\gamma_{r2}(P_2 \Box P_n)$-function, it is easy to verify that $a_0 \geq 2$.

Claim 1. For each $j \in \{2, 3, \ldots, n-1\}$, $b_j \geq 4$.

Proof of Claim 1. Suppose, to the contrary, that there exists some $j_0 \in \{2, 3, \ldots, n-1\}$ such that $b_{j_0} \leq 3$. Note that for each $j \in \{0, 1, \ldots, n-1\}$, $a_j \geq 1$. Therefore, we have $b_{j_0} = 3$ and hence $a_{j_0-2} = a_{j_0-1} = a_{j_0} = 1$. Assume that $|f((1, j_0))| = 1$, implying that $f((0, j_0)) = \emptyset$. Since f is a $\gamma_{r2}(P_2 \Box P_n)$-function, $f((0, j_0-1)) = \{1, 2\}$. Then we have $a_{j_0-1} \geq |f((0, j_0-1))| = 2$, which is a contradiction. Assume next that $|f((0, j_0))| = 1$, implying that $f((1, j_0)) = \emptyset$. Since f is a $\gamma_{r2}(P_2 \Box P_n)$-function, $\{1, 2\} \backslash f((0, j_0)) \subseteq f((1, j_0-1))$ and hence $|f((1, j_0-1))| = 1$. This implies that

$f((0,j_0-1))=\emptyset$ and hence $f((0,j_0-2))=\{1,2\}$. Thus, $a_{j_0-2}\geq |f((0,j_0-2))|=2$, which is also a contradiction. So, this claim is true.

Therefore, if $n\equiv 0(\mathrm{mod}3)$, then we have

$$\gamma_{r2}(P_2\Box P_n)=\sum_{k=0}^{\frac{n-3}{3}} b_{3k+2}\geq \frac{4n}{3}=\left\lceil\frac{4n}{3}\right\rceil,$$

and if $n\equiv 1(\mathrm{mod}3)$, then we have

$$\gamma_{r2}(P_2\Box P_n)=a_0+\sum_{k=1}^{\frac{n-1}{3}} b_{3k}\geq 2+\frac{4(n-1)}{3}=\frac{4n+2}{3}=\left\lceil\frac{4n}{3}\right\rceil.$$

In both cases, we provide a 2RDF $f':V(P_2\Box P_n)\to\mathcal{P}(\{1,2\})$ defined by

$$f'(v)=\begin{cases}\{1,2\}, & \text{if } v=(0,3k) \text{ for } 0\leq k\leq \lfloor\frac{n-1}{3}\rfloor,\\ \{1\}, & \text{if } v=(1,3k+1) \text{ for } 0\leq k\leq \lfloor\frac{n-2}{3}\rfloor,\\ \{2\}, & \text{if } v=(0,3k+2) \text{ for } 0\leq k\leq \lfloor\frac{n-2}{3}\rfloor,\\ \emptyset, & \text{otherwise,}\end{cases}$$

and hence

$$\gamma_{r2}(P_2\Box P_n)\leq\omega(f')=2(\lfloor\frac{n-1}{3}\rfloor+\lfloor\frac{n-2}{3}\rfloor+2)=\left\lceil\frac{4n}{3}\right\rceil.$$

If $n\equiv 2(\mathrm{mod}3)$, then we have

$$\begin{aligned}\gamma_{r2}(P_2\Box P_n)&=a_0+a_1+\sum_{k=1}^{\frac{n-2}{3}} b_{3k+1}\geq 3+\frac{4(n-2)}{3}\\ &=\frac{4n+1}{3}=\left\lceil\frac{4n}{3}\right\rceil.\end{aligned}$$

In this case, we also provide a 2RDF $f'':V(P_2\Box P_n)\to\mathcal{P}(\{1,2\})$ defined by

$$f''(v)=\begin{cases}\{1,2\}, & \text{if } v=(0,3k) \text{ for } 0\leq k\leq \frac{n-2}{3},\\ \{1\}, & \text{if } v=(1,3k+1) \text{ for } 0\leq k\leq \frac{n-2}{3},\\ \{2\}, & \text{if } v=(0,3k+2) \text{ for } 0\leq k\leq \frac{n-5}{3},\\ \emptyset, & \text{otherwise,}\end{cases}$$

and hence

$$\begin{aligned}\gamma_{r2}(P_2\Box P_n)&\leq \omega(f'')\\ &= 2(\frac{n-2}{3}+1)+(\frac{n-2}{3}+\frac{n-5}{3}+2)\\ &= \frac{4n+1}{3}=\left\lceil\frac{4n}{3}\right\rceil,\end{aligned}$$

which completes our proof. $\square$

We shall determine the exact value of $\gamma_{r2}(P_m \Box P_n)$ for any integers $m, n \geq 3$. For this purpose, we need some observations.

Observation 2.4. *Let $m, n \geq 3$ be any integers and let f be a $\gamma_{r2}(P_m \Box P_n)$-function. Then*

$$f((0,0)) + \sum_{i=1}^{m-1} |f((i,0))| + \sum_{j=1}^{n-1} |f((0,j))| \geq m + n - 2$$

and the equality holds if $f((0,0)) = \{1,2\}$, $f((1,0)) = f((0,1)) = \emptyset$ and $|f((i,0))| = |f((0,j))| = 1$ for $i, j \geq 2$.

Proof. If $f((i,0)) \neq \emptyset$ for each i, then $\sum_{i=0}^{m-1} |f((i,0))| \geq m$. If there exists some vertex, say $(i,0)$, such that $f((i,0)) = \emptyset$, then by the definition of $\gamma_{r2}(P_m \Box P_n)$-function, we have $f((i-1,0)) = \{1,2\}$, which implies that $\sum_{i=0}^{m-1} |f((i,0))| \geq m$. Similarly, we get $\sum_{j=0}^{n-1} |f((0,j))| \geq n$. Therefore, it is easy to verify that

$$f((0,0)) + \sum_{i=1}^{m-1} |f((i,0))| + \sum_{j=1}^{n-1} |f((0,j))| \geq m + n - 2,$$

establishing the desired lower bound.

If $f((0,0)) = \{1,2\}$, $f((1,0)) = f((0,1)) = \emptyset$ and $|f((0,j))| = 1 = |f((i,0))|$ for $i, j \geq 2$, then clearly

$$f((0,0)) + \sum_{i=1}^{m-1} |f((i,0))| + \sum_{j=1}^{n-1} |f((0,j))| = m + n - 2,$$

which completes our proof. □

Observation 2.5. *Let $m, n \geq 3$ be any integers and let f be a $\gamma_{r2}(P_m \Box P_n)$-function such that $f((0,0)) = \{1,2\}$, $f((1,0)) = f((0,1)) = \emptyset$ and $|f((i,0))| = |f((0,j))| = 1$ for $i, j \geq 2$. Then*

(i) $f((i,1)) \cup f((i+1,1)) \neq \emptyset$ *for* $i \geq 0$.

(ii) $\sum_{i=1}^{m-1} |f((i,1))| \geq \lfloor \frac{m}{2} \rfloor$.

(iii) The equality in (ii) holds if $|f((2k-1,1))| = 1$ *for* $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ *and* $|f((2k,1))| = 0$ *for* $1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$.

Proof. (i) If $f((i,1)) \cup f((i+1,1)) = \emptyset$ for some $i \geq 0$, then $f((i+1,0)) = \{1,2\}$, a contradiction to our assumption. Consequently, $f((i,1)) \cup f((i+1,1)) \neq \emptyset$ for $i \geq 0$.

(ii) According to (i), (ii) is trivial.

(iii) By our assumption, we have $f((i,0)) = \{1\}$ for $i \geq 2$. Let $f((2k-1,1)) = \{2\}$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and let $f((2k,1)) = \emptyset$ for $1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$. Then clearly $\sum_{i=1}^{m-1} |f((i,1))| = \lfloor \frac{m}{2} \rfloor$. □

The following observation can be deduced by a similar discussion to that for Observation 2.5.

Observation 2.6. *Let $m, n \geq 3$ be any integers and let f be a $\gamma_{r2}(P_m \Box P_n)$-function such that f satisfies the conditions of Observation 2.5 and (iii) of Observation 2.5. Then*

(i) $|f((i,2)) \cup f((i+1,2))| \neq 0$ *for* $i \geq 0$.

(ii) $\sum_{i=1}^{m-1} |f((i,2))| \geq \lfloor \frac{m-1}{2} \rfloor$.

(iii) The equality of (ii) holds if $|f((2k,2))| = 1$ *for* $1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$ *and* $|f((2k-1,2))| = 0$ *for* $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$.

Now by using induction we have the following result.

Observation 2.7. *Let $m, n \geq 3$ be any integers and let f be a $\gamma_{r2}(P_m \Box P_n)$-function such that f satisfies the conditions of Observations 2.5. Then*

(i) $|f((i,j)) \cup f((i+1,j))| \neq 0$ *for* $i \geq 0$ *and* $j \geq 1$.

(ii) $\sum_{i=1}^{m-1} |f((i,j))| \geq \lfloor \frac{m}{2} \rfloor$ *for odd* $j \geq 1$, *and* $\sum_{i=1}^{m-1} |f((i,j))| \geq \lfloor \frac{m-1}{2} \rfloor$ *for even* $j \geq 1$.

(iii) The first equality of (ii) holds if $|f((2k-1,j))| = 1$ *for* $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ *and* $|f((2k,j))| = 0$ *for* $1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$ *when j is odd, and the second equality of (ii) holds if* $|f((2k,j))| = 1$ *for* $1 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$ *and* $|f((2k-1,j))| = 0$ *for* $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ *when* $j \geq 1$ *is even.*

Now we may derive the following theorem.

Theorem 2.8. *For any integers $m, n \geq 3$,*

$$\gamma_{r2}(P_m \Box P_n) = \left\lceil \frac{m+1}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil - 2.$$

Proof. Let f be a $\gamma_{r2}(P_m \Box P_n)$-function such that f satisfies the conditions of Observation 2.5. Therefore, by Observations 2.4 and 2.7, we obtain

$$\begin{aligned}
\gamma_{r2}(P_m \Box P_n) &= f((0,0)) + \sum_{i=1}^{m-1} |f((i,0))| + \sum_{j=1}^{n-1} |f((0,j))| + \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} |f((i,j))| \\
&\geq m + n - 2 + \sum_{l=1}^{\lfloor n/2 \rfloor} \sum_{i=1}^{m-1} |f((i,2l-1))| + \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} \sum_{i=1}^{m-1} |f((i,2l))| \\
&\geq m + n - 2 + \sum_{l=1}^{\lfloor n/2 \rfloor} \left\lfloor \frac{m}{2} \right\rfloor + \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} \left\lfloor \frac{m-1}{2} \right\rfloor \\
&= (n+m-2) + \frac{n-1}{2}(m-1) \\
&= \left\lceil \frac{m+1}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil - 2.
\end{aligned}$$

To show the upper bound, we now provide a 2RDF $g : V(P_m \Box P_n) \to \mathcal{P}(\{1,2\})$ defined by

$$g((i,j)) = \begin{cases} \{1,2\}, & \text{if } i = j = 0, \\ \{1\}, & \text{if } i = 2k-1 \text{ for } 1 \le k \le \lfloor \frac{m}{2} \rfloor \text{ and} \\ & j = 2l-1 \text{ for } 1 \le l \le \lfloor \frac{n}{2} \rfloor, \\ \{2\}, & \text{if } i = 0 \text{ and } 2 \le j \le n-1, \text{ or} \\ & \text{if } 2 \le i \le m-1 \text{ and } j = 0, \text{ or} \\ & \text{if } i = 2k \text{ for } 1 \le k \le \lceil \frac{m-2}{2} \rceil \text{ and} \\ & j = 2l \text{ for } 1 \le l \le \lceil \frac{n-2}{2} \rceil, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} \gamma_{r2}(P_m \Box P_n) &\le \omega(g) \\ &= 2 + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + (n-2) + (m-2) + \left\lceil \frac{m-2}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil \\ &= \left\lceil \frac{m+1}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil - 2, \end{aligned}$$

which completes our proof. □

3 The 3-rainbow domination number

In this section, we will derive the exact value of the 3-rainbow domination number in Cartesian products of directed paths.

Lemma 3.1. *For any integers $m, n \ge 2$,*

$$\gamma_{r3}(P_m \Box P_n) \ge \begin{cases} mn - \frac{m-1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor, & \text{if } m \text{ is odd,} \\ \frac{3mn+2m}{4}, & \text{if both } m \text{ and } n \text{ are even with } m \ge n. \end{cases}$$

Proof. Let f be a $\gamma_{r3}(P_m \Box P_n)$-function such that $|\{(i,j) \in V(P_m \Box P_n) : f((i,j)) = \emptyset\}|$ is minimum and let $a_j = \sum_{i=0}^{m-1} |f((i,j))|$ for each $j \in \{0,1,\ldots,n-1\}$.

Claim 2. For each $i \in \{0,1,\ldots,m-1\}$ and $j \in \{0,1,\ldots,n-1\}$,

$$f((i,0)) \ne \emptyset \text{ and } f((0,j)) \ne \emptyset.$$

Proof of Claim 2. If there exists some $i \in \{1,2,\ldots,m-1\}$ such that $f((i,0)) = \emptyset$, then by the definition of $\gamma_{r3}(P_m \Box P_n)$-function, we have $f((i-1,0)) = \{1,2,3\}$. Define a function $g : V(P_m \Box P_n) \to \mathcal{P}(\{1,2,3\})$ by

$$g(v) = \begin{cases} \{1\}, & \text{if } v = (i-1,0), (i,0), \\ f(v) \cup \{1\}, & \text{if } v = (i-1,1), \\ f(v), & \text{otherwise.} \end{cases}$$

Then it is easy to see that g is a 3RDF of $P_m\Box P_n$ with weight $\omega(g) \leq \omega(f) = \gamma_{r3}(P_m\Box P_n)$, implying that g is also a $\gamma_{r3}(P_m\Box P_n)$-function. Moreover, clearly $|\{(i,j) \in V(P_m\Box P_n) : f((i,j)) = \emptyset\}| - |\{(i,j) \in V(P_m\Box P_n) : g((i,j)) = \emptyset\}| = 1$ or 2, a contradiction to the choice of f. Note that $f((0,0)) \neq \emptyset$ since $d^-((0,0)) = 0$. Therefore, we have $f((i,0)) \neq \emptyset$ for each $i \in \{0,1,\ldots,m-1\}$. Similarly, we have $f((0,j)) \neq \emptyset$ for each $j \in \{0,1,\ldots,n-1\}$. So, this claim is true.

Therefore, by Claim 2, we have $a_0 = \sum_{i=0}^{m-1} |f((i,0))| \geq m$.

Claim 3. There do not exist two vertices (i,j) and $(i+1,j)$, where $i,j \geq 1$, of $P_m\Box P_n$ such that
$$f((i,j)) = f((i+1,j)) = \emptyset.$$

Proof of Claim 3. Suppose, to the contrary, that there exist two vertices (i,j) and $(i+1,j)$, where $i,j \geq 1$, of $P_m\Box P_n$ such that $f((i,j)) = f((i+1,j)) = \emptyset$. Note that f is a $\gamma_{r3}(P_m\Box P_n)$-function. Therefore, $f((i+1,j-1)) = \{1,2,3\}$. Define a function $g : V(P_m\Box P_n) \to \mathcal{P}(\{1,2,3\})$ by

$$g(v) = \begin{cases} \{1\}, & \text{if } v = (i+1,j-1), (i+1,j), \\ f(v) \cup \{1\}, & \text{if } v = (i+2,j-1), \\ f(v), & \text{otherwise.} \end{cases}$$

Then it is easy to see that g is a 3RDF of $P_m\Box P_n$ and $\omega(g) \leq \omega(f) = \gamma_{r3}(P_m\Box P_n)$, implying that g is also a $\gamma_{r3}(P_m\Box P_n)$-function. Moreover, clearly $|\{(i,j) \in V(P_m\Box P_n) : f((i,j)) = \emptyset\}| - |\{(i,j) \in V(P_m\Box P_n) : g((i,j)) = \emptyset\}| = 1$ or 2, a contradiction to the choice of f. So, this claim is true.

For each $j \in \{0,1,\ldots,n-1\}$, let $b_j = |\{(i,j) \in V(P_m^j) : f((i,j)) = \emptyset\}|$. Then by Claims 2 and 3, we have that $b_0 = 0$ and $b_j \leq \lfloor \frac{m}{2} \rfloor$ for each $j \in \{1,2,\ldots,n-1\}$.

Claim 4. $a_0 + a_1 \geq 2m$ and $a_{j-1} + a_j \geq \lceil \frac{3m}{2} \rceil$ for each $j \in \{2,3,\ldots,n-1\}$.

Proof of Claim 4. Let $j \in \{1,2,\ldots,n-1\}$. If $f((i,j)) \neq \emptyset$ for each i, then clearly $a_j = \sum_{i=0}^{m-1} |f((i,j))| \geq m$ and hence if $j = 1$, then $a_0 + a_1 = a_0 + a_j \geq m + m = 2m$; if $j \in \{2,3,\ldots,n-1\}$, then

$$a_{j-1} + a_j \geq (m - b_{j-1}) + a_j \geq (m - \lfloor \frac{m}{2} \rfloor) + m = \lceil \frac{3m}{2} \rceil.$$

Hence we may assume that there exists some i such that $f((i,j)) = \emptyset$, implying that $b_j \geq 1$. For each $k \in \{1,2,\ldots,b_j\}$, let $f((i_k,j)) = \emptyset$. Then it is easy to see that $\sum_{k=1}^{b_j} |f((i_k,j))| = 0$ and by Claim 3, we have that for each $k \in \{1,2,\ldots,b_j\}$, $f((i_k-1,j)) \neq \emptyset$. Note that f is a $\gamma_{r3}(P_m\Box P_n)$-function. Therefore, for each $k \in \{1,2,\ldots,b_j\}$, $f((i_k,j-1)) \cup f((i_k-1,j)) = \{1,2,3\}$ and hence $|f((i_k,j-1))| +$

$|f((i_k-1,j))| \geq 3$. Thus,

$$\begin{aligned}
a_{j-1}+a_j &= \sum_{i=0}^{m-1}|f((i,j-1))|+\sum_{i=0}^{m-1}|f((i,j))| \\
&= \sum_{k=1}^{b_j}[|f((i_k,j-1))|+|f((i_k-1,j))|]+\sum_{i\in X_j}|f((i,j-1))| \\
&\quad +\sum_{i\in Y_j}|f((i,j))|+\sum_{k=1}^{b_j}|f((i_k,j))| \\
&\geq 3b_j+(m-b_{j-1}-b_j)+(m-2b_j) \\
&= 2m-b_{j-1},
\end{aligned}$$

where $X_j=\{0,1,\ldots,m-1\}\backslash\{i_1,i_2,\ldots,i_{b_j}\}$, $Y_j=X_j\backslash\{i_1-1,i_2-1,\ldots,i_{b_j}-1\}$ and $\sum_{i\in X_j}|f((i,j-1))| \geq m-b_{j-1}-b_j$. Note that $b_0=0$ and $b_j \leq \lfloor \frac{m}{2} \rfloor$ for each $j\in\{1,2,\ldots,n-1\}$. Therefore, we have $a_0+a_1 \geq 2m-b_0=2m$ and for each $j\in\{2,3,\ldots,n-1\}$, $a_{j-1}+a_j \geq 2m-b_{j-1} \geq 2m-\lfloor \frac{m}{2} \rfloor = \lceil \frac{3m}{2} \rceil$. So, this claim is true.

Therefore, it follows from Claim 4 that if n is odd, then

$$\begin{aligned}
\gamma_{r3}(P_m \Box P_n) = \omega(f) &= a_0+\sum_{k=1}^{\frac{n-1}{2}}(a_{2k-1}+a_{2k}) \\
&\geq m+\frac{n-1}{2}\left\lceil \frac{3m}{2} \right\rceil,
\end{aligned}$$

implying that if both m and n are odd, then

$$\begin{aligned}
\gamma_{r3}(P_m \Box P_n) &\geq m+\frac{(3m+1)(n-1)}{4} \\
&= mn-\frac{m-1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor;
\end{aligned}$$

if n is even, then

$$\begin{aligned}
\gamma_{r3}(P_m \Box P_n) = \omega(f) &= (a_0+a_1)+\sum_{k=1}^{\frac{n-2}{2}}(a_{2k}+a_{2k+1}) \\
&\geq 2m+\frac{n-2}{2}\left\lceil \frac{3m}{2} \right\rceil,
\end{aligned}$$

implying that if m is odd and n is even, then

$$\begin{aligned}
\gamma_{r3}(P_m \Box P_n) &\geq 2m+\frac{(3m+1)(n-2)}{4} \\
&= mn-\frac{m-1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor.
\end{aligned}$$

As a consequence, we have that if m is odd, then

$$\gamma_{r3}(P_m\Box P_n) \geq mn - \frac{m-1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor;$$

and if both m and n are even, then

$$\gamma_{r3}(P_m\Box P_n) \geq 2m + \frac{n-2}{2}\left\lceil\frac{3m}{2}\right\rceil = \frac{3mn+2m}{4},$$

which completes our proof. □

Note that $\gamma_{r3}(P_m\Box P_n) = \gamma_{r3}(P_n\Box P_m)$. Therefore, we have the following corollary.

Corollary 3.2. *For any integers $m, n \geq 2$,*

$$\gamma_{r3}(P_m\Box P_n) \geq \begin{cases} mn - \lfloor\frac{m-1}{2}\rfloor\frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ \frac{3mn+2n}{4}, & \text{if both } m \text{ and } n \text{ are even with } n \geq m. \end{cases}$$

As an immediate consequence of Lemma 3.1 and Corollary 3.2, we have the following result.

Corollary 3.3. *For integers $m, n \geq 2$,*

$$\gamma_{r3}(P_m\Box P_n) \geq \begin{cases} mn - \lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor, & \text{if } m \text{ is odd, or if } n \text{ is odd,} \\ \frac{3mn+2\max\{m,n\}}{4}, & \text{if both } m \text{ and } n \text{ are even.} \end{cases}$$

Lemma 3.4. *For integers $m, n \geq 2$,*

$$\gamma_{r3}(P_m\Box P_n) \leq \begin{cases} mn - \lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor, & \text{if } m \text{ or } n \text{ is odd,} \\ \frac{3mn+2\max\{m,n\}}{4}, & \text{if both } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. If m or n is odd, then we provide a 3RDF $f : V(P_m\Box P_n) \to \mathcal{P}(\{1,2,3\})$ defined by

$$f((i,j)) = \begin{cases} \{1,2\}, & \text{if } i = 2k-1 \text{ for } 1 \leq k \leq \lfloor\frac{m}{2}\rfloor \text{ and} \\ & j = 2l-1 \text{ for } 1 \leq l \leq \lfloor\frac{n}{2}\rfloor, \\ \emptyset, & \text{if } i = 2k-1 \text{ for } 1 \leq k \leq \lfloor\frac{m}{2}\rfloor \text{ and} \\ & j = 2l \text{ for } 1 \leq l \leq \lfloor\frac{n-1}{2}\rfloor, \text{ or} \\ & \text{if } i = 2k \text{ for } 1 \leq k \leq \lfloor\frac{m-1}{2}\rfloor \text{ and} \\ & j = 2l-1 \text{ for } 1 \leq l \leq \lfloor\frac{n}{2}\rfloor, \\ \{3\}, & \text{otherwise,} \end{cases}$$

and hence

$$\begin{aligned} \gamma_{r3}(P_m\Box P_n) &\leq \omega(f) \\ &= 2\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor + mn - \left(\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\right) \\ &= mn - \left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor. \end{aligned}$$

If both m and n are even, then we also provide a 3RDF $g : V(P_m \Box P_n) \to \mathcal{P}(\{1,2,3\})$ defined by

$$g((i,j)) = \begin{cases} \{1,2\}, & \text{if } i \in \{1,3,\ldots,\min\{n-3,m-1\}\} \\ & \text{and } j = 2k-1 \; for \; 1 \le k \le \frac{n-i-1}{2}, \text{ or} \\ & \text{if } i \in \{2,4,\ldots,\min\{n-2,m-1\}\} \\ & \text{and } j = 2k \text{ for } \frac{n-i}{2} \le k \le \frac{n-2}{2}, \text{ or} \\ & \text{if } m \ge n+1, \; i = 2l \text{ for } \frac{n}{2} \le l \le \frac{m-2}{2} \\ & \text{and } j = 2k \text{ for } 0 \le k \le \frac{n-2}{2}; \\ \{3\}, & \text{if } i = 0 \text{ and } 0 \le j \le n-1, \text{ or} \\ & \text{if } j = 0 \text{ and } 1 \le i \le \min\{n-1,m-1\}, \text{ or} \\ & \text{if } i \in \{2,4,\ldots,\min\{n-4,m-4\}\} \\ & \text{and } j = 2k \text{ for } 1 \le k \le \frac{n-i-2}{2}, \text{ or} \\ & \text{if } i \in \{1,3,\ldots,\min\{n-1,m-1\}\} \\ & \text{and } j = 2k-1 \text{ for } \frac{n-i+1}{2} \le k \le \frac{n}{2}, \text{ or} \\ & \text{if } m \ge n+1, i = 2l+1 \text{ for } \frac{n}{2} \le l \le \frac{m-2}{2} \\ & \text{and } j = 2k+1 \text{ for } 0 \le k \le \frac{n-2}{2}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

and hence

$$\omega(g) = \begin{cases} 2n + (\frac{m-2}{2})(\frac{3n}{2}) = \frac{3mn+2n}{4}, & \text{if } n \ge m \\ 2n + (\frac{n-2}{2})(\frac{3n}{2}) + (\frac{m-n}{2})(\frac{3n+2}{2}) = \frac{3mn+2m}{4}, & \text{if } m \ge n \end{cases}$$

Therefore, if both m and n are even, then $\gamma_{r3}(P_m \Box P_n) \le \omega(g) = \frac{3mn+2\max\{m,n\}}{4}$, which completes our proof. □

Using Corollary 3.3 and Lemma 3.4, we can derive the following result.

Theorem 3.5. *For any integers $m, n \ge 2$,*
(1) *If m is odd, or n is odd, then*

$$\gamma_{r3}(P_m \Box P_n) = mn - \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

(2) *If both m and n are even, then*

$$\gamma_{r3}(P_m \Box P_n) = \frac{3mn + 2\max\{m,n\}}{4}.$$

4 The k-rainbow domination number $(k \ge 4)$

We now determine the exact value of $\gamma_{rk}(P_m \Box P_n)$ for $k \ge 4$.

Theorem 4.1. *For any integers $m, n \ge 1$ and $k \ge 4$,*

$$\gamma_{rk}(P_m \Box P_n) = mn.$$

Proof. Let f be a $\gamma_{rk}(P_m\Box P_n)$-function such that $|\{(i,j)\in V(P_m\Box P_n): f((i,j))=\emptyset\}|$ is minimum and let $a_j=\sum_{i=0}^{m-1}|f((i,j))|$ for each $j\in\{0,1,\ldots,n-1\}$.

Claim 5. If $f((i,j))=\emptyset$, where $i,j\geq 1$, then $|f((i-1,j))|\geq 2$.

Proof of Claim 5. Suppose, to the contrary, that $|f((i-1,j))|\leq 1$. Without loss of generality, we may assume that $f((i-1,j))=\emptyset$ or $\{1\}$. Note that f is a $\gamma_{rk}(P_m\Box P_n)$-function. Therefore, $f((i,j-1))=\{1,2,\ldots,k\}$ or $\{2,3,\ldots,k\}$. Define a function $g: V(P_m\Box P_n)\to\mathcal{P}(\{1,2,\ldots,k\})$ by

$$g(v)=\begin{cases}\{1\}, & \text{if } v=(i,j-1),(i,j),\\ f(v)\cup\{1\}, & \text{if } v=(i+1,j-1),\\ f(v), & \text{otherwise.}\end{cases}$$

We observe that g is a kRDF of $P_m\Box P_n$ with weight $\omega(g)\leq\omega(f)-(k-4)\leq\omega(f)=\gamma_{rk}(P_m\Box P_n)$, implying that g is also a $\gamma_{rk}(P_m\Box P_n)$-function. Moreover, clearly $|\{(i,j)\in V(P_m\Box P_n): f((i,j))=\emptyset\}|-|\{(i,j)\in V(P_m\Box P_n): g((i,j))=\emptyset\}|=1$ or 2, a contradiction to the choice of f. So, this claim is true.

Using the similar method to Claim 2 of Lemma 3.1, we have that for each i and j, $f((i,0))\neq\emptyset$ and $f((0,j))\neq\emptyset$. This implies that $a_0\geq m$. Moreover, it follows from Claim 5 that for each $j\in\{1,2,\ldots,n-1\}$, $a_j\geq m$. Therefore,

$$\gamma_{rk}(P_m\Box P_n)=\omega(f)=\sum_{j=0}^{n-1}a_j\geq mn.$$

On the other hand, it is easy to see that $\gamma_{rk}(P_n\Box P_m)\leq mn$. As a result, we have $\gamma_{rk}(P_m\Box P_n)=mn$, which completes our proof. □

Acknowledgments

This work was supported by Research Foundation of Education Bureau of Jiangxi Province of China (No. GJJ150561), Doctor Fund of East China University of Technology (No. DHBK2015319), Natural Science Foundation of Fujian Province (No. 2016J01025), Science Foundation for the Education Department of Fujian Province (No. JA15295) and Science Foundation of Minjiang University (No. MYK15004).

References

[1] J. Amjadi, A. Bahremandpour, S. M. Sheikholeslami and L. Volkmann, The rainbow domination number of a digraph, *Kragujevac J. Math.* 37 (2013), 257–268.

[2] B. Brešar, M. A. Henning and D. F. Rall, Rainbow domination in graphs, *Taiwanese J. Math.* 12 (2008), 213–225.

[3] N. Dehgardi, S. M. Sheikholeslami and A. Khodkar, Bounding the paired-domination number of a tree in terms of its annihilation number, *Filomat* 28 (2014), 523–529.

[4] O. Favaron, Signed domination in regular graphs, *Discrete Math.* 158 (1996), 287–293.

[5] S. Fujita and M. Furuya, Rainbow domination numbers on graphs with given radius, *Discrete Appl. Math.* 166 (2014), 115–122.

[6] G. Hao and J. Qian, On the rainbow domination number of digraphs, *Graphs Combin.* 32 (2016), 1903–1913.

[7] G. Hao and J. Qian, Bounds on the domination number of a digraph, *J. Comb. Optim.* 35 (2018), 64–74.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[9] S. M. H. Moghaddam, A. Khodkar and B. Samadi, New bounds on the signed domination numbers of graphs, *Australas. J. Combin.* 61 (2015), 273–280.

[10] Z. Stepień and M. Zwierzchowski, 2-rainbow domination number of Cartesian products: $C_n \Box C_3$ and $C_n \Box C_5$, *J. Comb. Optim.* 28 (2014), 748–755.

[11] Y. Wu and N. Jafari Rad, Bounds on the 2-rainbow domination number of graphs, *Graphs Combin.* 29 (2013), 1125–1133.

(Received 12 June 2017; revised 17 Jan 2018)