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A note on the restricted arc connectivity of oriented graphs of girth four

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Abstract

Let D be a strongly connected digraph. An arc set S of D is a restricted arc-cut of D if $D - S$ has a non-trivial strong component D_1 such that $D - V(D_1)$ contains an arc. The **restricted arc-connectivity** $\lambda'(D)$ of a digraph D is the minimum cardinality over all restricted arc-cuts of D. A strongly connected digraph D is **λ'-connected** when $\lambda'(D)$ exists. This paper presents a family $\mathcal{F}$ of strong digraphs of girth four that are not λ'-connected and for every strong digraph $D \notin \mathcal{F}$ with girth four it follows that it is λ'-connected. Also, an upper and lower bound for $\lambda'(D)$ are given.

1 Terminology and introduction

All the digraphs considered in this work are finite oriented graphs; that is, they are digraphs with no symmetric arcs or loops. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. If v is a vertex of D, the sets of **out-neighbors** and **in-neighbors** of v are denoted by $N^+(v)$ and $N^-(v)$, respectively. If (u, v) is an arc of D, then it is said that u **dominates** v (or v is **dominated by** u) and this is denoted by $u \to v$. Two vertices u and v of a digraph are **adjacent** if $u \to v$ or $v \to u$. The numbers

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$d^+(v) = |N^+(v)|$ and $d^-(u) = |N^-(u)|$ are the **out-degree** and the **in-degree** of the vertex v. By a **cycle** of a digraph we mean a directed cycle. A p**-cycle** is a cycle of length p. The minimum integer p for which D has a p-cycle is the **girth** of D, denoted by $g(D)$. Given a digraph D, the subdigraph of D induced by a set of vertices X is denoted by $D[X]$. For any subset S of $A(D)$, the subdigraph obtained by deleting all the arcs of S is denoted by $D - S$. A digraph D is **strongly connected** or simply **strong** if for every pair u, v of vertices there exists a directed path from u to v in D. A **strong component** of a digraph D is a maximal induced subdigraph of D which is strong. A digraph D is called k**-arc-connected** if for any set S of at most $k-1$ arcs the subdigraph $D - S$ is strong. The **arc-connectivity** $\lambda(D)$ of a digraph D is the largest value of k such that D is k-arc-connected. For a pair X, Y of vertex sets of a digraph D, we define $(X, Y) = \{x \to y \in A(D) : x \in X, y \in Y\}$. Let X^c be the complement of X. If $Y = X^c$ we write (X, X^c) as $\partial^+(X)$ or $\partial^-(Y)$. Let D be a digraph with girth g. If $C = (v_1, v_2, \ldots, v_g)$ is a g-cycle of D, then let

$$\xi(C) = \min\left\{\sum_{i=1}^{g} d^+(v_i) - g, \sum_{i=1}^{g} d^-(v_i) - g\right\}$$

and

$$\xi(D) = \min\{\xi(C) : C \text{ is a } g\text{-cycle of } D\}.$$

We follow the book of Bang-Jensen and Gutin [4] for terminology and definitions not given here.

As is well known, a digraph is a mathematical object modeling networks. An important parameter in the study of networks is the fault tolerance: it is desirable that if some nodes (respectively links) are unable to work, the message can still be always transmitted. There are measures that indicate the fault tolerance of a network (modeled by a digraph D); for instance, the arc-connectivity of D measures how easily and reliably a packet sent by a vertex can reach another vertex. Since digraphs with the same arc-connectivity can have large differences in the fault tolerance of the corresponding networks, one might be interested in defining more refined reliability parameters in order to provide a more accurate measure of fault tolerance in networks than the arc-connectivity (see [6]). In this context, Volkmann [11] introduced the concept of restricted arc-connectivity of a digraph, which is closely related to the similar concept of restricted edge-connectivity in graphs proposed by Esfahanian and Hakimi [7].

Definition 1 (Volkmann [11]) *Let D be a strongly connected digraph. An arc set S of D is a* ***restricted arc-cut*** *of D if $D - S$ has a non-trivial strong component D_1 such that $D - V(D_1)$ contains an arc. The* ***restricted arc-connectivity*** *$\lambda'(D)$ of D is the miminum cardinality over all restricted arc-cuts. A strongly connected digraph D is said to be* ***λ'-connected*** *if $\lambda'(D)$ exists.*

Observe that $\lambda'(D)$ does not exist for every digraph with fewer than $g(D)+2$ vertices. Volkmann [11] proved that each strong digraph D of order $n \geq 4$ and girth $g(D) = 2$ or $g(D) = 3$ except for some families of digraphs is λ'-conncected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$. Moreover, he proved the following characterization.

Theorem 1 *[11] A strongly connected digraph D with girth g is λ'-connected if and only if D has a cycle of length g such that $D - V(C)$ contains an arc.*

Concerning the arc-restricted connectivity of digraphs, Meierling, Volkmann and Winzen [10] studied the restricted arc-connectivity of generalizations of tournaments. Balbuena, García-Vázquez, Hansberg and Montejano [1, 2] studied the restricted arc connectivity for some families of digraphs and introduced the concept of super-λ' digraphs. Results on restricted arc-connectivity of digraphs can be found in, e.g. Balbuena and García-Vázquez [3], Chen, Liu and Meng [5], Grüter, Guo and Holtkamp [8], Grüter, Guo, Holtkamp and Ulmer [9] and Wang and Lin [12].

In this paper we present a family $\mathcal{F}$ of strong digraphs of girth four that are not λ'-connected and for every strong digraph $D \notin \mathcal{F}$ with girth four it follows that it is λ'-connected and $\lambda(D) \leq \lambda'(D) \leq \xi(D)$.

2 Main result

Let D be a strong digraph of girth 4. In this section it is proved that D is λ'-connected with the exception of the case that D is a member of the following seven families (see Figure 1).

Let H_1 be the digraphs having the 4-cycle (u, v, w, z, u) and the following vertex sets: $A = \{a_1, a_2, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_q\}$, $C = \{c_1, c_2, \ldots, c_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$ such that $u \to a_i \to v$ for $1 \leq i \leq p$, $v \to b_i \to w$ for $1 \leq i \leq q$, $w \to c_i \to z$ for $i \leq i \leq r$ and $z \to d_i \to u$ for $1 \leq i \leq s$. The cases that A, B, C or D are empty sets are also allowed.

Let H_2 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u), and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$ such that $w \to a_i \to u$ for $1 \leq i \leq p$ and $u \to b_i \to w$ for $1 \leq i \leq q$. The cases that A or B are empty sets are also allowed.

Let H_3 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u) and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$, $B = \{b_1, b_2, \ldots, b_q\}$ and $C = \{c_1, c_2, \ldots, c_r\}$ such that $u \to a_i \to v$, for $1 \leq i \leq p$, $v \to b_i \to w$ for $1 \leq i \leq q$ and $w \to c_i \to u$ for $1 \leq i \leq r$. The cases that A, B or C are empty sets are also allowed.

Let H_4 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u), a vertex y such that $u \to y \to w$ and y is adjacent to v, and the vertex set $A = \{a_1, a_2, \ldots, a_p\}$ such that $w \to a_i \to u$ for $1 \leq i \leq p$. The case that A is an empty set is also admissible.

Let H_5 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u) such that x is adjacent to z, and the vertex set $A = \{a_1, a_2, \ldots, a_p\}$ such that $u \to a_i \to w$ for $1 \leq i \leq p$.

Let H_6 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u) such that x is adjacent to z, and the vertex sets $A = \{a_1, a_2, \ldots, a_p\}$, and $B = \{b_1, b_2, \ldots, b_q\}$ such that $u \to a_i \to v$ for $1 \leq i \leq p$ and $v \to b_i \to w$ for $1 \leq i \leq q$. The cases that A and B are empty sets are also allowed.

Let H_7 be the digraphs having the 4-cycles (u, v, w, z, u) and (u, v, w, x, u) such that x is adjacent to z, and a vertex y adjacent to v such that $u \to y \to w$.

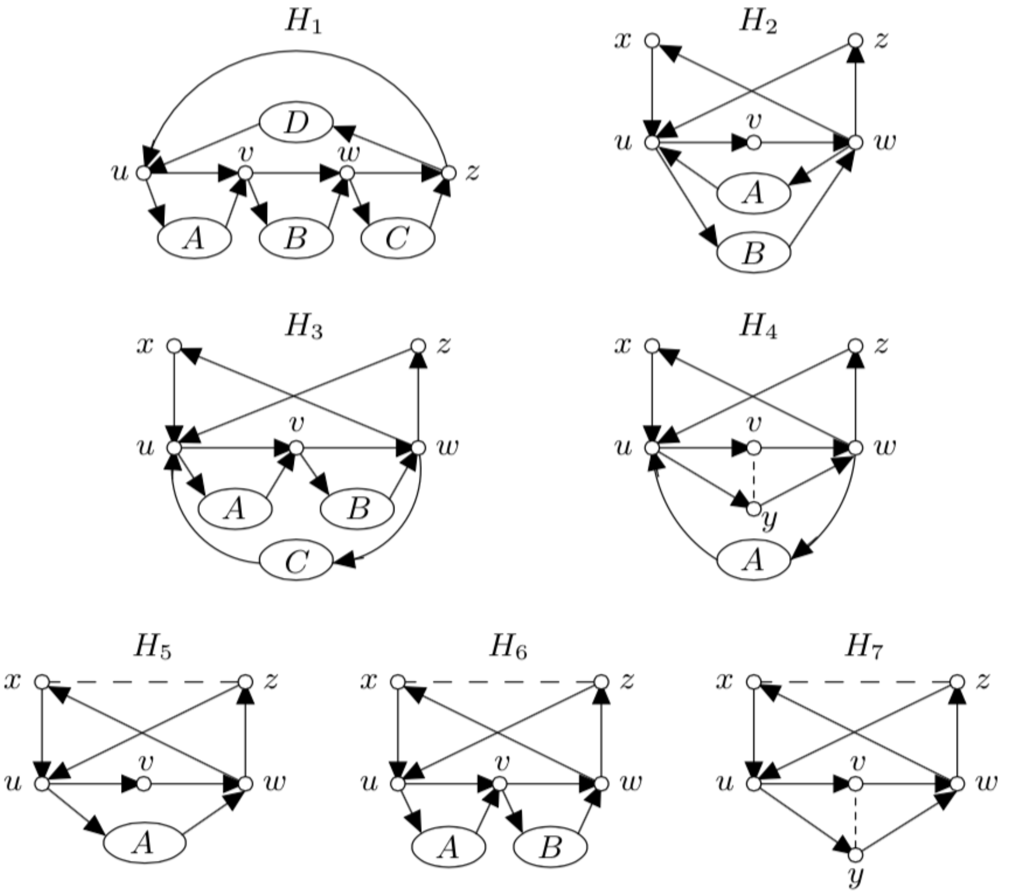

Figure 1: Families of digraphs that are not λ'-connected. Dotted line indicates adjacency.

Observe that by Theorem 1, the digraphs of $H_1, H_2, \ldots, H_7$ are not λ'-connected.

Theorem 2 *Let D be a strong digraph of girth 4 and $|V(D)| \geq 6$. If D is not isomorphic to a member of the families $H_1, H_2, \ldots, H_7$, then D is λ'-connected and*

$$\lambda(D) \leq \lambda'(D) \leq \xi(D).$$

Proof. To prove the left inequality, since every restricted cut is a cut, it follows that $\lambda(D) \leq \lambda'(D)$.

Next, we prove the right hand inequality. Let $C = (u, v, w, z, u)$ be a 4-cycle of D such that $\xi(D) = \xi(C)$. Suppose without loss of generality that $\xi(C) = d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4$. If $D - \{u, v, w, z\}$ contains an arc, then D is λ'-connected and $\lambda'(D) \leq \xi(D)$. Hence suppose that $D - \{u, v, w, z\}$ consists of a set of isolated vertices. Since D is not isomorphic to a member of H_1, D has to contain a 4-cycle C' containing two arcs of C. Let $C' = (u, v, w, x, u)$. We continue the proof by distinguishing three cases.

Case 1 Assume that $d^+(x) = d^-(x) = 1$.

Subcase 1.1 If $d^+(z) = d^-(z) = 1$. Since D is not isomorphic to any member of H_2, H_3 and H_4, it follows that $|V(D)| \geq 7$ impliying that there exists a set of vertices $a_1, a_2, \ldots, a_m$, $m \geq 2$, such that $a_i \notin \{u, v, w, x, z\}$ for $1 \leq i \leq m$. If $d^+(v) = d^-(v) = 1$. Since D is strong, it follows that $d^+(a_i) = d^-(a_i) = 1$ for every $1 \leq i \leq m$ impliying that D is isomorphic to a member of H_2, a contradiction. Therefore, either $d^+(v) \geq 2$ or $d^-(v) \geq 2$. Suppose that $d^+(v) \geq 2$ and $d^-(v) = 1$, then there exists a vertex a_1 such that $v \to a_1$. Since D is strong and has girth 4, it follows that $a_1 \to w$. Moreover, since D is not a member of the families H_3 and H_4, there exists a vertex a_2, $a_2 \neq a_1$ such that $u \to a_2 \to w$. Also, as $d^-(v) = 1$, it follows that $d^+(a_2) = 1$. Consider the 4-cycle $C_1 = (u, a_2, w, z, u)$, therefore

$$\begin{aligned} \xi(C_1) &\leq d^+(u) + d^+(a_2) + d^+(w) + d^+(z) - 4 \\ &< d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D), \end{aligned}$$

giving a contradiction. Hence $d^+(v) = 1$ and $d^-(v) \geq 2$ or $d^+(v) \geq 2$ and $d^-(v) \geq 2$.

First suppose that $d^+(v) = 1$ and $d^-(v) \geq 2$, then there exists a vertex a_1 such that $a_1 \to v$. Further, since D is strong and has girth 4, it follows that $u \to a_1$. As D is not isomorphic to any member of families H_3 and H_4, there exists a vertex a_2, $a_2 \neq a_1$ such that $u \to a_2 \to w$. Let $S = \{ua_1, vw, wx\} \subset A(D)$. The digraph $D - S$ has a strong component D_1 containing the 4-cycle (u, a_2, w, z, u) and $D - S$ contains the arc a_1v. Therefore D is λ'-connected and

$$\lambda'(D) \leq |S| \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

Now, suppose that $d^+(v) \geq 2$ and $d^-(v) \geq 2$, then there exist two vertices a_1, a_2, such that $a_1 \to v$ and $v \to a_2$. Since D is strong and has girth 4, $u \to a_1$ and $a_2 \to w$. Since D is not isomorphic to any member of the family H_3, there exists a vertex a_3 such that $u \to a_3 \to w$. Let $S = \partial^+(\{u, a_3, w, z\})$, then S is a restricted arc-cut of D and

$$\begin{aligned} \lambda'(D) &\leq |S| \leq d^+(u) + d^+(a_3) + d^+(w) + d^+(z) - 4 \\ &\leq d^+(u) + 2 + d^+(w) + d^+(z) - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 \\ &= \xi(D), \end{aligned}$$

Subcase 1.2 Assume that either $d^+(z) \geq 2$ or $d^-(z) \geq 2$. This implies that there exists a vertex a, different from u, v, w, x in $N^+(z) \cup N^-(z)$. Suppose first that $z \to a$. Therefore

$$\begin{aligned} \xi((u, v, w, x, u)) &\leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4 \\ &< d^+(u) + d^+(v) + d^+(w) + 2 - 4 \leq \xi(D), \end{aligned}$$

giving a contradiction. Now suppose that $a \rightarrow z$. Let $S = \partial^+(\{u, v, w, x\})$. Note that $D - S$ has a strong component D_1 containing the 4-cycle (u, v, w, x, u) and $D - V(D_1)$ contains the arc az. Hence S is a λ'-restricted arc cut and

$$\begin{aligned}\lambda'(D) \leq |S| &\leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + 1 - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),\end{aligned}$$

and the result follows.

Case 2 Assume that $d^+(x) = 1$ and $d^-(x) = 2$. This implies that $z \rightarrow x$ and therefore $d^+(z) \geq 2$. Since (u, v, w, x, u) is a 4-cycle, it follows that

$$\begin{aligned}\xi((u, v, w, x, u)) &\leq d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4 \\ &< d^+(u) + d^+(v) + d^+(w) + 2 - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),\end{aligned}$$

yielding a contradiction.

Case 3 Assume that $d^+(x) = 2$ and $d^-(x) = 1$. This implies that $x \rightarrow z$.

Subcase 3.1 If $d^+(z) = 1$ and $d^-(z) = 2$. Suppose first that $d^+(v) = d^-(v) = 1$. Since D is not isomorphic to any member of the family H_5, it follows that there exists a vertex a_1 such that $w \rightarrow a_1 \rightarrow u$. Let $S = \partial^+(\{u, v, w, a_1\})$. The digraph $D - S$ has a strong component D_1 containing the 4-cycle (u, v, w, a_1, u) and $D - V(D_1)$ contains the arc xz. Hence D is λ'-connected and

$$\begin{aligned}\lambda'(D) &\leq d^+(u) + d^+(v) + d^+(w) + d^+(a_1) - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + 1 - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).\end{aligned}$$

Now, suppose that either $d^+(v) \geq 2$ or $d^-(v) \geq 2$. If $d^+(v) \geq 2$ and $d^-(v) = 1$, then there exists a vertex a_1 such that $v \rightarrow a_1$. Further, as D is strong, it follows that $a_1 \rightarrow w$. Since D is not isomorphic to any member of the families H_6 and H_7, the order of D is at least 7 and there exists a vertex a_2 adjacent to u and w. If $u \rightarrow a_2 \rightarrow w$, then a_2 is not adjacent to v and $d^+(a_2) = 1$. Since (u, a_2, w, z, u) is a 4-cycle, it follows that

$$\begin{aligned}\xi((u, a_2, w, z, u)) &\leq d^+(u) + d^+(a_2) + d^+(w) + d^+(z) - 4 \\ &= d^+(u) + 1 + d^+(w) + d^+(z) - 4 \\ &< d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),\end{aligned}$$

giving a contradiction.

If that $w \to a_2 \to u$. Let $S = \partial^+(\{u, v, w, a_2\})$. The digraph $D - S$ has a strong component D_1 containing the 4-cycle (u, v, w, a_2, u) and $D - V(D_1)$ has the arc xz. Therefore D is λ'-connected and

$$\begin{aligned}\lambda'(D) &\leq d^+(u) + d^+(v) + d^+(w) + d^+(a_2) - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + 1 - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).\end{aligned}$$

Now, suppose that $d^+(v) = 1$ and $d^-(v) \geq 2$, then there exists a vertex a_1 such that $a_1 \to v$, and since D is strong it follows that $u \to a_1$. Since D is not isomorphic to any member of the families H_6 and H_7, then $|V(D)| \geq 7$ and there exists a vertex a_2 such that a_2 and w are adjacent. If $u \to a_2 \to w$, let $S = \{ua_1, vw, wx\} \subset A(D)$, then $D - S$ has a strong componononent D_1 containing the 4-cycle (u, a_2, w, z, u) and $D - V(D_1)$ has the arc a_1v. Therefore D is λ'-connected and

$$\lambda'(D) \leq 3 \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).$$

If $w \to a_2 \to u$, let $S = \partial^+(\{u, v, w, a_2\}) \subset A(D)$, then $D - S$ is a restricted arc cut of D such that $D - S$ has a strong component D_1 containing the 4-cycle (u, v, w, a_2, u) and $D - V(D_1)$ has the arc xz. Therefore,

$$\begin{aligned}\lambda'(D) &\leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(a_2) - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + 1 - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).\end{aligned}$$

Now, suppose that $d^+(v) \geq 2$ and $d^-(v) \geq 2$, then there are two vertices a_1 and a_2 such that $a_1 \to v$ and $v \to a_2$. Since D is strong and has girth 4 it follows that $u \to a_1$ and $a_2 \to w$ (note that this may be the case where $a_1 \to w$ or $u \to a_2$). Since D is not isomorphic to any member of the family H_6 there exists a vertex a_3 adjacent to u and w. If $u \to a_3 \to w$ (note that this may be the case where $a_3 = a_1$ or $a_3 = a_2$ or a_3 is adjacent to v). Let $S = \partial^+(\{u, a_3, w, z\})$. Then the digraph $D - S$ has a strong component D_1 containing de 4-cycle (u, a_3, w, z, u) and $D - V(D_1)$ has the arc a_1v or va_2, according to the case. Therefore D is λ'-connected and

$$\begin{aligned}\lambda'(D) &\leq |S| = d^+(u) + d^+(a_3) + d^+(w) + d^+(z) - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).\end{aligned}$$

If $w \to a_3 \to u$. Let $S = \partial^+(\{u, v, w, a_3\}) \subset A(D)$, then the digraph $D - S$ has a strong component D_1 containing de 4-cycle (u, v, w, a_3, u) and $D - V(D_1)$ has the arc xz. Therefore D is λ'-connected and

$$\begin{aligned}\lambda'(D) &\leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(a_3) - 4 \\ &= d^+(u) + 2 + d^+(w) + 1 - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D).\end{aligned}$$

Subcase 3.2 If $d^+(z) \geq 2$ or $d^-(z) \geq 3$. Then there exists a vertex $a \notin \{u, v, w, x\}$ such that a and z are adjacent. Suppose first that $z \to a$, then consider the set of arcs $S = \partial^+(\{u, v, w, x\})$. Therefore the digraph $D - S$ has a strong component D_1 containing de 4-cycle (u, v, w, x, u) and $D - V(D_1)$ has the arc az. Consequently, D is λ'-connected and

$$\begin{aligned} \lambda'(D) &\leq |S| = d^+(u) + d^+(v) + d^+(w) + d^+(x) - 4 \\ &= d^+(u) + d^+(v) + d^+(w) + 2 - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D). \end{aligned}$$

Now, suppose that $a \to z$. Since D is strong it follows that either $v \to a$ or $w \to a$. Suppose first that $v \to a$ and let $S = \partial^+(\{u, v, a, z\})$. Therefore $D - S$ has a strong component D_1 containing de 4-cycle (u, v, a, z, u) and $D - V(D_1)$ has the arc wx. Therefore D is λ'-connected and

$$\begin{aligned} \lambda'(D) &\leq |S| = d^+(u) + d^+(v) + d^+(a) + d^+(z) - 4 \\ &\leq d^+(u) + +d^+(v) + 2 + d^+(z) - 4 \\ &\leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D). \end{aligned}$$

Now, suppose that $w \to a$. If either $v \to a$ or there exists a vertex $a' \neq a$ such that $z \to a'$, then this case is reduced to one of the two previous subcases. Otherwise observe that the condition on the girth implies that neither $a \to v$ nor $u \to a$. Suppose that $a \to u$. Let $S = \{zu, au\}$, then the digraph $D - S$ has a strong component D_1 containing de 4-cycle (u, v, w, x, u) and $D - V(D_1)$ has the arc az.Therefore D is λ'-connected and

$$\lambda'(D) \leq 2 \leq d^+(u) + d^+(v) + d^+(w) + d^+(z) - 4 = \xi(D),$$

concluding the proof. ■

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