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Stirling numbers of the first kind for graphs

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Abstract

For a simple graph G, let G° be the graph obtained from G by adding a loop at each vertex. Let $\mathfrak{S}_G$ be the set of all permutations σ on $V(G)$ such that $v\sigma(v) \in E(G^\circ)$. In other words, each $\sigma \in \mathfrak{S}_G$ partitions the graph G into vertex-disjoint cycles, where a 1-cycle is a single vertex, a 2-cycle is a single edge, and orientation for cycles of length three or higher matters. We define the graphical factorial of a graph G, denoted by $G!$, as the cardinality of $\mathfrak{S}_G$. The Stirling numbers of the first kind for G, denoted by $\left[{G \atop k}\right]$, is the number of permutations $\sigma \in \mathfrak{S}_G$ such that σ partitions $V(G)$ into exactly k cycles. In this paper, we will find the Stirling numbers of the first kind of elements of some families of graphs, such as paths, cycles, complete bipartite graphs, wheels, fans, and ladders.

1 Introduction

The Stirling numbers of the second kind, denoted by $\left\{{n \atop k}\right\}$, count the number of set partitions of $[n] = \{1, \ldots, n\}$ into k parts. These numbers satisfy the recurrence relation

$$\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\},$$

with the initial conditions $\left\{{n \atop 0}\right\} = \left\{{0 \atop k}\right\} = 0$ for all $n, k > 0$, and $\left\{{0 \atop 0}\right\} = 1$. The Bell numbers, denoted by B_n, are defined as $B_n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}$.

Now consider E_n the empty graph on n vertices. It is easy to check that the number of partitions of $V(E_n)$ into k parts, where the vertices in each part form an independent set in $\left\{{n \atop k}\right\}$. By generalizing this idea, the k-th Stirling number of the second kind for a graph G, denoted by $\left\{{G \atop k}\right\}$, is defined as the number of partitions of $V(G)$ into k independent sets. According to Galvin and Tanh [5], this definition was

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first explicitly introduced by Tomescu in 1971 [11]. Consequently, the Bell number for a graph G is defined as $B_G = \sum_{k=0}^{n} \left\{ {G \atop k} \right\}$, where $n = |V(G)|$. For a more recent discussions about these numbers, see [3, 4, 7].

On the other hand, the (unsigned) Stirling numbers of the first kind, denoted by $\left[{n \atop k} \right]$, count the number of cyclic partitions of $[n] = \{1, \ldots, n\}$ into k cycles. In other words, $\left[{n \atop k} \right]$ counts the number of permutations in $\mathfrak{S}_n$, the symmetric group on $[n]$, that can written as the product of k disjoint cycles, where 1-cycles are included in the count. The Stirling numbers of the first kind satisfy the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

with the initial conditions $\left[{n \atop 0} \right] = \left[{0 \atop k} \right] = 0$ for all $n, k > 0$, and $\left[{0 \atop 0} \right] = 1$. Clearly, $n! = \sum_{k=0}^{n} \left[{n \atop k} \right]$.

Using a similar idea that motivated the definition of the Stirling numbers of the second kind for graphs, we will define the graphical Stirling numbers of the first kind: For a simple graph G, let G° be the graph obtained from G by adding a loop at each vertex. Let $\mathfrak{S}_G$ be the set of all permutations σ on $V(G)$ such that $v\sigma(v) \in E(G^{\circ})$. In other words, each $\sigma \in \mathfrak{S}_G$ partitions the graph G into vertex-disjoint cycles, where a 1-cycle is a single vertex, a 2-cycle is a single edge, and orientation for cycles of length three or higher matters. To avoid any confusion, from now on, by "cycle" we mean this more general definition. We will call an element in $\mathfrak{S}_G$ a cyclic partition of G. We define the graphical factorial for a graph G, denoted by $G!$, as the number of distinct cyclic partitions of G. The k-th Stirling number of the first kind of a graph G, denoted by $\left[{G \atop k} \right]$, is the number of permutations $\sigma \in \mathfrak{S}_G$ such that σ partitions $V(G)$ into exactly k cycles. It is easy to see that $G! = \sum_{k=0}^{n} \left[{G \atop k} \right]$, where $n = |V(G)|$. It is noteworthy that this definition of a factorial of a graph coincides with that of seating rearrangements with stays on a graph that was discussed by DeFord in [1].

In this paper, K_n, C_n, and P_n denote the complete graph, the cycle, and the path on n vertices, respectively. Also, $K_{n,m}$ denotes the complete bipartite graph with one part having n and the other part having m vertices. The join of two simple graphs G and H, denoted by $G \bowtie H$, is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set $E(G) \cup E(H) \cup \{uv \mid u \in G, v \in H\}$. We call the graph $K_1 \bowtie C_n$ the wheel on $(n+1)$ vertices and denote it by W_n, and call the graph $K_1 \bowtie P_n$ the fan on $(n+1)$ vertices and denote it by F_n.

In addition to the join of two graphs, we will use two graphical products: For simple graphs G and H, the Cartesian product of G and H, denoted by $G \square H$, is a graph with the vertex set $V(G) \times V(H)$ such that any two vertices (u, u') and (v, v') are adjacent in $G \square H$ if and only if $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. The strong product if G and H, denoted by $G \boxtimes H$, is a graph with the vertex set $V(G) \times V(H)$ such that any two vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$, or $uv \in E(G)$ and $u'v' \in E(H)$.

Throughout this paper, we will use Iverson bracket which is defined as

$$[Q] = \begin{cases} 1 & \text{if } Q \text{ is true;} \\ 0 & \text{otherwise,} \end{cases}$$

where Q is a statement that can be true or false. Moreover, we will use the notation $x^{\underline{k}}$ for the falling factorial power $x(x-1)\cdots(x-k+1)$, where $x \in \mathbb{R}$ and $k \in \mathbb{N}$.

The following result was proven in [1] and [2]. In this theorem, f_k is the k-th Fibonacci number defined by the recurrence relation $f_k = f_{k-1} + f_{k-2}$ and the initial conditions $f_1 = f_2 = 1$. By convention, we assume that $P_0! = 1 = f_1$, where P_0 is the path on zero vertices, i.e., the empty set.

Theorem 1.1. *Let* $n, m \in \mathbb{N}$. *Then*

1. $K_n! = n!$;
2. $P_n! = f_{n+1}$;
3. $C_n! = f_{n+1} + f_{n-1} + 2$, *for* $n \geq 3$;
4. $K_{n,m}! = \sum_{i=0}^{k} m^{\underline{i}}\, n^{\underline{i}}$, *where* $k = \min\{m, n\}$;
5. $W_n! = (2n+1)f_{n+1} + nf_n + f_{n-1} - 2(n-1)$, *for* $n \geq 3$;
6. $F_n! = f_{n+1} + \sum_{l=1}^{n}[f_{n-l+1}(f_{l+2} - 1) + f_l(f_{n-l+2} - 1)]$, *for* $n \geq 2$.

We will finish this section with the following observation whose proof is obvious:

Observation 1.1. *Let* G *be a simple graph with* $n \geq 3$ *vertices. Then* $\left[{G \atop 1}\right] = 2|\mathcal{H}(G)|$, *where* $\mathcal{H}(G)$ *is the set of undirected Hamiltonian cycles of* G. □

2 Stirling Numbers of the First Kind for Basic Graphs

The first result in this section is obvious and we will omit its proof.

Proposition 2.1. *For* $n \in \mathbb{N}$, $\left[{K_n \atop k}\right] = \left[{n \atop k}\right]$. □

In our next two theorems, $\left[{P_n \atop k}\right]$ and $\left[{C_n \atop k}\right]$ are given in closed-form expressions:

Theorem 2.1. *Let* n, k *be positive integers. Then* $\left[{P_n \atop k}\right] = \binom{k}{n-k}$.

Proof. It is easy to see that $\left[{P_n \atop k}\right]$ is the number of integer solutions to the Diophantine equation

$$x_1 + \cdots + x_k = n,$$

where $x_i \in \{1, 2\}$ for all $1 \leq i \leq k$. Starting from one of the leaves of P_n, say v, each x_i represents whether the i-th cycle from v along P_n is a 1-cycle or a 2-cycle. We know that the number of integer solutions to this equation is $\binom{k}{n-k}$. Note that in any such cyclic partition the number of 1-cycles and 2-cycles is $2k-n$ and $n-k$, respectively. $\square$

Using Theorem 2.1 and the well-known Fibonacci identity

$$\sum_{k=0}^{n}\binom{k}{n-k}=\sum_{j=0}^{n}\binom{n-j}{j}=f_{n+1},$$

we confirm part (2) of Theorem 1.1.

Theorem 2.2. *Let $n,k\in\mathbb{N}$. If $n\geq 3$ and $k\geq 2$, then $\left[{C_n \atop k}\right]=\binom{k-1}{n-k}+2\binom{k-1}{n-k-1}$. If $n\geq 3$, then $\left[{C_n \atop 1}\right]=2$.* $\square$

Proof. We know that for $n\geq 3$, $\left[{C_n \atop 1}\right]=2$ since the only cycle of length three or higher is the graph itself. On the other hand, for $n\geq 3$ and $k\geq 2$, we have

$$\begin{bmatrix}C_n\\ k\end{bmatrix}=\begin{bmatrix}P_{n-1}\\ k-1\end{bmatrix}+2\begin{bmatrix}P_{n-2}\\ k-1\end{bmatrix}. \tag{1}$$

The reason for the above identity is this: Let v be any vertex in C_n and let u and w be its two neighbors. Three cases can happen: v is in a 1-cycle, vu is a 2-cycle or vw is a 2-cycle. In the first case, we are left with P_{n-1} when we remove v. In the second case, we are left with P_{n-2} when we remove vu—and similarly in the third case, when we remove vw. Using Theorem 2.1 and equation (1), we have $\left[{C_n \atop k}\right]=\binom{k-1}{n-k}+2\binom{k-1}{n-k-1}$. $\square$

Theorem 2.3. *Let assume that $n,m\in\mathbb{N}$ and $m\leq n$. Then for $k\in\mathbb{N}$,*

$$\begin{bmatrix}K_{n,m}\\ k\end{bmatrix}=\sum_{i=0}^{m}\binom{m}{i}\binom{n}{i}\begin{bmatrix}i\\ k+2i-(m+n)\end{bmatrix}i!.$$

Proof. We will prove this theorem by a combinatorial argument. Since $K_{n,m}$ does not contain any odd cycles beside the 1-cycles, in cycles of length 2 or higher, each vertex in one part can be matched with a vertex in the other part, where all the matchings are pairwise disjoint. Due to this observation, we will first choose i vertices in each part that will be in an even cycle. The remaining vertices in each part then become 1-cycles, and there are $m+n-2i$ such cycles. As a result, we are left with $l=k-(m+n-2i)$ even cycles in a graph H isomorphic to $K_{i,i}$. Let us assume that the vertices in one part of H are labeled $v_1,\ldots,v_i$, and we call this part H_1. We will label the vertices in H_2, the other part of H, by $u_1,\ldots,u_i$. Suppose C is an even cycle in H and let v be a vertex in C that belongs to H_1. Starting with v and traversing C, we will visit the vertices in H_1 at every other step until we return to v, given that C is of length 4 or higher. This creates a cycle of labels of vertices in C whose length is half of the length of C. On the other hand, if C is a 2-cycle, the label of v creates a 1-cycle among the labels of vertices in H_1. Based on this observation, we will first partition the labels in H_1 into l disjoint cycles, which can be done in $\left[{i \atop l}\right]$ many ways. Then we rearrange the vertices in H_2 in $i!$ ways and create even cycles in H accordingly: Let $C_1, \ldots, C_l$ be a cyclic partition of the labels of vertices in H_1 into l cycles arranged in a left-to-right order. Starting with the leftmost vertex v' in the leftmost cycle C' not yet being used, we will match v' with the leftmost available vertex in a fixed rearrangements of $u_1, \ldots, u_i$, and then by moving back and forth between H_2 and H_1, we match the vertex at each step with the leftmost available vertex in the other part, until all the vertices in C' has been visited and we are back at v'. We then move on to the next available cycle in $C_1, \ldots, C_l$, and repeat this process. $\square$

Finally, by summing over $k = 0, \ldots, m+n$, we have

$$\begin{aligned} K_{n,m}! = \sum_{k=0}^{m+n} \begin{bmatrix} K_{n,m} \\ k \end{bmatrix} &= \sum_{k=0}^{m+n} \sum_{i=0}^{m} \binom{m}{i}\binom{n}{i} \begin{bmatrix} i \\ k+2i-(m+n) \end{bmatrix} i! \\ &= \sum_{i=0}^{m} \binom{m}{i}\binom{n}{i} i! \left(\sum_{k=0}^{m+n} \begin{bmatrix} i \\ k+2i-(m+n) \end{bmatrix} \right) \\ &= \sum_{i=0}^{m} m^{\underline{i}} \binom{n}{i} \left(\sum_{k=m+n-2i}^{m+n-i} \begin{bmatrix} i \\ k+2i-(m+n) \end{bmatrix} \right) \\ &= \sum_{i=0}^{m} m^{\underline{i}} \binom{n}{i} \left(\sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} \right) = \sum_{i=0}^{m} m^{\underline{i}} \binom{n}{i} i! = \sum_{i=0}^{m} m^{\underline{i}} n^{\underline{i}}, \end{aligned}$$

which confirms part (4) of Theorem 1.1 with the assumption that $m = \min\{m, n\}$.

We will finish this section with theorems regarding wheels and fans.

Theorem 2.4. *For $n \geq 3$, let $W_n = K_1 \bowtie C_n$. Then*

1. $\begin{bmatrix} W_n \\ k \end{bmatrix} = \binom{k-1}{n-k} + 2\binom{k-1}{n-k-1} + n\binom{k-1}{n-k} + 2n \sum_{j=0}^{n-k-1} \binom{k-1}{j}$ *for $k \geq 3$;*
2. $\begin{bmatrix} W_3 \\ 2 \end{bmatrix} = 11$ *and* $\begin{bmatrix} W_n \\ 2 \end{bmatrix} = 2(1+2n)$ *for $n \geq 4$;*
3. $\begin{bmatrix} W_n \\ 1 \end{bmatrix} = 2n$.

Proof. Let v be the vertex representing K_1 in $K_1 \bowtie C_n$ and we will refer to the copy of C_n in this graphs as C. We will consider three cases: 1) v is in a 1-cycle; 2) v is in a 2-cycle; 3) v is in a cycle of length 3 or higher. It is easy to see what happens when we have the first two cases. In the third case, since v needs to be in a cycle C' of length three or higher, there will be two vertices in C, say u and w, that are adjacent to v in C'. Let us assume that we choose u first, for which we have n possibilities. Then traversing C clockwise, we will choose w and add the path P that thus connects u to w to form C', and then decide on the orientation on C'. Assuming that l is the number of vertices along P (including u and w), what is left of W_n when we remove C is a path with $n - l$ vertices. As a result, $n - l$ needs to be greater than or equal to $k - 1$ so that we can have a cyclic partition of $K_1 \bowtie C_n$ into k cycles. This means that $2 \leq l \leq n - k + 1$. Consequently, for each choice of u,

l, and the orientation on C', we have to find the number of cyclic partitions of P_{n-l} into $k-1$ cycles. Following this argument, when $k \geq 3$, we have

$$\begin{bmatrix} W_n \\ k \end{bmatrix} = \begin{bmatrix} C_n \\ k-1 \end{bmatrix} + n \begin{bmatrix} P_{n-1} \\ k-1 \end{bmatrix} + 2n \left(\begin{bmatrix} P_{n-2} \\ k-1 \end{bmatrix} + \cdots + \begin{bmatrix} P_{k-1} \\ k-1 \end{bmatrix} \right),$$

and as a result,

$$\begin{bmatrix} W_n \\ k \end{bmatrix} = \binom{k-1}{n-k} + 2\binom{k-1}{n-k-1} + n\binom{k-1}{n-k} + 2n \sum_{j=0}^{n-k-1} \binom{k-1}{j},$$

which proves part(1) using Theorems 2.1 and 2.2. When $n \geq 4$ and $k = 2$, v is either in a 1-cycle, or in a cycle of length 3 or higher. In the former case, the other cycle is C; in the latter case, the other cycle is either a 1-cycle or a 2-cycle. Therefore, $\begin{bmatrix} W_n \\ 2 \end{bmatrix} = 2 + 2n + 2n$. The case where $n = 3$ and $k = 2$ can easily be checked by hand. Finally, part (3) follows from the fact that $K_1 \bowtie C_n$ has n undirected Hamiltonian cycles. $\square$

Corollary 2.1. *For $n, k \geq 3$, if $n > 2k$, then $\begin{bmatrix} W_n \\ k \end{bmatrix} = n2^k$.* $\square$

A combinatorial argument for the above corollary is this: Let v and C be the same as in the previous proof. It is not hard see that v cannot be in a 1-cycle or a 2-cycle, because we will not be able to partition C into $k-1$ cycles, even if we use only 2-cycles to do so. As a result, v is in a cycle of length 3 or higher. We choose a vertex in C, say u. Starting with u, we traverse clockwise along C and partition the graph into $k-1$ cycles of size 1 or 2. The remaining vertices along with v form a cycle that can be oriented in two different ways. Therefore, $\begin{bmatrix} W_n \\ k \end{bmatrix} = n2^k$.

Theorem 2.5. *For $n \geq 2$, let $F_n = K_1 \bowtie P_n$. Then for $k \geq 1$,*

$$\begin{bmatrix} F_n \\ k \end{bmatrix} = \binom{k-1}{n-k+1} + (2k-n)\binom{k}{n-k} + 2k \sum_{i=2}^{n-k+1} \binom{k-1}{n-i-k+1}.$$

Proof. Let v be the vertex representing K_1 in $K_1 \bowtie P_n$ and we will refer to the copy of P_n in this graphs as P. We will consider three cases: 1) v is in a 1-cycle; 2) v is in a 2-cycle; 3) v is in a cycle of length 3 or higher. It is easy to see what happens in the first case. In the second case, we will first partition P into k cycles, which can be done in $\begin{bmatrix} P_n \\ k \end{bmatrix} = \binom{k}{n-k}$ ways. We know from the proof of Theorem 2.1 that the number of 1-cycles in any cyclic partition of P into k cycles is $2n-k$. We choose one them and make it a 2-cycle in F_n by adding v to it. In the third case, let us assume that the number of vertices in C the cycle containing v, v itself excluded, is i. It is easy to see that $2 \leq i \leq n-k+1$, since we need at least $k-1$ vertices not in C in order to partition F_n into k cycles. It follows that the number of these partition is equal to the number of integer solutions to the Diophantine equation

$$x_1 + \cdots + x_k = n,$$

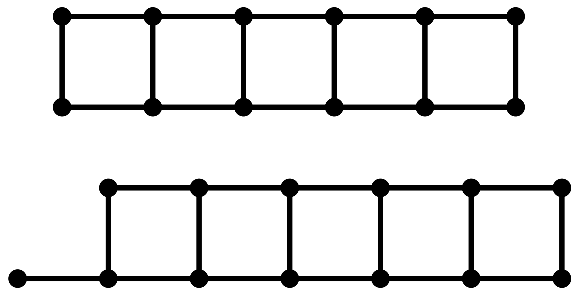

Figure 1: L_n and L_n^*

where $x_j \in \{1,2\}$ for all $1 \leq j \leq k$, with the exception of one them that needs to be equal to i. The reasoning is this: Starting from one of the leaves of P, say u, each x_j represents whether the j-th cycle from u along P is a 1-cycle or a 2-cycle, with the exception of one representing the number of vertices in C that belong to P. We will first choose which x_j is equal to i. Then by removing it from the above Diophantine equation, we are left with the equation

$$y_1 + \cdots + y_{k-1} = n - i,$$

where $y_j \in \{1,2\}$ for all $1 \leq j \leq k-1$. We know the number of integer solutions to this equation is $\binom{k-1}{n-i-k+1}$. Keeping in mind that C has two distinct orientations, we finish the proof. □

Corollary 2.2. *For $n \geq 2$, if $n \geq 2k$, then $\left[{F_n \atop k}\right] = k2^k$.* □

A combinatorial argument, similar to the one we have for Corollary 2.1, can be made for this corollary as well. We leave this to the reader.

3 Stirling Numbers of the First Kind for Ladders

In this section, we will find the ordinary generating functions for the Stirling numbers of the first kind for three different families of ladders. The first ladder that we will consider is $L_n = P_2 \square P_n$, where $n \in \mathbb{N}$. We assume that $\left[{L_0 \atop k}\right] = [k = 0]$. Also, it is clear that for $n \in \mathbb{N}$, $\left[{L_n \atop 0}\right] = 0$, and for $k > 2n$, $\left[{L_n \atop k}\right] = 0$. Now let $l(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[{L_n \atop k}\right] y^k x^n$.

Theorem 3.1. *For $n \in \mathbb{N}$, let $L_n = P_2 \square P_n$. With the assumption that $\left[{L_0 \atop k}\right] = [k = 0]$, the ordinary generating function for $\left[{L_n \atop k}\right]$ is*

$$l(x,y) = \frac{(1-yx)(1-x)}{1-(1+y)^2 x - (-1+y)y^2 x^2 + 2y^2(1+y)x^3 - y^3 x^4}.$$

Proof. In order to find the ordinary generating function of $\left[{L_n \atop k}\right]$, we need to consider the family of graphs L_n^* obtained from L_n by linking a new vertex to one of the vertices on one of the boundary copies of P_2 in L_n (see Figure 1). We assume that $L_0^* = P_1$ and let $l^*(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}L_n^*\\k\end{bmatrix}y^kx^n$.

Suppose v is one the vertices in one of the boundary copies of P_n in L_n. By considering the cases where v is in a 1-cycle, a 2-cycle, or a cycle of length 3 or higher, for $n \geq 3$ and $k \geq 2$, we have the recurrence

$$\begin{bmatrix}L_n\\k\end{bmatrix} = \begin{bmatrix}L_{n-1}^*\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-1}\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-2}^*\\k-2\end{bmatrix} + \begin{bmatrix}L_{n-2}\\k-2\end{bmatrix} + 2\sum_{i=1}^{n-1}\begin{bmatrix}L_{n-1-i}\\k-1\end{bmatrix}, \tag{2}$$

and consequently,

$$\begin{bmatrix}L_{n-1}\\k\end{bmatrix} = \begin{bmatrix}L_{n-2}^*\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-2}\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-3}^*\\k-2\end{bmatrix} + \begin{bmatrix}L_{n-3}\\k-2\end{bmatrix} + 2\sum_{i=1}^{n-2}\begin{bmatrix}L_{n-2-i}\\k-1\end{bmatrix}. \tag{3}$$

Note that the above equation holds when $n = 3$ and $k \geq 2$. By subtracting (3) from (2), we have

$$\begin{aligned}\begin{bmatrix}L_n\\k\end{bmatrix} - \begin{bmatrix}L_{n-1}\\k\end{bmatrix} = &\begin{bmatrix}L_{n-1}^*\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-1}\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-2}^*\\k-2\end{bmatrix} + \begin{bmatrix}L_{n-2}\\k-2\end{bmatrix}\\ &+ \begin{bmatrix}L_{n-2}\\k-1\end{bmatrix} - \begin{bmatrix}L_{n-2}^*\\k-1\end{bmatrix} - \begin{bmatrix}L_{n-3}^*\\k-2\end{bmatrix} - \begin{bmatrix}L_{n-3}\\k-2\end{bmatrix}.\end{aligned} \tag{4}$$

On the other hand, for $n \geq 1$ and $k \geq 1$, we have

$$\begin{bmatrix}L_n^*\\k\end{bmatrix} = \begin{bmatrix}L_n\\k-1\end{bmatrix} + \begin{bmatrix}L_{n-1}^*\\k-1\end{bmatrix}. \tag{5}$$

Noting that $\begin{bmatrix}L_n\\1\end{bmatrix} = 2$ for $n \geq 2$, $\begin{bmatrix}L_n^*\\1\end{bmatrix} = 0$ for $n \geq 1$,

$$l(x,y) = 1 + yx + y^2x + 2yx^2 + 2y^2x^2 + 4y^3x^2 + y^4x^2 + \sum_{n=3}^{\infty}\sum_{k=1}^{\infty}\begin{bmatrix}L_n\\k\end{bmatrix}y^kx^n,$$

and

$$l^*(x,y) = y + 2y^2x + y^3x + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\begin{bmatrix}L_n^*\\k\end{bmatrix}y^kx^n,$$

and by using (4) and (5), we have the following matrix equation

$$\begin{bmatrix} 1-x-yx-yx^2-y^2x^2+y^2x^3 & -yx+yx^2-y^2x^2+y^2x^3 \\ -y & 1-yx \end{bmatrix}\begin{bmatrix} l(x,y) \\ l^*(x,y)\end{bmatrix} = \begin{bmatrix} 1-x \\ 0 \end{bmatrix}.$$

By solving this matrix equation, we have

$$\begin{bmatrix} l(x,y) \\ l^*(x,y)\end{bmatrix} = \frac{\begin{bmatrix} 1-yx & yx-yx^2+y^2x^2-y^2x^3 \\ y & 1-x-yx-yx^2-y^2x^2+y^2x^3 \end{bmatrix}\begin{bmatrix} 1-x \\ 0 \end{bmatrix}}{1-(1+y)^2x-(-1+y)y^2x^2+2y^2(1+y)x^3-y^3x^4},$$

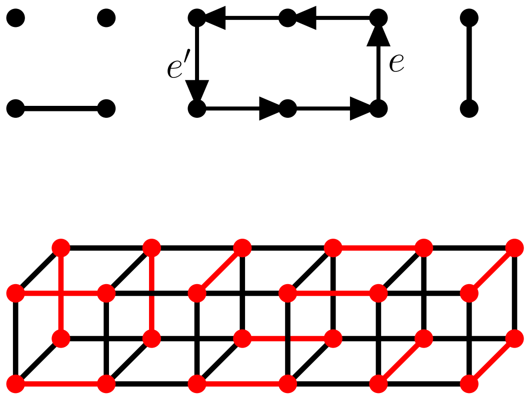

Figure 2: A cyclic partition of L_6 and the associated domino tiling of $P_2 \square L_6$

which gives us

$$l^*(x, y) = \frac{y(1 - x)}{1 - (1 + y)^2 x - (-1 + y)y^2 x^2 + 2y^2(1 + y)x^3 - y^3 x^4}$$

and

$$l(x, y) = \frac{(1 - yx)(1 - x)}{1 - (1 + y)^2 x - (-1 + y)y^2 x^2 + 2y^2(1 + y)x^3 - y^3 x^4}. \qquad \square$$

Corollary 3.1. *For $n \in \mathbb{N}$, let $L_n = P_2 \square P_n$. With the assumption that $L_0! = 1$, the ordinary generating function for $L_n!$ is*

$$\hat{l}(x) = l(x, 1) = \frac{1 - x}{1 - 3x - 3x^2 + x^3}. \qquad \square$$

According to the Online Encyclopedia of Integer Sequences, the sequence $L_n!$ is the number of perfect matchings (or domino tilings) of $C_4 \square P_n = P_2 \square L_n$ [9]. (For more on domino tilings of $P_2 \square P_n \square P_{2m}$, where $n, m \in \mathbb{N}$, see [1].) Here is how the bijection between the graph factorial of L_n and the perfect matchings of $P_2 \square L_n$ works: 1) When a vertex v is a 1-cycle in L_n, in the associated domino tiling we place a vertical domino between the two copies of v in $C_4 \square P_n$. 2) When u and v form a 2-cycle, in the associated domino tiling, we place two horizontal dominoes between the two copies of u and v in each copy of L_n. 3) We know that L_n does not contain any odd cycles other than the 1-cycles, since it is bipartite. Let C be an even cycle of length four or higher. Since L_n has only two copies of P_n, there are two edges in C that each link a vertex in one copy of P_n in L_n to a vertex in the other copy of L_n. Let e be the edge on the right and e' the one on the left. If C is traversed clockwise in L_n, then we put a domino on the upper copy of L_n in $C_4 \square P_n$ along e. Then we place two dominoes in the lower copy of L_n, perpendicular to the first domino and to its left, so that the projections of these three dominoes

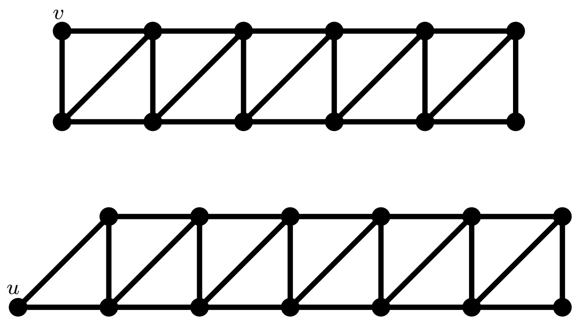

Figure 3: T_{2n} and T_{2n+1}

into the plane $\mathcal{P}$ containing one of the copies of L_n give us three consecutive edges in C. Now we will go to the upper copy and place two more parallel dominoes directly to the left of the two previously-placed dominoes so that their projections into $\mathcal{P}$ overlap at two vertices. We will continue in this fashion until the projections of the placed dominoes into $\mathcal{P}$ reaches e'. (See Figure 2.) On whichever copy of L_n the last two dominoes are placed, we will place a domino along e' in the other copy. If C is traversed counter-clockwise, we will do the same process, but will place the first domino on the lower copy.

The second family of ladders we are considering is defined as follows: For $n \in \mathbb{N}$, we define the triangular ladder on $2n$ vertices, denoted by T_{2n}, to be the graph obtained from L_n by adding parallel diagonals in each face of L_n. In order to find $\left[{T_{2n} \atop k}\right]$, we need to define the triangular ladder graphs on $2n+1$ vertices, denoted by T_{2n+1}: these graphs are obtained from T_{2n} by adding a vertex and linking it to the two adjacent boundary vertices in T_{2n}. We call this vertex the corner vertex of T_{2n+1}. (See Figure 3.) As we will demonstrate below, regardless of whether the number of vertices is even or odd, the number of cycle decompositions of these families of graphs satisfy the same recurrence relation. Because of this, we will combine to two families and find the ordinary generating function for this new family, $t(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left[{T_n \atop k}\right]y^k x^n$. We assume that $\left[{T_0 \atop k}\right] = [k=0]$. It is easy to see that for $n \in \mathbb{N}$, $\left[{T_n \atop 0}\right] = 0$, and for $k > n$, $\left[{T_n \atop k}\right] = 0$.

Theorem 3.2. *For $n \in \mathbb{N}$, let T_n be the triangular ladder graph on n vertices. With the assumption that $\left[{T_0 \atop k}\right] = [k=0]$, the ordinary generating function for $\left[{T_n \atop k}\right]$ is*

$$t(x,y) = \frac{1-x}{1-(1+y)x - y(1+y)x^3 + y^2x^5}.$$

Proof. Suppose v is a degree two vertex in one of the boundary copies of P_n in T_{2n} (see Figure 3). By considering the cases where v is in a 1-cycle, a 2-cycle, a 3-cycle, or a cycle of length 4 or higher, for $n \geq 2$ and $k \geq 2$, we have the recurrence

$$\begin{bmatrix} T_{2n} \\ k \end{bmatrix} = \begin{bmatrix} T_{2n-1} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{2n-2} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{2n-3} \\ k-2 \end{bmatrix} + \begin{bmatrix} T_{2n-4} \\ k-2 \end{bmatrix} + 2\sum_{i=1}^{2n-3} \begin{bmatrix} T_i \\ k-1 \end{bmatrix}.$$

On the other hand, when u is the corner vertex in T_{2n+1} (see Figure 3), the cases where u is in a 1-cycle, a 2-cycle, a 3-cycle, or a cycle of length 4 or higher, for $n \geq 2$ and $k \geq 2$, we have

$$\begin{bmatrix} T_{2n+1} \\ k \end{bmatrix} = \begin{bmatrix} T_{2n} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{2n-1} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{2n-2} \\ k-2 \end{bmatrix} + \begin{bmatrix} T_{2n-3} \\ k-2 \end{bmatrix} + 2\sum_{i=1}^{2n-2} \begin{bmatrix} T_i \\ k-1 \end{bmatrix}.$$

Note that regardless of whether the number of vertices is odd or even, for $n \geq 4$ and $k \geq 2$, the number of cyclic partitions of this family of graphs into k cycles satisfies the recurrence relation

$$\begin{bmatrix} T_n \\ k \end{bmatrix} = \begin{bmatrix} T_{n-1} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{n-2} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{n-3} \\ k-2 \end{bmatrix} + \begin{bmatrix} T_{n-4} \\ k-2 \end{bmatrix} + 2\sum_{i=1}^{n-3} \begin{bmatrix} T_i \\ k-1 \end{bmatrix}. \tag{6}$$

It follows that

$$\begin{bmatrix} T_{n-1} \\ k \end{bmatrix} = \begin{bmatrix} T_{n-2} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{n-3} \\ k-1 \end{bmatrix} + \begin{bmatrix} T_{n-4} \\ k-2 \end{bmatrix} + \begin{bmatrix} T_{n-5} \\ k-2 \end{bmatrix} + 2\sum_{i=1}^{n-4} \begin{bmatrix} T_i \\ k-1 \end{bmatrix}. \tag{7}$$

By subtracting equation (7) from equation (6), for $n \geq 5$ and $k \geq 2$, we have

$$\begin{bmatrix} T_n \\ k \end{bmatrix} - \begin{bmatrix} T_{n-1} \\ k \end{bmatrix} - \begin{bmatrix} T_{n-1} \\ k-1 \end{bmatrix} - \begin{bmatrix} T_{n-3} \\ k-1 \end{bmatrix} - \begin{bmatrix} T_{n-3} \\ k-2 \end{bmatrix} + \begin{bmatrix} T_{n-5} \\ k-2 \end{bmatrix} = 0. \tag{8}$$

Noting that $\left[\begin{smallmatrix} T_n \\ 1 \end{smallmatrix}\right] = 2$ for $n \geq 2$, and

$$\begin{aligned} t(x,y) \;&=\; 1 + yx + yx^2 + y^2x^2 + 2yx^3 + 3y^2x^3 + y^3x^3 + 2yx^4 + 6y^2x^4 \\ &\quad + 5y^3x^4 + y^4x^4 + \sum_{n=5}^{\infty}\sum_{k=1}^{\infty} \begin{bmatrix} T_n \\ k \end{bmatrix} y^k x^n, \end{aligned}$$

by using (8), we have

$$t(x,y) = \frac{1-x}{1-(1+y)x - y(1+y)x^3 + y^2x^5}. \qquad \square$$

Corollary 3.2. *For $n \in \mathbb{N}$, let T_n be the triangular ladder graph on n vertices. With the assumption that $T_0! = 1$, the ordinary generating function for $T_n!$ is*

$$\hat{t}(x) = t(x,1) = \frac{1-x}{1-2x-2x^3+x^5}. \qquad \square$$

According to the Online Encyclopedia of Integer Sequences, the sequence $T_n!$ is the number permutations of length n within distance 2 of a fixed permutation [10]. Without loss of generality, we may assume that the fixed permutation used is the identity permutation $1\,2\,\cdots\,n$. If π is a permutation on $[n]$, then its distance from the identity permutation is defined as $\max_{i\in[n]} |p_i - i|$. In general, $V(d,n)$ denotes

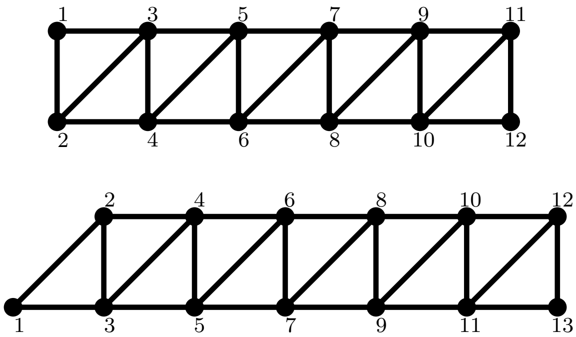

Figure 4: Labeling the vertices of T_{12} and T_{13}

the number of permutations of length n with distance d of a fixed permutation [8]. Here we are dealing with $V(2,n)$.

Here is how the bijection between the number of cycle decomposition of T_n and $V(2,n)$ works: When n is even, label the boundary vertex of degree two in T_n with 1 and then, alternating between the two copies of P_n, label the rest of the vertices with numbers 2 through n. When n is odd, label the corner vertex of T_n with 1 and then, by alternating between the two copies of P_n, label the rest of the vertices with numbers 2 through n such that 2 is a degree three vertex while 3 is a degree four vertex. In either case, any label is only adjacent to labels that are at distance at most two from it. In a cyclic partition of this graph, if a label is in a 1-cycle, then in the bijection, that label is a fixed point of the associated permutation. If two labels form a 2-cycle, then in the bijection, these two labels are switched in the associated permutation. Finally, if $l_1, l_2, \ldots, l_k$ is a cycle of length three or higher, then in the associated permutation, l_1 is in l_2's position, l_2 is in l_3's position, etc. It is not hard to see that because of how the labeling is constructed, the distance of the resulting permutation from the identity permutation is at most two. (See Figure 4.)

Now we will find the Stirling number of the first kind of the strong ladder graph $S_n = P_2 \boxtimes P_n$. We assume that $\left[{S_0 \atop k}\right] = [k=0]$. Also, it is easy to see that for $n \geq 1$, $\left[{S_n \atop 0}\right] = 0$, and for $k > 2n$, $\left[{S_n \atop k}\right] = 0$. Now let $s(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[{S_n \atop k}\right] y^k x^n$.

Theorem 3.3. *For $n \in \mathbb{N}$, let $S_n = P_2 \boxtimes P_n$, where $\boxtimes$ is the strong graph product. With the assumption that $\left[{S_0 \atop k}\right] = [k=0]$, the ordinary generating function for $\left[{S_n \atop k}\right]$ is*

$$s(x,y) = \frac{1 - 2x - 2yx}{1 - (2+3y+y^2)x - 2y(2+3y+y^2)x^2 + 4y(1+y)^2x^3}.$$

Proof. In order to find the ordinary generating function of $\left[{S_n \atop k}\right]$, we need to consider the family of graphs S_n^* obtained from S_n by adding a vertex and linking it to the two vertices in one of the boundary copies of P_n in S_n. Let $s^*(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[{S_n^* \atop k}\right] y^k x^n$ be the ordinary generating function for the Stirling numbers of the first kind of the members of this family of graphs. To make computing $s(x,y)$ and $s^*(x,y)$ easier, we

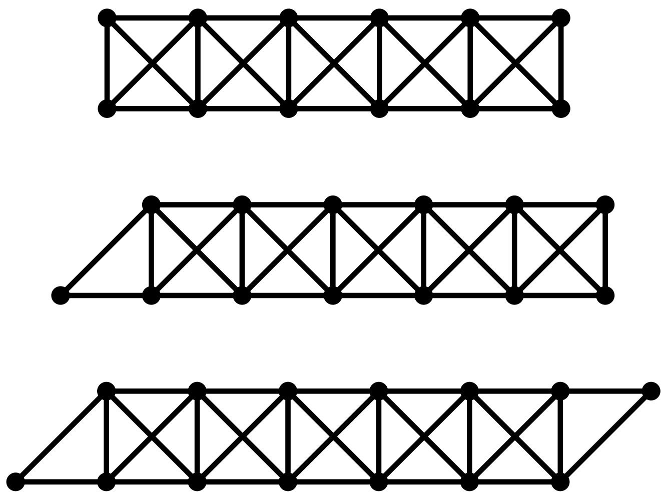

Figure 5: S_n, S_n^*, and $S_n^\diamond$

will denote S_n and S_n^* by $\mathcal{S}_{2n}$ and $\mathcal{S}_{2n+1}$, respectively, based the number of vertices in each graph. Hence,

$$s(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\mathcal{S}_{2n}\\ k\end{bmatrix}y^k x^n$$

and

$$s^*(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}\mathcal{S}_{2n+1}\\ k\end{bmatrix}y^k x^n.$$

Suppose v is one the vertices in one of the boundary copies of P_n in $\mathcal{S}_{2n}$. By considering the cases where v is in a 1-cycle, a 2-cycle, a 3-cycle, or a cycle of length 4 or higher, for $n \geq 3$ and $k \geq 2$, we have the recurrence

$$\begin{aligned}\begin{bmatrix}\mathcal{S}_{2n}\\ k\end{bmatrix} = \begin{bmatrix}\mathcal{S}_{2n-1}\\ k-1\end{bmatrix} + \begin{bmatrix}\mathcal{S}_{2n-2}\\ k-1\end{bmatrix} + 4\begin{bmatrix}\mathcal{S}_{2n-3}\\ k-1\end{bmatrix} + 2\begin{bmatrix}\mathcal{S}_{2n-3}\\ k-2\end{bmatrix} + 2\begin{bmatrix}\mathcal{S}_{2n-4}\\ k-1\end{bmatrix} + 2\begin{bmatrix}\mathcal{S}_{2n-4}\\ k-2\end{bmatrix}\\ + 4\sum_{i=0}^{n-2}2^i\begin{bmatrix}\mathcal{S}_{2n-4-2i}\\ k-1\end{bmatrix} + 2\sum_{i=0}^{n-2}2^i\begin{bmatrix}\mathcal{S}_{2n-4-2i}\\ k-2\end{bmatrix} + 8\sum_{i=0}^{n-3}2^i\begin{bmatrix}\mathcal{S}_{2n-5-2i}\\ k-1\end{bmatrix}\\ + 4\sum_{i=0}^{n-3}2^i\begin{bmatrix}\mathcal{S}_{2n-5-2i}\\ k-2\end{bmatrix}.\end{aligned} \tag{9}$$

In the above sums, the reason for 2^i in each term is that cycles of length 4 or higher can either use two parallel or two crossing edges in each $\boxtimes$ of this strong product. On the other hand, for $n \geq 2$ and $k \geq 1$, we have

$$\begin{bmatrix}\mathcal{S}_{2n+1}\\ k\end{bmatrix} = \begin{bmatrix}\mathcal{S}_{2n}\\ k-1\end{bmatrix} + 2\begin{bmatrix}\mathcal{S}_{2n-1}\\ k-1\end{bmatrix} + 2\sum_{i=0}^{n-1}2^i\begin{bmatrix}\mathcal{S}_{2n-2-2i}\\ k-1\end{bmatrix} + 4\sum_{i=0}^{n-2}2^i\begin{bmatrix}\mathcal{S}_{2n-3-2i}\\ k-1\end{bmatrix}. \tag{10}$$

Consequently, for $n \geq 3$ and $k \geq 2$, we have

$$\begin{bmatrix} \mathcal{S}_{2n-1} \\ k \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{2n-2} \\ k-1 \end{bmatrix} + 2\begin{bmatrix} \mathcal{S}_{2n-3} \\ k-1 \end{bmatrix} + 2\sum_{i=0}^{n-2} 2^i \begin{bmatrix} \mathcal{S}_{2n-4-2i} \\ k-1 \end{bmatrix} + 4\sum_{i=0}^{n-3} 2^i \begin{bmatrix} \mathcal{S}_{2n-5-2i} \\ k-1 \end{bmatrix} \quad (11)$$

and

$$\begin{bmatrix} \mathcal{S}_{2n-1} \\ k-1 \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{2n-2} \\ k-2 \end{bmatrix} + 2\begin{bmatrix} \mathcal{S}_{2n-3} \\ k-2 \end{bmatrix} + 2\sum_{i=0}^{n-2} 2^i \begin{bmatrix} \mathcal{S}_{2n-4-2i} \\ k-2 \end{bmatrix} + 4\sum_{i=0}^{n-3} 2^i \begin{bmatrix} \mathcal{S}_{2n-5-2i} \\ k-2 \end{bmatrix}. \quad (12)$$

By subtracting the sum of (12) and two times (11) from (9), for $n \geq 3$ and $k \geq 2$, we have

$$\begin{bmatrix} \mathcal{S}_{2n} \\ k \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-1} \\ k \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-1} \\ k-1 \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{2n-2} \\ k-1 \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{2n-2} \\ k-2 \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-4} \\ k-1 \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-4} \\ k-2 \end{bmatrix} = 0. \quad (13)$$

On the other hand, by subtracting two times (11) from (10), for $n \geq 3$ and $k \geq 2$, we have

$$\begin{bmatrix} \mathcal{S}_{2n+1} \\ k \end{bmatrix} - \begin{bmatrix} \mathcal{S}_{2n} \\ k-1 \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-1} \\ k \end{bmatrix} - 2\begin{bmatrix} \mathcal{S}_{2n-1} \\ k-1 \end{bmatrix} = 0. \quad (14)$$

Noting that $\left[\begin{smallmatrix} \mathcal{S}_{2n} \\ 1 \end{smallmatrix}\right] = 2^n$ for $n \geq 3$, $\left[\begin{smallmatrix} \mathcal{S}_{2n+1} \\ 1 \end{smallmatrix}\right] = 2^n$ for $n \geq 1$,

$$s(x,y) = 1 + yx + y^2x + 6yx^2 + 11y^2x^2 + 6y^3x^2 + y^4x^2 + \sum_{n=3}^{\infty}\sum_{k=1}^{\infty} \begin{bmatrix} \mathcal{S}_{2n} \\ k \end{bmatrix} y^k x^n,$$

and $s^*(x,y) =$

$$y + 2yx + 3y^2x + y^3x + 4yx^2 + 16y^2x^2 + 19y^3x^2 + 9y^4x^2 + y^5x^2 + \sum_{n=3}^{\infty}\sum_{k=1}^{\infty} \begin{bmatrix} \mathcal{S}_{2n+1} \\ k \end{bmatrix} y^k x^n,$$

by using (13) and (14), we have the following matrix equation

$$\begin{bmatrix} 1 + yx + y^2x - 2yx^2 - 2y^2x^2 & -2x - 2yx \\ -y & 1 - 2x - 2yx \end{bmatrix} \begin{bmatrix} s(x,y) \\ s^*(x,y) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By solving this matrix equation, we have

$$\begin{bmatrix} s(x,y) \\ s^*(x,y) \end{bmatrix} = \frac{\begin{bmatrix} 1 - 2x - 2yx & 2x + 2yx \\ y & 1 + yx + y^2x - 2yx^2 - 2y^2x^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1 - (2 + 3y + y^2)x - 2y(2 + 3y + y^2)x^2 + 4y(1+y)^2x^3},$$

which gives us

$$s^*(x,y) = \frac{y}{1 - (2 + 3y + y^2)x - 2y(2 + 3y + y^2)x^2 + 4y(1+y)^2x^3}$$

and

$$s(x,y) = \frac{1 - 2x - 2yx}{1 - (2 + 3y + y^2)x - 2y(2 + 3y + y^2)x^2 + 4y(1+y)^2x^3}. \qquad \square$$

Corollary 3.3. *For $n \in \mathbb{N}$, let $S_n = P_2 \boxtimes P_n$, where $\boxtimes$ is the strong graph product. With the assumption that $S_0! = 1$, the ordinary generating functions for $S_n^*!$ and $S_n!$ are*

$$\hat{s}^*(x) = s^*(x,1) = \frac{1}{1-6x-12x^2+16x^3}$$

and

$$\hat{s}(x) = s(x,1) = \frac{1-4x}{1-6x-12x^2+16x^3},$$

respectively. □

According to the Online Encyclopedia of Integer Sequences, the sequence $S_n!$ is the number of $2 \times n$ array permutations with each element making zero or one king moves [6].

Finally, we will find the factorial of the family $S_n^\diamond$ obtained from S_{n-1} by adding two vertices to this graph and then linking one of the new vertices to the two adjacent vertices in a boundary copy of P_2 in S_{n-1} and linking the other new vertex to the other two. (See Figure 5.) We will call this graph on $2n$ vertices, the strong diamond ladder. We will omit the following theorem's proof as it is very similar to that of Theorem 3.3.

Theorem 3.4. *For $n \in \mathbb{N}$, let $S_n^\diamond$ be the strong diamond graph on $2n$ vertices. With the assumption that $\left[{S_0^\diamond \atop k}\right] = [k=0]$ and $\left[{S_1^\diamond \atop k}\right] = [k=1]$, the ordinary generating function for $\left[{S_n^\diamond \atop k}\right]$ is $d(x,y) = P(x,y)/Q(x,y)$, where*

$$\begin{aligned} P(x,y) = 1 &- (4+3y+y^2)x + (4+2y+3y^3+y^4)x^2 - 2y(-6-8y-y^2+y^3)x^3 \\ &- 4y(1+y)^2(2+3y^2+y^3)x^4 + 8y^3(1+y)^3x^5 \end{aligned}$$

and

$$Q(x,y) = (1-2x-2yx)(1-(2+3y+y^2)x - 2y(2+3y+y^2)x^2 + 4y(1+y)^2x^3).$$ □

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