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Sufficient conditions for graphs to be maximally 4-restricted edge connected*

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Abstract

For a subset S of edges in a connected graph G, the set S is a k-restricted edge cut if $G-S$ is disconnected and every component of $G-S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A connected graph G is said to be λ_k-connected if G has a k-restricted edge cut. Let $\xi_k(G) = \min\{|[X,\bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$, where $\bar{X} = V(G)\backslash X$. A graph G is said to be maximally k-restricted edge connected if $\lambda_k(G) = \xi_k(G)$. In this paper we show that if G is a λ_4-connected graph with $\lambda_4(G) \leq \xi_4(G)$ and the girth satisfies $g(G) \geq 8$, and there do not exist six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$, $(1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge connected.

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1 Terminology and introduction

We consider finite, undirected and simple graphs. For graph-theoretical terminology and notation not defined here we follow [5]. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Given a nonempty vertex subset V' of V, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. For two disjoint vertex sets X and Y of V, let $[X, Y]$ be the set of edges with one endpoint in X and the other one in Y. The order of G is the number of vertices in G. The degree of a vertex v in G, denoted by $d_G(v)$, is the number of edges of G incident with v. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. A (v_0, v_k)-path, denoted by $P = v_0v_1 \dots v_k$, is a sequence of adjacent vertices where all the vertices are distinct. Likewise, a cycle is a path that begins and ends with the same vertex. The length of a path or a cycle is the number of edges contained in the path or cycle. The distance between two vertices x and y is, denoted by $d(x, y)$, the length of a shortest path between x and y in G. The girth $g = g(G)$ is the length of a shortest cycle in G.

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. A classical measurement of the fault tolerance of a network is the edge connectivity $\lambda(G)$. The edge connectivity $\lambda(G)$ of a connected graph G is the minimum cardinality of an edge cut of G. As a more refined index than the edge connectivity, Fàbrega and Fiol [10] proposed the more general concept of the k-restricted edge connectivity of G as follows.

Definition 1.1 *[10] For a subset S of edges in a connected graph G, S is a k-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A minimum k-restricted edge cut is called a λ_k-cut. A connected graph G is said to be λ_k-connected if G has a k-restricted edge cut.*

There is a significant amount of research on k-restricted edge connectivity [2, 4, 7–11, 13, 18–21, 27]. In view of recent studies on k-restricted edge connectivity, it seems that the larger $\lambda_k(G)$ is, the more reliable the network G is [3, 14, 22]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first and so the optimization of k-restricted edge connectivity draws a lot of attention. For any positive integer k, let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$, where $\bar{X} = V(G) \backslash X$. It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [1, 6, 12, 15, 28].

Let $G_1, \dots, G_n$ be n copies of K_t. Add a new vertex u and let u be adjacent to every vertex in $V(G_i)$, $i = 1, \dots, n$. The resulting graph is denoted by $G^*_{n,t}$. It can be verified that $G^*_{n,t}$ has no $(\delta(G^*_{n,t}) + 1)$-restricted edge cuts and $G^*_{n,t}$ is the only exception for the existence of k-restricted edge cuts of a connected graph G when $k \leq \delta(G) + 1$.

Theorem 1.2 *[28]. Let G be a connected graph with order at least $2(\delta(G)+1)$ which is not isomorphic to any $G^*_{n,t}$ with $t = \delta(G)$. Then for any $k \le \delta(G) + 1$, G has k-restricted edge cuts and $\lambda_k(G) \le \xi_k(G)$.*

A λ_k-connected graph G is said to be maximally k-restricted edge connected if $\lambda_k(G) = \xi_k(G)$. When $k = 2$, the k-restricted edge connectivity of G is the restricted edge connectivity of G; a maximally k-restricted edge connected graph is a maximally restricted edge connected graph. There has been much research on maximally restricted edge connected graphs. See [13,17,22–24]. Let G be a λ_k-connected graph and let S be a λ_k-cut of G.

In 1989, Plesník and Znám [16] gave the following sufficient condition for a graph to be maximally edge connected.

Theorem 1.3 *[16] Let G be a connected graph. If there do not exist four vertices u_1, u_2, v_1, v_2 in G such that the distance $d(u_i, v_j) \ge 3$ $(1 \le i, j \le 2)$, then G is maximally edge connected.*

In 2013, Qin et al. [17] gave the following theorem.

Theorem 1.4 *[17] Let G be a λ_2-connected graph with the girth $g(G) \ge 4$. If there are not four vertices u_1, u_2, v_1, v_2 in G such that the distance $d(u_i, v_j) \ge 3$ $(1 \le i, j \le 2)$, then G is maximally restricted edge connected.*

In 2015, Wang et al. [25] gave the following theorem.

Theorem 1.5 *[25] Let G be a λ_3-connected graph with the girth $g(G) \ge 5$. If there are not five vertices $u_1, u_2, v_1, v_2, , v_3$ in G such that the distance $d(u_i, v_j) \ge 3$ $(1 \le i \le 2; 1 \le j \le 3)$, then G is maximally 3-restricted edge connected.*

In this article, we extend the above result to λ_4-connected graphs.

2 Main results

We first give an existing result.

Lemma 2.1 *[21] Let G be a λ_k-connected graph with $\lambda_k(G) \le \xi_k(G)$ and let $S = [X, Y]$ be a λ_k-cut of G. If there exists a connected subgraph H of order k in $G[X]$ with the property that*

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| \le \sum_{v \in X \setminus V(H)} |N(v) \cap Y|,$$

then G is maximally k-restricted edge connected.

Theorem 2.2 *Let G be a λ_4-connected graph with $\lambda_4(G) \le \xi_4(G)$ and let the girth $g(G) \ge 8$. If there are not six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \ge 3$ $(1 \le i, j \le 3)$, then G is maximally 4-restricted edge connected.*

Proof: We suppose, on the contrary, that G is not maximally 4-restricted edge connected. Let $S = [X, Y]$ be a λ_4-cut of G. Denote $X_1 = \{x \in X : N(x) \cap Y \neq \emptyset\}$ and $Y_1 = \{y \in Y : N(y) \cap X \neq \emptyset\}$. Let $X_0 = X \setminus X_1$, $Y_0 = Y \setminus Y_1$, and let $m_0 = |X_0|$, $m_1 = |X_1|$, $n_0 = |Y_0|$ and $n_1 = |Y_1|$. If $|X| = 4$ or $|Y| = 4$, then $\lambda_4(G) \leq \xi_4(G) \leq |S| = \lambda_4(G)$, i.e., G is maximally 4-restricted edge connected, a contradiction. Therefore $|X| \geq 5$ and $|Y| \geq 5$.

Claim 1. $m_0 \geq 2$ and $n_0 \geq 2$.

By contradiction. Without loss of generality, assume $m_0 \leq 1$. Let $m_0 = 0$. By [26], there is a connected subgraph H of order 4 such that $X_0 \subseteq V(H)$ in $G[X]$. Let $m_0 = 1$ and $X_0 = \{x\}$. Since $G[X]$ is connected, there is a spanning tree T in $G[X]$. Therefore $x \in V(T)$. Since T has two vertices of degree 1, there is a vertex v of degree 1 such that $v \neq x$. Then $T - v$ is a tree and $x \in V(T - v)$. Since there is a vertex v_2 of degree 1 such that $v_2 \neq x$, $T - v - v_2$ is a tree and $x \in V(T - v - v_2)$. Continuing this process, we can obtain a tree T' of order 4 such that $x \in V(T')$. Let $H = (G[X])[V(T')]$. Therefore, in $G[X]$, there is a connected subgraph H of order 4 such that $X_0 \subseteq V(H)$. Let $u \in X\backslash V(H)$. Then $|[\{u\}, Y]| \geq 1$. Since $|V(T')| = 4$, the maximum cardinality of paths is less than or equal to 3. Since $g(G) \geq 8$, $|[\{u\}, V(H)]| \leq 1$ holds. Therefore, we have that

$$\begin{aligned} \sum_{u \in X\backslash V(H)} |N(u) \cap V(H)| &= |[X\backslash V(H), V(H)]| \\ &\leq |X\backslash V(H)| \\ &\leq |[X\backslash V(H), Y]| \\ &= \sum_{u \in X\backslash V(H)} |N(u) \cap Y|. \end{aligned} \tag{2.1}$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Therefore $m_0 \geq 2$. Similarly, we have $n_0 \geq 2$. The proof of Claim 1 is complete.

Claim 2. $m_0 = 2$ or $n_0 = 2$.

Suppose that $m_0 \geq 3$ and $n_0 \geq 3$. Then there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that $u_1, u_2, u_3 \in X_0$ and $v_1, v_2, v_3 \in Y_0$. By the definition of X_0 and Y_0, we have that $|N(u_i) \cap Y| = 0 = |N(v_j) \cap X|$ for $1 \leq i \leq 3; 1 \leq j \leq 3$. It follows that $d(u_i, v_j) \geq 3 \quad (i, j \in \{1, 2, 3\})$, a contradiction. Combining this with Claim 1, we have that $m_0 = 2$ or $n_0 = 2$. The proof of Claim 2 is complete.

Claim 3. In $G[X]$, let H be a connected subgraph of order 4 such that it contains X_0 as most as possible and let $V(H) = \{x_1, x_2, x_3, x_4\}$. If $X_0 = \{u_1, u_2\}$, then
(1) $|X_0 \cap V(H)| = 1$;
(2) $H = u_1x_2x_3x_4$ is a path of length 3, where $u_1 = x_1$, if $u_1 \in V(H)$; and $u_1x_2x_3x_4u_2$ is a path of length 4 in $G[X]$;
(3) $(N(u_1) \cap X)\backslash V(H) = \emptyset$ and $(N(u_2) \cap X)\backslash V(H) = \emptyset$.

Since $|X_0| = 2$, $1 \leq |X_0 \cap V(H)| \leq 2$ holds. We consider the following two cases.

Case 1. $|X_0 \cap V(H)| = 2$.

Since $g(G) \geq 8$, $|[\{u\}, V(H)]| \leq 1$ for $u \in X \backslash V(H)$. Note that $X_0 = \{u_1, u_2\} \subseteq V(H)$. Then $|[\{u\}, Y]| \geq 1$ for $u \in X \backslash V(H)$. By (2.1), we have that

$$\sum_{u \in X \backslash V(H)} |N(u) \cap V(H)| \leq \sum_{u \in X \backslash V(H)} |N(u) \cap Y|.$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction.

Case 2. $|X_0 \cap V(H)| = 1$.

In this case, suppose $u_1 \in V(H)$. Since $g(G) \geq 8$, H is a tree of order 4, and $|[\{u\}, V(H)]| \leq 1$ for $u \in X \backslash V(H)$. If $|N(u_2) \cap V(H)| = 0$, then $|[\{u\}, V(H)]| \leq |[\{u\}, Y]|$ for $u \in X \backslash V(H)$. Therefore, we have that

$$\sum_{u \in X \backslash V(H)} |N(u) \cap V(H)| \leq \sum_{u \in X \backslash V(H)} |N(u) \cap Y|.$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Then $|N(u_2) \cap V(H)| = 1$. Suppose that H is not a path. Then H has at least three vertices of degree 1. Let u_2 be adjacent to a vertex y of H. Then there is a vertex v of degree 1 such that $v \neq u_1$ and y in H. Therefore, $(G[X])[V(H - v) \cup \{u_2\}]$ is a connected graph of order 4, a contradiction to H. Then H is a path P of length 3. If u_1 is not a vertex of degree 1, then there is a connected subgraph of order 4 such that it contains u_1, u_2 in $G[V(H) \cup \{u_2\}]$, a contradiction to H. Therefore u_1 is a vertex of degree 1 in P. Let $P = u_1 x_2 x_3 x_4$. Suppose that $N(u_2) \cap V(H) = \emptyset$. Then $|[\{u\}, V(H)]| \leq |[\{u\}, Y]|$ for $u \in X \backslash V(H)$. Therefore, we have that

$$\sum_{u \in X \backslash V(H)} |N(u) \cap V(H)| \leq \sum_{u \in X \backslash V(H)} |N(u) \cap Y|.$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Therefore, $|N(u_2) \cap V(H)| = 1$. If $N(u_2) \cap \{x_2, x_3\} \neq \emptyset$, a contradiction to H. Then u_2 is adjacent to x_4.

Suppose, on the contrary, that $x \in (N(u_1) \cap X) \backslash V(H)$. Then $P' = x u_1 x_2 x_3$ is a path of length 3 in $G[X]$. Since $g(G) \geq 8$, $|N(u) \cap V(P')| \leq 1$ for $u \in X \backslash V(P')$. If $N(u_2) \cap V(P') \neq \emptyset$, then there is a connected subgraph H' of order 4 in $G[X]$ with $u_1, u_2 \in V(H')$, a contradiction to that $|X_0 \cap V(H)| = 1$. Therefore, we have that $|N(u_2) \cap V(P')| = 0$ and $|N(u) \cap V(P')| \leq |N(u) \cap Y|$ for $u \in X \backslash V(P')$. Thus,

$$\sum_{u \in X \backslash V(P')} |N(u) \cap V(P')| \leq \sum_{u \in X \backslash V(P')} |N(u) \cap Y|.$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. So $(N(u_1) \cap X) \backslash V(H) = \emptyset$ and $d(u_1) = 1$ in $G[X]$.

Suppose, on the contrary, that $x \in (N(u_2) \cap X) \backslash V(H)$. By Claim 3 (2), $P' = x_3 x_4 u_2 x$ is a path of length 3 in $G[X]$. Since $g(G) \geq 8$, $|N(u) \cap V(P')| \leq 1$ for $u \in X \backslash V(P')$.

Since $d(u_1) = 1$ in $G[X]$ and $u_1x_2 \in E(G[Y])$, we have $N(u_1) \cap V(P') = \emptyset$. Therefore, we have that $|N(u) \cap V(P')| \leq |N(u) \cap Y|$ for $u \in X \backslash V(P')$. Thus,

$$\sum_{u \in X \backslash V(P')} |N(u) \cap V(P')| \leq \sum_{u \in X \backslash V(P')} |N(u) \cap Y|.$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. So $(N(u_2) \cap X) \backslash V(H) = \emptyset$. The proof of Claim 3 is complete.

Similarly to Claim 3, we have that the following claim.

Claim 4. In $G[Y]$, let H^* be a connected subgraph of order 4 such that it contains Y_0 as most as possible and let $V(H^*) = \{y_1, y_2, y_3, y_4\}$. If $Y_0 = \{v_1, v_2\}$, then
(1) $|Y_0 \cap V(H^*)| = 1$;
(2) $H^* = v_1y_2y_3y_4$ is a path of length 3, where $v_1 = y_1$, if $v_1 \in V(H^*)$; and $v_1y_2y_3y_4v_2$ is a path of length 4 in $G[Y]$;
(3) $(N(v_1) \cap Y) \backslash V(H^*) = \emptyset$ and $(N(v_2) \cap Y) \backslash V(H^*) = \emptyset$.

Without loss of generality, suppose $m_0 = 2$. We consider the following cases.

Case 1. $n_0 = 2$.

Claim 5. $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| \leq 1$ in G (See Fig 1).

Suppose $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| \geq 2$. It is sufficient to show that $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 2$. Since $x_2x_3x_4$ and $y_2y_3y_4$ are paths, and $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 2$, we have that there is a cycle of G whose length is at most 6, a contradiction to $g(G) \geq 8$. The proof of Claim 5 is complete.

Suppose, first, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 1$ and $x_{i_0}y_{j_0} \in E(G)$ $(2 \leq i_0 \leq 4, 2 \leq j_0 \leq 4)$. Let $x_i \in \{2, 3, 4\} \backslash \{i_0\}$ with $x_ix_{i_0} \in E(H)$ and $y_j \in \{2, 3, 4\} \backslash \{j_0\}$ with $y_jy_{j_0} \in E(H^*)$. By Claim 5, $d(x_i, y_j) \neq 1$. If $d(x_i, y_j) = 2$, then there is a vertex y in $G[Y]$ such that $x_iy, yy_j \in E(G)$ or there is a vertex x in $G[X]$ such that $x_ix, xy_j \in E(G)$. Without loss of generality, suppose that there is a vertex y in $G[Y]$ such that $x_iy, yy_j \in E(G)$. Then there is a cycle C in G, and $x_{i_0}, y_{j_0}, x_i, y_j, y \in V(C)$ and the length of C is 5, a contradiction to $g(G) \geq 8$. Therefore, $d(x_i, y_j) \geq 3$. By Claim 4 (3), $d(x_i, v_i) \geq 3$ for $\{1, 2\}$. Similarly to the discussion on x_i, we have that $d(y_j, u_k) \geq 3$ for $k \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_j\}$, a contradiction.

Suppose, second, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 0$. Since there is no $d(x, y) \geq 3$ for every $x \in \{x_2, x_3, x_4\}$ and $y \in \{y_2, y_3, y_4\}$, there are two vertices $x_{i_0} \in \{x_2, x_3, x_4\}$ and $y_{j_0} \in \{y_2, y_3, y_4\}$ such that $d(x_{i_0}, y_{j_0}) = 2$. Let $i \in \{2, 3, 4\} \backslash \{i_0\}$ with $x_ix_{i_0} \in E(H)$ and $j \in \{2, 3, 4\} \backslash \{j_0\}$ with $y_jy_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i, y_j) \geq 3$ holds. By Claim 4 (3), $d(x_i, v_j) \geq 3$ for $j \in \{1, 2\}$. Similarly, $d(y_j, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_j\}$, a contradiction.

Case 2. $n_0 \geq 3$.

Let $Y_0 = \{y_0, v_1, v_2, v_3, \ldots\}$. By Claim 3 (2), we have that $H = u_1x_2x_3x_4$ and $u_1x_2x_3x_4u_2$ is a path in $G[X]$. Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for

$v \in Y \backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0 \backslash V(H^*)$, by Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0 \backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$.

Case 2.1. $|Y_0 \cap V(H^*)| = 1$.

Let $Y_0 \cap V(H^*) = \{v_1\}$. Note that H^* is a path of length 3 or a $K_{1,3}$. Similarly to the discussion on H, we have that $G[V(H^*) \cup \{y_0\}]$ is a path of length 4, denoted by $P_1 = y_1y_2y_3y_4y_5$, where $v_1 = y_1, y_5 = y_0$. Similarly to Case 1, there is a contradiction.

Case 2.2. $|Y_0 \cap V(H^*)| = 2$.

Let $Y_0 \cap V(H^*) = \{v_1, v_2\}$. Since H^* is a path of length 3 or a $K_{1,3}$, we have that $1 \leq d_{H^*}(v_1, v_2) \leq 3$.

Case 2.2.a. $d_{H^*}(v_1, v_2) = 3$.

In this case, H^* is a path of length 3, denoted by $H^* = y_1y_2y_3y_4$, where $v_1 = y_1, v_2 = y_4$. Similarly to the proof of Claim 5, we have the following claim.

Claim 6. $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| \leq 1$ in G (See Fig 2).

Suppose, first, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| = 1$ Without loss of generality, we consider the following cases.

Case 2.2.a.1. $x_2y_2 \in E(G)$.

In this case, $x_3x_2y_2y_3$ is a path in G. Since $g(G) \geq 8$ and Claim 6, $d(x_3, y_3) = 3$ holds. Assume $d(x_3, v_1) = 2$. Since $N(v_1) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_3y, yv_1 \in E(G)$. Thus, $x_3yv_1y_2x_2x_3$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3, v_1) = 3$. Similarly, $d(x_3, v_2) \geq 3$. By Claim 3, $d(y_3, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_3\}$ and $y \in \{v_1, v_2, y_3\}$, a contradiction.

Case 2.2.a.2. $x_3y_2 \in E(G)$.

In this case, $x_2x_3y_2y_3$ is a path in G. By Claim 6, $x_2y_3 \notin E(G)$. If $d(x_2, y_3) = 2$, then there is a vertex y in $G[Y]$ such that $x_2y, yy_3 \in E(G)$ or there is a vertex x in $G[X]$ such that $x_2x, xy_3 \in E(G)$. Without loss of generality, suppose that there is a vertex y in $G[Y]$ such that $x_2y, yy_3 \in E(G)$. Note that $x_3y_2y_3yx_2x_3$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, y_3) = 3$. Assume $d(x_2, v_1) = 2$. Since $N(v_1) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_2y, yv_1 \in E(G)$. Thus, $x_2yv_1y_2x_3x_2$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_1) = 3$. Assume $d(x_2, v_2) = 2$. Since $N(v_2) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_2y, yv_2 \in E(G)$. Thus, $x_2yv_2y_3y_2x_3x_2$ is a 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_2) \geq 3$. By Claim 3, $d(y_3, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_2\}$ and $y \in \{v_1, v_2, y_3\}$, a contradiction.

Suppose, second, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| = 0$. Assume $d(x, y) \geq 3$ for every $x \in \{x_2, x_3, x_4\}$ and $y \in \{y_2, y_3\}$. If $d(x_{i_0}, v_1) = 2$ for $x_{i_0} \in \{x_2, x_3, x_4\}$, then $d(x_i, v_1) \geq 3$ for $i \in \{2, 3, 4\} \backslash \{i_0\}$ by $g(G) \geq 8$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, y_1, y_2\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2, x_3, x_4\}$ and $y_{j_0} \in \{y_2, y_3\}$ such that $d(x_{i_0}, y_{j_0}) = 2$. Let $i \in \{2,3,4\}\backslash\{i_0\}$ with $x_i x_{i_0} \in E(H)$, and $j \in \{2,3\}\backslash\{j_0\}$ with $y_j y_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i, y_j) \geq 3$ holds. Since $d(x_{i_0}, y_{j_0}) = 2$, $d(x_i, v_j) \geq 3$ for $j \in \{1,2\}$ by $g(G) \geq 8$. By Claim 3, $d(y_j, u_i) \geq 3$ for $i \in \{1,2\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_j\}$, a contradiction.

Case 2.2.b. $d_{H^*}(v_1, v_2) = 2$.

Suppose, first, that $H^* \cong K_{1,3}$, where $V(H^*) = \{v_1, v_2, y_1, y_2\}$ and $d_{H^*}(y_2) = 3$. Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for $v \in Y\backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0\backslash V(H^*)$, by Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0\backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$. If y_0 is adjacent to v_i ($i \in \{1,2\}$), then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to H^*. If y_0 is adjacent to y_2, then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to H^*. Therefore, y_0 is adjacent to y_1 (See Fig. 3). Similarly to the proof of Claim 5, we have the following claim.

Claim 7. $|[\{x_2, x_3, x_4\}, \{y_1, y_2\}]| \leq 1$ in G.

Suppose, first, that $|[\{x_2, x_3, x_4\}, \{y_1, y_2\}]| = 1$ and $x_{i_0} y_{j_0}$ is an edge in G, where $i_0 \in \{2,3,4\}$ and $j_0 \in \{2,3\}$. Without loss of generality, we consider the following cases.

Case 2.2.b.1. $x_2 y_2 \in E(G)$.

If $d(x_3, v_i) = 2$ for $1 \leq i \leq 2$ or $d(x_3, y_0) = 2$, then there is a vertex y in $G[Y]$ such that $x_3 y, y v_i \in E(G)$ or $x_3 y, y y_0 \in E(G)$. Thus, there is a at most 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3, v_i) \geq 3$ and $d(x_3, y_0) \geq 3$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_3\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Case 2.2.b.2. $x_2 y_1 \in E(G)$.

The proof of this case is the same as Case 2.2.b.1.

Case 2.2.b.3. $x_3 y_2 \in E(G)$.

If $d(x_2, v_i) = 2$ for $1 \leq i \leq 2$ or $d(x_2, y_0) = 2$, then there is a vertex y in $G[Y]$ such that $x_2 y, y v_i \in E(G)$ or $x_2 y, y y_0 \in E(G)$. Thus, there is a at most 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_i) \geq 3$ and $d(x_2, y_0) \geq 3$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_2\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Case 2.2.b.4. $x_3 y_1 \in E(G)$.

The proof of this case is the same as Case 2.2.b.3.

Suppose, secondly, that $|[\{x_2, x_3, x_4\}, \{y_1, y_2\}]| = 0$. Assume $d(x,y) \geq 3$ for every $x \in \{x_2, x_3, x_4\}$ and $y \in \{y_1, y_2\}$. If $d(x_{i_0}, v_1) = 2$ for $2 \leq i_0 \leq 4$, then $d(x_i, v_1) \geq 3$ for $i \in \{2,3,4\}\backslash\{i_0\}$ by $g(G) \geq 8$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, y_1, y_2\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2, x_3, x_4\}$ and $y_{j_0} \in \{y_2, y_3\}$ such that $d(x_{i_0}, y_{j_0}) = 2$. Let $i \in \{2,3,4\}\backslash\{i_0\}$ with $x_i x_{i_0} \in E(H)$, and $j \in \{2,3\}\backslash\{j_0\}$ with $y_j y_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i, y_j) \geq 3$ holds. Since $d(x_{i_0}, y_{j_0}) = 2$, $d(x_i, v_j) \geq 3$ for $j \in \{1,2\}$ by $g(G) \geq 8$.

By Claim 3, $d(y_j, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_j\}$, a contradiction.

Suppose, secondly, that H^* is a path of length 3, denoted $H^* = y_1y_2y_3y_4$. Without loss of generality, suppose $v_1 = y_1, v_2 = y_3$.

Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for $v \in Y \backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0 \backslash V(H^*)$, by Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0 \backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$. If y_0 is adjacent to v_i ($i \in \{1, 2\}$), then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to H^*. If y_0 is adjacent to y_2, then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to H^*. Therefore, y_0 is adjacent to y_4 (See Fig. 4). Similarly to the proof of Claim 5, we have the following claim.

Claim 8. $|[\{x_2, x_3, x_4\}, \{y_2, y_4\}]| \leq 1$ in G.

Suppose, first, that $|[\{x_2, x_3, x_4\}, \{y_2, y_4\}]| = 1$ Without loss of generality, we consider the following cases.

Case 2.2.b.5. $x_2y_2 \in E(G)$.

Assume $d(x_3, v_{j_0}) = 2$ for $v_{j_0} \in \{v_1, v_2, y_0\}$. Since $N(v_i) \cap X = \emptyset$ and $N(y_0) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_3y, yv_i(y_0) \in E(G)$. Thus, there is a cycle C in G whose length of C is at most 7, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3, v_j) \geq 3$ and $d(x_3, y_0) \geq 3$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_3\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Case 2.2.b.6. $x_3y_2 \in E(G)$.

Similarly, we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_2\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Suppose, secondly, that $|[\{x_2, x_3, x_4\}, \{y_2, y_4\}]| = 0$.

Assume $d(x, y) \geq 3$ for every $x \in \{x_2, x_3, x_4\}$ and $y \in \{y_2, y_4\}$. Since $g(G) \geq 8$, there is one x_i of x_2, x_3 such that $d(x_i, v_1) \geq 3$. Therefore, by Claim 3, we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, y_2, y_4\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2, x_3, x_4\}$ and $y_{j_0} \in \{y_2, y_3\}$ such that $d(x_{i_0}, y_{j_0}) = 2$. Let $x_{i_0}x_i \in E(H)$. Without loss of generality, we consider the following cases.

Case 2.2.b.7. $d(x_{i_0}, y_2) = 2$.

Since $g(G) \geq 8$, $d(x_i, v_j) \geq 3$ for $j \in \{1, 2\}$ and $d(x_i, y_4) \geq 3$ hold. Therefore, by Claim 3, we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_4\}$, a contradiction.

Case 2.2.b.8. $d(x_{i_0}, y_4) = 2$.

Similarly, we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_2, y_0, y_2\}$, a contradiction.

Case 2.2.c. $d_{H^*}(v_1, v_2) = 1$.

Suppose, first, that H^* is a path of length 3, denoted by $P_3 = y_1y_2y_3y_4$. If $v_1 = y_1, v_2 = y_2$, then $N(y_0) \cap V(H^*) = \{y_4\}$. Otherwise, there is a connected subgraph

G^* of order 4 in $G[V(H^*) \cup \{y_0\}]$ such that $v_1, v_2, y_0 \in V(G^*)$, a contradiction to H^*. Since $d_{H^*}(v_2, y_0) = 3$, Similarly to Case 2.2.a, we have that there are six vertices x_1, x_2, x_3, z_1, z_2 and z_3 in G such that the distance $d(x_i, z_j) \geq 3$ $(1 \leq i, j \leq 3)$, a contradiction.

Suppose that $H^* \cong K_{1,3}$, where $d_{H^*}(v_1) = 3$. Then there is a connected subgraph G^* of order 4 in $G[V(H^*) \cup \{y_0\}]$ such that $v_1, v_2, y_0 \in V(G^*)$, a contradiction to H^*.

Case 2.3. $|Y_0 \cap V(H^*)| = 3$.

Let $Y_0 = \{v_1, v_2, v_3, \ldots\}$. Suppose that $n_0 = 3$. Since $g(G) \geq 8$, $|[\{y\}, V(H^*)]| \leq 1$ for $y \in Y \backslash V(H^*)$. Since $Y_0 \subseteq V(H^*)$, we have that

$$\begin{aligned} \sum_{y \in Y \backslash V(H^*)} |N(y) \cap V(H^*)| &= |[Y \backslash V(H^*), V(H^*)]| \\ &\leq |Y \backslash V(H^*)| \\ &\leq |[Y \backslash V(H^*), X]| \\ &= \sum_{y \in Y \backslash V(H^*)} |N(y) \cap X|. \end{aligned} \tag{2.2}$$

By Lemma 2.1, G is maximally 4-restricted edge connected, a contradiction. Then $n_0 \geq 4$. Suppose that $v_1, v_2, v_3 \in Y_0 \cap V(H^*)$. Since H^* is a path of length 3 or a $K_{1,3}$, there is at least a vertex of degree 1 in v_1, v_2, v_3. Without loss of generality, suppose $d_{H^*}(v_1) = 1$ and $v_1 = y_1$.

Case 2.3.1. $H^* = y_1y_2y_3y_4$ is a path of length 3.

Since $|Y_0 \cap V(H^*)| = 3$, we have that $H^* = v_1v_2v_3y_4$ (See Fig. 5) or $H^* = v_1v_2y_3v_3$. We consider the following cases.

Case 2.3.1.1. $H^* = v_1v_2v_3y_4$.

Since $g(G) \geq 8$, we have the following claim.

Claim 9. $|[\{x_2, x_3, x_4\}, \{y_4\}]| \leq 1$ in G.

Suppose, first, that $|[\{x_2, x_3, x_4\}, \{y_4\}]| = 1$ and $x_{i_0}y_4 \in E(G)$ for $x_{i_0} \in \{x_2, x_3, x_4\}$. Let $x_ix_{i_0} \in E(H)$. Since $g(G) \geq 8$, we have $d(x_i, v_j) \geq 3$ for $j \in \{1, 2, 3\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, v_3\}$, a contradiction.

Suppose, secondly, that $|[\{x_2, x_3, x_4\}, \{y_4\}]| = 0$.

Since there is no $d(x_i, v_j) \geq 3$ for every $i \in \{2, 3, 4\}$ and every $j \in \{1, 2, 3\}$, there is one $d(x_{i_0}, v_{j_0}) = 2$ for $i_0 \in \{2, 3, 4\}$ and $j_0 \in \{1, 2, 3\}$. Let $x_ix_{i_0} \in E(H)$. Since $g(G) \geq 8$, $d(x_i, v_j) \geq 3$ for every $j \in \{1, 2, 3\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, v_3\}$, a contradiction.

Case 2.3.1.2. $H^* = v_1v_2y_3v_3$.

Similarly to Case 2.3.1.1, we have that there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(1 \leq i, j \leq 3)$, a contradiction.

Case 2.3.2. $H^* \cong K_{1,3}$.

Let $d(y_2) = 3$ in H^*. Since $|Y_0 \cap V(H^*)| = 3$, we have that $y_2 = v_2$ and $y_2 \neq v_2$ or v_3 or v_3. Similarly to Case 2.3.1, we have that there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(1 \leq i, j \leq 3)$, a contradiction.

Case 2.4. $|Y_0 \cap V(H^*)| \geq 4$.

If $d(x_i, v_j) \geq 3$ for every $i \in \{2,3,4\}$ and every $j \in \{1,2,3,4\}$, then there are six vertices u_1, u_2, x_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(i, j \in \{1,2,3\})$, a contradiction. Then $d(x_{i_0}, v_{j_0}) = 2$ for $i_0 \in \{2,3,4\}$ and $j_0 \in \{1,2,3,4\}$. Since $g(G) \geq 8$, $d(x_{i_0}, v_j) \geq 3$ for every $j \in \{1,2,3,4\} \backslash \{j_0\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_{i_0}\}$ and $y \in \{v_j : j \in \{1,2,3,4\} \backslash \{j_0\}\}$, a contradiction.

Summarizing Cases 1 and 2, we obtain that G is maximally 4-restricted edge connected. □

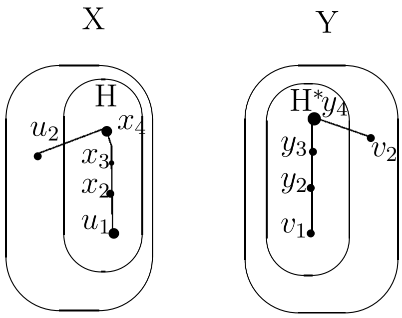

Fig. 1. The structure of $G[X]$ and $G[Y]$

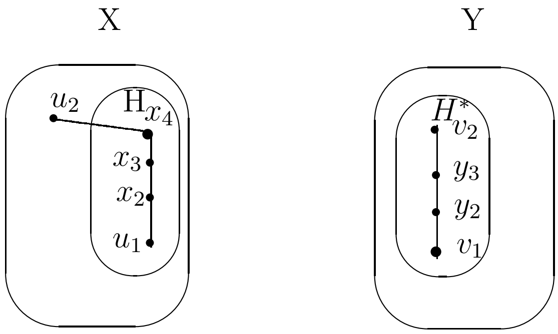

Fig. 2. The structure of $G[X]$ and $G[Y]$

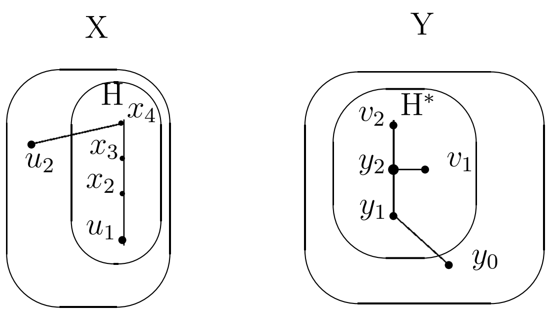

Fig. 3. The structure of $G[X]$ and $G[Y]$

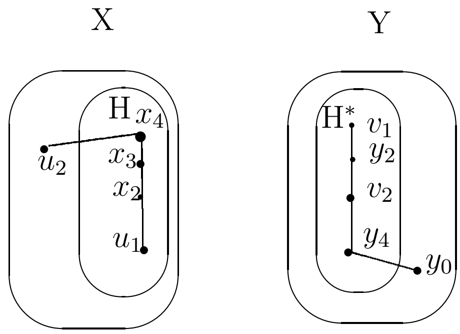

Fig. 4. The structure of $G[X]$ and $G[Y]$

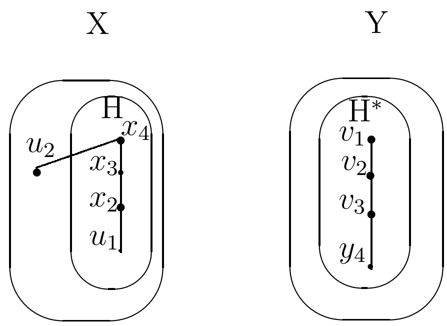

Fig. 5. The structure of $G[X]$ and $G[Y]$

3 Conclusion

In this paper, we have investigated the problem of the maximally 4-restricted edge connected graph and shown a sufficient condition for graphs to be maximally 4-restricted edge connected, i.e., if G is a λ_4-connected graph with $\lambda_4(G) \leq \xi_4(G)$ and the girth satisfies $g(G) \geq 8$, and there do not exist six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$, $(1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge connected. Our further work aims to investigate the problem of the maximally k-restricted edge connected graph.

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