New Bounds For Pairwise Orthogonal Diagonal Latin Squares

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Abstract

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. Let d_r be the least integer such that for all $n > d_r$ there exist r pairwise orthogonal diagonal Latin squares of order n. In a previous paper Wallis and Zhu gave several bounds on the d_r. In this paper we shall present some constructions of pairwise orthogonal diagonal Latin squares and consequently obtain new bounds for five pairwise orthogonal diagonal Latin squares.

1 Introduction

A *Latin square* of order n is an $n \times n$ array such that every row and every column is a permutation of an n–set. A *transversal* in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A *symmetric transversal* in a Latin square is a transversal which is a set of symmetric positions. A *transversal Latin square* is a Latin square whose main diagonal is a transversal. A *diagonal Latin square* is a transversal Latin square whose back diagonal also forms a transversal. It is easy to see that the existence of a transversal Latin square with a symmetric transversal implies the existence of a diagonal Latin square.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. t *pairwise orthogonal diagonal (transversal) Latin squares* of order n, denoted briefly by t PODLS(n) (POILS(n)) are t pairwise orthogonal Latin squares each of which is a diagonal (transversal) Latin square of order n. We let $N(n)$ ($D(n), I(n)$) denote the maximum number of pairwise orthogonal (diagonal, transversal) Latin squares of order n.

For $t = 2$, it has been shown (see [1, 5, 6, 7, 10]) that a pair of orthogonal diagonal Latin squares exists for all n with the 3 exceptions $n \in \{2, 3, 6\}$.

For $t = 3$, Wallis and Zhu [8] showed that $d_3 \leq 446$. Then Zhu [11] and Du [2] showed that three pairwise orthogonal diagonal Latin squares of order n exist for all n with the 5 exceptions $n \in E = \{2,3,4,5,6\}$ and 11 possible exceptions, of which 46 is the largest number. Thus $d_3 \leq 46$.

For $t = 4$, Wallis and Zhu [8] showed that $d_4 \leq 510$. Then Du [3] showed that four pairwise orthogonal diagonal Latin squares of order n exists for all n with the 5 exceptions $n \in E$ and 29 possible exceptions, of which 69 is the largest number. Thus $d_4 \leq 69$.

For $t = 5$, Wallis and Zhu [8] showed that $d \leq 2724$. It is our purpose to improve this result. We shall present some constructions of pairwise orthogonal diagonal Latin squares and consequently deduce that five pairwise orthogonal diagonal Latin squares of order n exist for all n with the 6 exceptions $n \in \{2,3,4,5,6,7\}$ and 43 possible exceptions, of which 164 is the largest number. Thus $d_5 \leq 164$.

Theorem 1.1 *There exist five pairwise orthogonal diagonal Latin squares of every order n where $n > 164$. Order $2 \leq n \leq 7$ are impossible; the only orders for which the existence is undecided are:*

10	12	14	15	18	20	21	22	24	26
28	30	33	34	35	36	38	39	40	42
44	45	46	48	50	51	52	54	55	60
62	66	68	69	70	74	76	82	84	90
98	106	164							

For our purpose, let $IA_t(v,k)$ denote t pairwise orthogonal Latin squares of order v (briefly t POLS(v)) with t sub-POLS(k) missing. Usually we leave the size k hole in the lower right corner. Further denote by $IA_t^*(v,k)$ an $IA_t(v,k)$ in which the first $v-k$ elements in the main diagonal of every square are distinct and different from the missing elements. It is easy to see that the existence of an $IA_{t+1}(v,k)$ implies the existence of an $IA_t^*(v,k)$, and that $IA_t^*(v,1)$ exists if there exist t pairwise orthogonal transversal Latin squares of order v.

Finally, we denote by $A_t(v,k)$ the t POILS(v) in which the cells $\{(v-k+i, v-i+1) : 1 \leq i \leq k\}$ is a common transversal about elements $x_1, x_2, \cdots, x_k$. It is clear that an $A_t(n,n)$ exists if $D(n) \geq t$.

For the following proof, we construct

Example 1.1. There exists a $A_{q-3}(q,k)$ for odd prime q, $0 \leq k \leq q$.

Proof. In $GF(q) = \{a_0 = 0, a_1 = 1, a_2 = -1, a_3, \cdots, a_{q-1}\}$, consider the $q \times q$ arrays

$$L_k = (h_{ij}^k) \qquad\qquad 3 \leq k \leq q-1$$

where $h_{ij}^k = \lambda_k a_i + \mu_k a_j$, $\lambda_k, \mu_k \in GF(q)\backslash\{0,1,-1\}$, $\lambda_k \neq \mu_k$ and $\lambda_k + \mu_k = 1$. It is easy to see that L_k, $3 \leq k \leq q-1$, are $q-3$ POILS(q) each of which has element i in the cells (i,i). Then we obtain $A_{q-3}(q,k)$ by the permutation σ:

$$\begin{pmatrix} 1 & 2 & \cdots & q-k-1 & q-k & q-k+1 & \cdots & q-1 & q \\ q-k & q-k-1 & \cdots & 2 & 1 & q & \cdots & q-k+2 & q-k+1 \end{pmatrix}$$

From [2] we have

Example 1.2. An $A_6(9,2)$ exists.

2 Some Constructions

We need the following new constructions. For simplicity we shall not state their most general form, but only the special case to meet the need of this paper.

First we let Q be a Latin square of order n based on the set $I_n = \{0, 1, \cdots, n-1\}$ and let S, T be transversals of Q. We form a permutation $\sigma_{S,T}$ on I_n as follows: $\sigma_{S,T}(s) = t$ where s and t are the entries of S and T, respectively, occuring in the same row. We denote by $Q(S,T)$ the Latin square obtained by renaming symbols using $\sigma_{S,T}$. Obviously we have

(a) If U is a transversal of Q then U is also a transversal of $Q(S,T)$;

(b) If V is a Latin square which is orthogonal to Q, then V is also orthogonal to $Q(S,T)$.

Let A, B be Latin squares and let h be a symbol. We denote by A_h the copy of A obtained by replacing each entry x of A with the ordered pair (h, x). Further we denote by (A, B) the Latin square $(A_{b_{ij}})$, where $B = (b_{ij})$.

Lemma 2.1. For a positive integer k, let $A_1, A_2, \cdots, A_5$ be 5 pairwise orthogonal Latin squares of order k which possess 12 disjoint common transversals $T_1, T_2, \cdots, T_{11}$ and the main diagonal D. Then there exist 5 pairwise orthogonal diagonal Latin squares of order $12k$.

Proof. Consider the 5 pairwise orthogonal Latin squares of order $12k$

$$\overline{A}_i = (A_i, B_i) \qquad\qquad 1 \leq i \leq 5$$

where the B_i $(1 \leq i \leq 5)$ are 5 pairwise orthogonal Latin squares of order 12.

We denote by $\tilde{A}_1$ the Latin square obtained by replacing each subsquare A_1 with $A_1(D, T_j)$ in the j-th block column of $\overline{A}_1$, $1 \leq j \leq 11$. As a result of such replacement the 0-th block column of $\tilde{A}_1$ contains the $12k$ entries from T_j which are just the same as the entries appearing in the main diagonals of blocks in the j-th block column. For each j $(1 \leq j \leq 11)$ exchange the two entries from the above two sets of entries appearing in the same row of $\tilde{A}_1$. From (a) it follows immediately that the resulting array $\hat{A}_1$ is a transversal Latin square with a symmetric transversal.

Do the same replacement and exchange of entries for $\overline{A}_2, \overline{A}_3, \cdots, \overline{A}_5$. By (b) the resulting squares $\hat{A}_2, \hat{A}_3, \cdots, \hat{A}_5$ together with $\hat{A}_1$ form 5 POILS with a common symmetric transversal, which consists of the main diagonals of those blocks appearing in the block back diagonal. By simultaneously permuting rows and columns we have 5 PODLS($12k$).

Lemma 2.2. For a positive integer k, let $A_1, A_2, \cdots, A_5$ be 5 pairwise orthogonal diagonal Latin squares of order k which possess three disjoint common transversals

T_1, T_2 and the main diagonal D. If the positions of T_1, T_2 are symmetric about the main diagonal, then there exist 5 pairwise orthogonal diagonal Latin squares of order $7k$.

Proof. Consider the 5 pairwise orthogonal Latin squares of order $7k$

$$\overline{A}_1 = (A_i, B_i) \quad 1 \leq i \leq 5$$

where the B_i ($1 \leq i \leq 5$) are 5 pairwise orthogonal transversal Latin squares of order 7.

Notice that the set $E_i^j(t)$ of the entries of the $(tk+j)$-th column of $\overline{A}_1$ which lie on the transversals T_i coincides with the set $\overline{E}_i^j(t)$ of the entries of the $[(7-t-1)k+j]$-th column of A_1 lying on the transversals T_i, $t = 0, 1, 2$. For each $i = 1, 2$, $j = 1, 2, \cdots, k$, exchange in $\overline{A}_1$ the elements of $E_i^j(t)$ and $\overline{E}_i^j(t)$ appearing on the same row, $t = 0, 1, 2$. It is clear that the resulting array $\hat{A}_1$ is a transversal Latin square with a transversal which consists of an element in the central cell and a set of elements in symmetric positions.

Do the same exchange of entries for $\overline{A}_2, \overline{A}_3, \cdots, \overline{A}_5$. It is easy to see that the resulting squares $\hat{A}_2, \hat{A}_3, \cdots, \hat{A}_5$ together with $\hat{A}_1$ form 5 POILS with a common transversal, which consists of the back diagonal in the central block and the T_1 in the upper right blocks and the T_2 in the lower left blocks of the block back diagonal. By simultaneously permuting rows and columns we have 5 PODLS(7k).

Next, we give the following three constructions which are generalizations of the main constructions in [11].

Lemma 2.3. Suppose there are $t+1$ POLS(q) such that t of them are t PODLS(q).

(1) Suppose $2 \mid qmk, D(k) \geq t$ and suppose $IA_t^*(m+k_i, k_i)$ exist for $0 \leq i \leq q-1$, where $k = k_0 + k_1 + \cdots + k_{q-1}$. Further suppose an $IA^{**}(m+k_0, k_0)$ exists if $2 \mid q$. Then $D(qm+k) \geq t$.

(2) Suppose $2 \nmid qmk, D(k) \geq t$ and suppose $IA_t^*(m+k_i, k_i)$ exist for $0 \leq i \leq q-1$, where $k = k_0 + k_1 + \cdots + k_{q-1}, k_0 = 1$. Then $D(m+1) \geq t$ implies $D(qm+k) \geq t$.

Proof. (1) is Lemma 3.3 in [11], and (2) is Lemma 2.5 in [2].

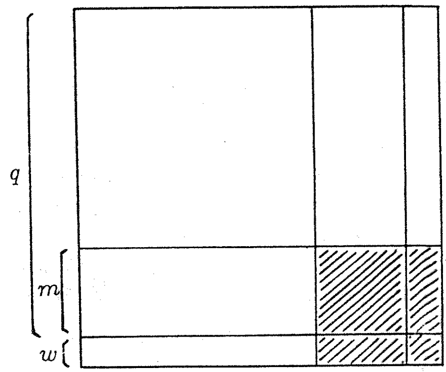
Lemma 2.4. Suppose there are $t+k$ POLS(q) such that t of them are t PODLS(q)

(1) Suppose $2 \nmid q$ and suppose there are $IA_t^*(m+w_i, w_i)$, $1 \leq i \leq k$, $w_1 + w_2 + \cdots + w_k = w$. Then $min\{I(m), D(m+w)\} \geq t$ implies $D(qm+w) \geq t$.

(2) Suppose $2 \mid q$ and suppose there are $IA_t^*(m+w_i, w_i)$, $1 \leq i \leq k$, $w_1 + w_2 + \cdots + w_k = w$. Further suppose there are $A_t(m+w, w)$. Then $I(m) \geq t$ implies $D(qm+w) \geq t$.

Proof. (1) is Lemma 3.7 in [11], and (2) is Lemma 2.3 in [3]. For the convenience of the reader we give here the proof of (2).

The cell (q,q)'s element in each of the k extra order q orthogonal matrices determines a common transversal in the t PODLS(q). For each of k such transversals intersecting in the cell (q,q), fill its cells with an $IA_t^*(m+w_i,w_i)$, $1 \le i \le k$, but leave the cell (q,q) empty. Fill the other cells with t POILS(m). Notice that the cells in the back diagonal of t PODLS(q) are filled with some modified $IA_t^*(m+w_i,w_i)$ or t POILS(m) whose back diagonal of the order m subarray is occupied by different elements. Label the elements and get the right and lower parts as we did in Lemma 2.5 in [4]. Then we get an $IA_t^*(qm+w,m+w)$ shown in the figure.



Now fill the size $m+w$ hole in the lower right corner of the $IA_t^*(qm+w,m+w)$ with the given $A_t(m+w,w)$, and permute rows and columns with permutation σ:

$$\begin{pmatrix} 1 & 2 & \cdots & \frac{qm}{2} & \frac{qm}{2}+1 & \frac{qm}{2}+2 & \cdots & qm & qm+1 & qm+2 & \cdots & qm+w \\ 1 & 2 & \cdots & \frac{qm}{2} & \frac{qm}{2}+w+1 & \frac{qm}{2}+w+2 & \cdots & qm+w & \frac{qm}{2}+1 & \frac{qm}{2}+2 & \cdots & \frac{qm}{2}+w \end{pmatrix}$$

then we obtain t PODLS$(qm+w)$.

The remaining verification is a routine matter and the proof is complete.

Lemma 2.5. Suppose there are $t+w+1$ POLS(q) such that t of them are t PODLS(q).

(1) Suppose $h_0 = 0$, $2 \nmid q$ and there exist $IA_t^*(m+h_i,h_i)$ and $IA_t^*(m+1+h_i,h_i)$, $0 \le i \le q-1$, $h = h_0 + h_1 + \cdots + h_{q-1}$. Then $\min\{D(m+w), D(h)\} \ge t$ implies $D(qm+w+h) \ge t$, provided $2 \mid (m+w)$ or $2 \mid h$.

(2) Suppose $h_0 = 0$, $2 \mid q$ and there exist $IA_t^*(m+h_i,h_i)$ and $IA_t^*(m+1+h_i,h_i)$, $0 \le i \le q-1$, $h = h_0 + h_1 + \cdots + h_{q-1}$. Further suppose there are $A_t(m+w,w)$. Then $D(h) \ge t$ implies $D(qm+w+h) \ge t$, provided $2 \mid w$ or $2 \mid h$.

Proof. (1) is Lemma 3.9 in [11], and (2) is Lemma 2.7 in [2].

We also need the following lemmas from [8].

Lemma 2.6. If $N(g) \geq r$, where r is odd, and $0 < x \leq g$, then

$$D(g(r+1)+x) \geq min\{I(g), I(r+1), I(r+2), D(x)\}.$$

Lemma 2.7. If $N(g) \geq r$, r being odd, and x is an odd number satisfying $0 < x \leq g$, then

$$D((r+1)g + x - 1) \geq min\{I(g), I(g-1), D(r+1), D(r+2), D(x)\}.$$

Lemma 2.8. If $N(g) \geq r$, where g is even, then

$$D(g(r+2)+1) \geq min\{D(g+1), I(r+2)\}$$

For the following lemmas we need pairwise balanced designs. A pairwise balanced design (briefly PBD) of index unity is a pair (X, A) where X is a set (of points) and A is a collection of some subsets a of X (called blocks) such that any pair of distinct points of X is contained in exactly one block of A. Denote by $(v, K, 1)$-PBD a PBD with v points, block sizes all in K, and index unity. We assume that the reader is familiar with the various composition constructions for PBDs. For details see [9].

Lemma 2.9. Suppose there is a $(v, K, 1)$-PBD and for every $k \in K$ there exist t pairwise orthogonal idempotent Latin squares of order k. Further suppose the v-set X can be partitioned into disjoint subsets:

$$X = S_1 \cup S_2 \cup \cdots \cup S_m$$

such that each S_i is a subset of some block B_i, at most one S_i has odd order, and for each i there exist $A_t(k_i, s_i)$. Then there exist t PODLS(v).

From Lemma 2.9 and composition constructions for PBDs, we have

Lemma 2.10. If $N(g) \geq 7$, $0 < u, v \leq g$, g, u odd, then $min\{D(g), D(u), D(v)\} \geq 5$ implies $D(7g + u + v) \geq 5$.

Finally, we need the following lemma from [4].

Lemma 2.11. For any prime power order $q \geq 3$,

$$D(q) = \begin{cases} q-3, & \text{if } q \text{ is odd} \\ q-2, & \text{if } q \text{ is even.} \end{cases}$$

3 The bound for $D(n) \geq 5$

From [8] we have

Lemma 3.1. $D(n) \geq 5$ for $n > 2724$.

In this section we give a better bound:

Theorem A. $D(n) \geq 5$ for $n \geq 519$.

We prove the theorem in three lemmas.

Lemma 3.2. $D(n) \geq 5$ for $n \in \{526, 527, 574, 575, 582, 583, 766, 767\}$.

Proof. Apply Lemma 2.10 with $(g, u, v) \in \{(71, 13, 16), (71, 19, 31), (79, 13, 8), (79, 13, 9), (79, 13, 16), (79, 13, 17), (107, 9, 8), (107, 9, 9)\}$.

Lemma 3.3. $D(n) \geq 5$ for $n \geq 8.79 + 36$.

Proof. Suppose g is odd and $g \geq 79$, $g \notin F = \{85, 87, 93, 95, 111, 119, 123, 159, 175, 183, 291, 295, 335\}$. Then $N(g) \geq 7$ and $N(g-1) \geq 6$. If $x \in S = \{1, 9, 11, 23, 29, 31, 37\}$, then $D(x) \geq 5$ and $g \geq x$. So there exist 5 PODLS $(8g + x)$ and 5 PODLS $(8g + x - 1)$ for all such x and g.

For $g \in F$, apply Lemmas 2.6 and 2.7 using Table 1. We obtain the lemma.

Table 1

n	8n+1	8n+9	8n+11	8n+19
85	8.79+49	8.83+25	8.83+27	8.79+67
87	8.81+49	8.88+1	8.83+43	8.88+11
93	8.88+41	8.88+49	8.89+43	8.88+59
95	8.89+49	8.96+1	8.89+59	8.89+67
111	8.109+17	8.109+25	8.109+27	8.103+83
119	8.117+17	8.117+25	8.117+27	8.113+67
123	8.121+17	8.121+25	8.121+27	8.117+67
159	8.157+17	8.157+25	8.157+27	8.153+67
175	8.173+17	8.176+1	8.173+27	8.176+11
183	8.181+17	8.184+1	8.181+27	8.184+11
291	8.289+17	8.289+25	8.289+27	8.288+43
295	8.293+17	8.293+25	8.293+27	8.289+67
335	8.333+17	8.333+25	8.333+27	8.329+67

n	8n+23	8n+29	8n+31	8n+37
85	8.79+71	8.81+61	8.83+47	8.83+53
87	8.81+71	8.89+13	8.88+23	8.88+29
93	Lemma 3.2	8.89+61	8.91+47	8.91+53
95	8.89+71	8.97+13	8.89+79	8.96+29
111	8.105+71	8.113+13	8.109+47	8.109+53
119	8.113+71	8.121+13	8.117+47	8.117+53
123	8.117+71	8.125+13	8.121+47	8.121+53
159	8.153+71	8.155+61	8.157+47	8.157+53
175	8.169+71	8.171+61	8.176+23	8.176+29
183	8.176+79	8.179+61	8.184+23	8.184+29
291	8.288+47	8.288+53	8.289+47	8.289+53
295	8.289+71	8.285+109	8.293+47	8.293+53
335	8.329+71	8.331+61	8.333+47	8.333+53

Lemma 3.4. $D(n) \geq 5$ for $519 \leq n \leq 8.79 + 36$.

Proof. Apply Lemmas 2.6 and 2.7 using Table 2.

Table 2

n	8n+1	8n+3	8n+5	8n+7
65	8.64+9	8.64+11	8.64+13	Lemma 3.2
66	8.64+17	8.64+19	8.65+13	8.64+23
67	8.64+25	8.64+27	8.64+29	8.64+31
68	8.65+25	8.65+27	8.65+29	8.65+31
69	8.64+41	8.64+43	8.65+37	8.64+47
70	8.65+41	8.65+43	8.65+53	8.65+47
71	8.71+1	8.64+59	8.65+61	Lemma 3.2
72	8.72+1	8.71+11	8.71+13	Lemma 3.2
73	8.73+1	8.72+11	8.72+13	8.71+23
74	8.73+9	8.73+11	8.73+13	8.72+23
75	8.73+17	8.73+19	8.72+29	8.73+23
76	8.73+25	7.73+27	8.73+29	8.73+31
77	8.72+41	8.72+43	8.73+37	8.72+47
78	8.73+41	8.73+43	8.72+53	8.73+47
79	8.79+1	8.72+59	8.73+53	8.71+71
80	8.79+9	8.79+11	8.79+13	8.72+71
81	8.81+1	8.87+19	8.80+13	8.79+23
82	8.81+9	8.81+11	8.81+13	8.80+23
83	8.83+1	8.81+19	8.79+37	8.81+23

4 The case n odd

In this section we prove Theorem 1.1 for n odd.

First, from Lemma 2.11 we have

Lemma 4.1. $D(n) \geq 5$ for $n \in \{59, 61, 71, 79, 83, 101, 103, 107, 121, 125, 127, 131, 173\}$.

Lemma 4.2. $D(n) \geq 5$ for $n \in \{57, 85\}$.

Proof. Apply Lemma 2.8 with $(g, r) \in \{(8, 5), (12, 5)\}$.

Lemma 4.3. $D(n) \geq 5$ for $n \in \{63, 77, 91, 119\}$.

Proof. Apply Lemma 2.2 with $k \in \{9, 11, 13, 17\}$.

Lemma 4.4. $D(n) \geq 5$ for $n \in \{75, 93, 123, 135, 177\}$.

Proof. Apply Lemma 2.4 (1) with $(g, m; w) \in \{(9, 8; 3), (11, 8; 5), (17, 7; 4), (19, 7; 2), (25, 7; 2)\}$.

Lemma 4.5. $D(n) \geq 5$ for $n \in \{87, 133, 175\}$.

Proof. Apply Lemma 2.5 (1) with $(g, m; w, h) \in \{(11,7;2,8),(11,7;1,3),(23,7;1,13)\}$.

We then have

Theorem B. Theorem 1.1 is true for n odd.

Proof. Apply Lemmas 2.6 and 2.10 using Table 3. Combining Lemmas 4.1-4.5, we obtain the desired result.

Table 3

n	8n+1	8n+3	8n+5	8n+7
7	Lemma 4.2	Lemma 4.1	Lemma 4.1	Lemma 4.3
8	8.8+1	Lemma 4.1	?	Lemma 4.1
9	8.9+1	Lemma 4.4	Lemma 4.3	Lemma 4.1
10	8.9+9	Lemma 4.1	Lemma 4.2	Lemma 4.5
11	8.11+1	Lemma 4.3	Lemma 4.4	(11,9,9)
12	8.11+9	8.11+11	Lemma 4.1	Lemma 4.1
13	8.13+1	Lemma 4.1	(13,9,9)	(13,11,9)
14	8.13+9	8.13+11	8.13+13	Lemma 4.3
15	Lemma 4.1	Lemma 4.4	Lemma 4.4	Lemma 4.1
16	8.16+1	Lemma 4.1	Lemma 4.5	Lemma 4.4
17	8.17+1	8.16+11	8.16+13	(17,13,11)
18	8.17+9	8.17+11	8.17+13	(19,9,9)
19	8.19+1	(19,11,11)	(19,13,11)	(19,17,9)
20	8.19+9	8.19+11	8.19+13	(19,17,17)
21	8.19+17	8.19+19	Lemma 4.1	Lemma 4.5
22	Lemma 4.4	(23,9,9)	(23,11,9)	(23,11,11)
23	8.23+1	(23,17,9)	23,17,11)	(23,17,13)
24	8.23+9	8.23+11	8.23+13	(25,13,11)
25	8.25+1	8.23+19	(25,19,11)	8.23+23
26	8.25+9	8.25+11	8.25+13	(27,17,9)
27	8.25+17	8.25+19	(27,23,9)	8.25+23
28	8.27+9	8.27+11	8.27+13	(27,25,17)
29	8.29+1	8.27+19	(31,11,9)	8.27+23
30	8.29+9	8.29+11	8.29+13	(31,,19,11)
31	8.31+1	8.29+19	(31,27,9)	8.29+23
32	8.32+1	8.31+11	8.31+13	(31,29,17)
33	8.32+9	8.32+11	8.32+13	8.31+23
34	8.32+17	8.32+19	8.31+29	8.32+23
35	8.32+25	8.32+27	8.32+29	8.32+31
36	(37,17,13)	(37,19,13)	(37,23,11)	(37,23,13)
37	8.37+1	(37,27,13)	(37,29,13)	(37,31,13)
38	8.37+9	8.37+11	8.37+13	(41,13,11)
39	8.37+17	8.37+19	(41,19,11)	8.37+23
40	8.37+25	8.37+27	8.37+29	8.37+31
41	8.41+1	(43,19,11)	8.37+37	(43,23,11)
42	8.41+9	8.41+11	8.41+13	(43,31,11)

43	8.43+1	8.41+19	(47,11,9)	8.43+23
44	8.43+9	8.43+11	8.43+13	8.41+31
45	8.43+17	8.43+19	8.41+37	8.43+23
46	8.43+25	8.43+27	8.43+29	8.43+31
47	8.47+1	(49,27,9)	8.43+37	(49,31,9)
48	8.47+9	8.47+11	8.47+13	(49,31,17)
49	8.47+17	8.47+19	(53,17,9)	8.47+23
50	8.47+25	8.47+27	8.47+29	8.47+31
51	(53,19,19)	(53,31,9)	(53,31,11)	8.49+23
52	8.49+25	8.49+27	8.49+29	8.49+31
53	8.53+1	(53,29,27)	8.49+37	(59,9,9)
54	8.53+9	8.53+11	8.53+13	8.49+47
55	8.53+17	8.53+19	(61,9,9)	8.53+23
56	8.56+1	8.53+27	8.53+29	8.53+31
57	8.57+1	8.56+11	8.56+13	(61,27,9)
58	8.57+9	8.57+11	8.57+13	8.56+23
59	8.57+17	8.57+19	8.56+29	8.57+23
60	8.57+25	8.57+27	8.57+29	8.57+31
61	8.59+17	8.59+19	8.57+37	8.59+23
62	8.57+41	8.57+43	8.59+29	8.57+47
63	8.57+49	8.61+19	8.57+53	8.61+23
64	8.64+1	8.59+43	8.61+29	8.61+31

5 The case for n even

In this section we prove Theorem 1.1 for n even.

Lemma 5.1. $D(n) \geq 5$ for $n \in \{58, 114, 116, 118, 122\}$.

Proof. Apply Lemma 2.4 (2) with $(g, m; w) \in \{(8,7;2),(16,7;2),(16,7;4),(16,7;6),(16,7;10)\}$. The conditions $A_5(9,2), A_5(11,4), A_5(13,6), A_5(17,10)$ come from Examples 1.2 and 1.1.

Lemma 5.2. $D(n) \geq 5$ for $n \in \{124, 126\}$.

Proof. Apply Lemma 2.5 (2) with $(g, m; w, h) \in \{(16,7;4,8),(16,7;6,8)\}$. The conditions $A_5(11,4), A_5(13,6)$ come from Example 1.1.

Lemma 5.3. $D(n) \geq 5$ for $n = 156$.

Proof. Apply Lemma 2.1 with $k = 13$.

Lemma 5.4. $D(n) \geq 5$ for $n = 168$.

Proof. Apply Lemma 2.6 with $(r, g, x) = (7, 19, 6)$.

Lemma 5.5. $D(n) \geq 5$ for $n \in \{266, 274\}$.

Proof. Apply Lemma 2.7 with $(r, g, x) \in \{(7,32,11),(7,32,19)\}$.

We then have

Theorem C. Theorem 1.1 is true for n even.

Proof. Apply Lemmas 2.3 (2) and 2.10 using Table 4.

Table 4

n	7n+1	7n+3	7n+5	7n+7
9	7.9+1	?	?	?
11	7.11+1	(9,9,8)	?	?
13	7.13+1	(11,9,8)	(11,11,8)	(13,13,8)
15	?	(13,9,8)	(13,11,8)	Lemma 5.2
17	7.17+1	Lemma 5.1	Lemma 5.2	(17,13,8)
19	7.19+1	7.17+17	(17,11,8)	(19,13,8)
21	(17,13,16)	7.19+17	7.19+19	(19,19,16)
23	7.23+1	?	(19,17,16)	Lemma 5.4
25	7.25+1	7.23+17	7.23+19	(23,13,8)
27	7.27+1	7.25+17	7.25+19	(23,19,16)
29	7.29+1	7.27+17	7.27+19	(25,19,16)
31	7.31+1	7.29+17	7.29+19	(27,19,16)
33	7.29+29	7.31+17	7.31+19	(29,19,16)
35	7.31+29	7.31+31	(31,25,8)	(31,19,16)
37	7.37+1	(31,29,16)	(31,31,16)	Lemma 5.5
39	Lemma 5.5	7.37+17	7.37+19	(37,13,8)
41	7.41+1	7.37+31	(37,17,16)	(37,19,16)
43	7.43+1	7.41+17	7.41+19	(41,13,8)
45	7.41+29	7.43+17	7.43+19	(41,19,16)
47	7.47+1	7.43+31	(41,31,16)	(41,17,32)
49	7.49+1	7.47+17	7.47+19	(47,13,8)
51	7.47+29	7.49+17	7.49+19	(47,27,8)
53	7.53+1	7.49+31	7.47+47	(47,41,8)
55	7.49+43	7.53+17	7.53+19	7.49+49
57	7.53+29	7.53+31	(53,25,8)	(53,27,8)
59	7.59+1	(53,37,8)	7.53+47	7.53+49
61	7.61+1	7.59+17	7.59+19	(53,47,16)
63	7.59+29	7.61+17	7.61+19	(61,13,8)
65	7.61+29	7.61+31	7.59+47	7.59+49
67	7.67+1	7.59+59	7.61+47	7.61+49
69	7.61+57	7.67+17	7.67+19	(67,13,8)
71	7.71+1	7.97+31	(67,17,16)	(67,19,16)
73	7.73+1	7.71+17	7.71+19	7.67+49

n	7n+9	7n+11	7n+13
9	7.9+9	?	?
11	7.11+9	7.11+11	?
13	7.13+9	7.13+11	7.13+13
15	Lemma 5.1	Lemma 5.1	Lemma 5.1
17	7.17+9	7.17+11	7.17+13
19	7.19+9	7.19+11	7.19+13
21	Lemma 5.3	(19,17,8)	(19,19,8)
23	7.23+9	7.23+11	7.23+13
25	7.25+9	7.25+11	7.25+13
27	7.27+9	7.27+11	7.27+13
29	7.29+9	7.29+11	7.29+13
31	7.31+9	7.31+11	7.31+13
33	7.31+23	7.31+25	7.31+27
35	(31,29,8)	(31,23,16)	(31,25,16)
37	7.37+9	7.37+11	7.37+13
39	7.37+23	7.37+25	7.37+27
41	7.41+9	7.41+11	7.41+13
43	7.43+9	7.43+11	7.43+13
45	7.43+23	7.43+25	7.43+27
47	7.47+9	7.47+11	7.47+13
49	7.49+9	7.49+11	7.49+13
51	7.49+23	7.49+25	7.49+27
53	7.53+9	7.53+11	7.53+13
55	7.53+23	7.53+25	7.53+27
57	7.53+37	(53,31,8)	7.53+41
59	7.59+9	7.59+11	7.59+13
61	7.61+9	7.61+11	7.61+13
63	7.61+23	7.61+25	7.61+27
65	7.61+37	7.59+53	7.61+41
67	7.67+9	7.67+11	7.67+13
69	7.67+23	7.67+25	7.67+27
71	7.71+9	7.71+11	7.71+13
73	7.73+9	7.73+11	7.73+13

Combining Theorems B and C, we obtain Theorem 1.1.

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References

[1] J.W. Brown, F. Cherry, L. Most, M. Most, E.T. Parker and W.D. Wallis. *The spectrum of orthogonal diagonal Latin squares*, in *Graphs, Matrices and Designs* (ed. R. Rees), Marcel Dekker, to appear.

[2] B. Du, *Some constructions of pairwise orthogonal diagonal Latin squares*, JCMCC, 9 (1991), 97-106.

[3] B. Du, *Four pairwise orthogonal diagonal Latin squares*, Utilitas Math., to appear.

[4] E. Gergely, *A remark on doubly diagonalized orthogonal Latin squares*, Discrete Math., 10 (1974), 185-188.

[5] K. Heinrich and A.J.W. Hilton, *Doubly diagonal orthogonal Latin squares*, Discrete Math., 46 (1983), 173-182.

[6] W.D. Wallis, *Three new orthogonal diagonal Latin squares*, in *Enumeration and Design*, Academic Press Canada, 1984, 313-315.

[7] W.D. Wallis and L. Zhu, *Four pairwise orthogonal diagonal Latin squares of order 12*, Utilits Math., 21C (1982), 205-207.

[8] W.D. Wallis and L. Zhu, *Some bounds for pairwise orthogonal diagonal Latin squares*, Ars Combinatoria 17A (1984), 353-366.

[9] R.M. Wilson, *Constructions and uses of pairwise balanced designs*, Math. Centre Tracts 55 (1974), 18-41.

[10] L. Zhu, *Orthogonal diagonal Latin squares of order fourteen*, J. Austral Math. Soc. (A), 36 (1984), 1-3.

[11] L. Zhu, *Three pairwise orthogonal diagonal Latin squares*, JCMCC 5 (1989), 27-40.

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