ORIENTABLE AND NON-ORIENTABLE MAPS WITH GIVEN AUTOMORPHISM GROUPS

JOZEF ŠIRÁŇ and MARTIN ŠKOVIERA

FACULTY OF MATHEMATICS AND PHYSICS
COMENIUS UNIVERSITY, 842 15 BRATISLAVA
CZECHOSLOVAKIA
e-mail: siran@mff.uniba.cs, skoviera@mff.uniba.cs

ABSTRACT. We prove that for every finite group G there exist orientable as well as non-orientable maps with automorphism group isomorphic to G.

1. Introduction

Whenever combinatorial or other mathematical structures are investigated, the question of their isomorphisms, and hence of their automorphisms, arises naturally. This justifies the study of their automorphism groups. There are a number of results about various classes of combinatorial structures saying that every finite group is the automorphism group of some member of the class. Examples are provided by graphs [5], cubic graphs [6], Steiner triple systems [8], "pictures" [1], and others. Results of this type indicate that a given class is, to some extent, rich. On the other hand, there are some very natural classes that do not have this property, for instance, trees [4].

Similar questions have been asked in connection with graph embeddings on surfaces. As was proved in [3], every finite group is the automorphism group of some map on an orientable surface. However, it is by no means obvious that the same holds for non-orientable maps. The purpose of this paper is to answer this question in the affirmative.

Throughout, a *map* is a connected graph cellularly embedded in some closed surface. The map is *orientable* (*non-orientable*) if so is the supporting surface. An *automorphism* of a map is a mapping which sends vertices to vertices, edges to edges, faces to faces and preserves their incidence; if the surface is orientable it also preserves the orientation. A mapping that preserves the incidence of vertices, edges, and faces of an orientable map but reverses the orientation is called a *reflection*.

2. The orientable case

The aim of this section is to prove that every finite group G is the group of (orientation preserving) automorphisms of some orientable map. As mentioned in Introduction, this result has already been proved by Cori and Machi [3]. Their proof is based on the idea of a hypermap which is then replaced by the corresponding bipartite map embedded in the same surface [11]. We adopt a more direct approach by modifying the original ideas of Frucht [5]: We take a suitable embedding of a Cayley graph of G and subsequently incorporate the features of direction and colouring in a new map without changing the automorphism group. Besides simplicity, this method has the advantage that it applies also to the non-orientable case.

Let us start by taking the (right) Cayley graph $C(G,\Omega)$ for the group G with respect to a generating set Ω of G. We always assume that Ω does not contain the group identity e and that $\Omega^{-1} = \Omega$. Recall that $C(G,\Omega)$ has vertex set G and arc set $G \times \Omega$. The initial vertex of an arc $(g,r) \in G \times \Omega$ is g and the reverse of (g,r) is the arc $(gr, r^{-1}) \in G \times \Omega$. The element r is referred to as the *colour* of the arc (g,r). It is well known that for each $h \in G$ the mapping φ_h defined by $\varphi_h(g,r) = (hg,r)$ is a colour-preserving automorphism of the graph $C(G,\Omega)$. Moreover, every colour-preserving automorphism of $C(G,\Omega)$ has this form, and hence the group of all colour-preserving automorphisms of $C(G,\Omega)$ is isomorphic to G (see, for instance [12]).

We now construct an orientable map with underlying graph $C(G,\Omega) = K$. In order to do this we specify for every vertex $g \in G$ the *local rotation* P_g at g, i.e., a cyclic permutation of arcs emanating from g. Choose P_e arbitrarily and for $g \in G$ set $P_g = \varphi_g P_e \varphi_g^{-1}$. The product $P = \prod_{g\in G} P_g$ is a *rotation* of K describing a 2-cell embedding $M = (K,P)$ of K on some orientable surface. It is easy to see that each local rotation P_g induces the same cyclic permutation p of Ω and therefore M coincides with the Cayley map $M(G,\Omega,p)$ in the sense of Biggs and White [2, p. 117]. The following observation can be found in [12, Theorem 5.3.4].

Proposition 1. *For every $h \in G$, the mapping φ_h is an automorphism of the map $M = (C(G,\Omega),P)$ described above. Consequently, the group of all colour-preserving automorphisms of M is isomorphic to G.*

Proof. Since the group of all colour-preserving automorphisms of the graph $K = C(G,\Omega)$ is isomorphic to G we only need to show that φ_h is a map automorphism, which is easily seen to be equivalent with the fact that $\varphi_h P = P\varphi_h$. The latter is proved by the following computation:

$$\varphi_h P(g,r) = \varphi_h(g,p(r)) = (hg,p(r)) = P(hg,r) = P\varphi_h(g,r). \quad \square$$

With help of this proposition it is now easy to establish the main result of this section.

Theorem 2. *For every finite group G there exists an orientable map M such that the group of (orientation-preserving) automorphisms of M is isomorphic to G.*

Proof. Clearly we may assume that G is non-trivial. Let $K = C(G, \Omega)$ be a Cayley graph of the group G and let M be a Cayley map constructed as above. Obviously, there exists a subset $\Omega' \subset \Omega$ with the following two properties: (1) for every $x \in \Omega$, either x or x^{-1} is in Ω', and (2) if both x and x^{-1} belong to Ω' then x is an involution. (Note that specifying the set Ω' results in assigning preferred orientation to edges that correspond to non-involutory generators.) Now, for every generator $r \in \Omega'$ choose a fixed 3-connected planar map H_r with two distinguished vertices u_r and v_r such that:

(i) u_r and v_r lie in the outer face of the map,

(ii) if r is not an involution then H_r has no non-trivial automorphisms fixing the set $\{u_r, v_r\}$,

(iii) if r is an involution then H_r has precisely one non-trivial map automorphism fixing the set $\{u_r, v_r\}$, and this automorphism interchanges u_r and v_r, and

(iv) the maps H_r, $r \in \Omega'$ are pairwise non-isomorphic.

One of the possibilities to define an infinite series of such H_r's is suggested in Figs. 1 and 2. Further, let us choose a fixed orientation of the supporting surface of the Cayley map M and, similarly, fix an orientation of every map H_r.

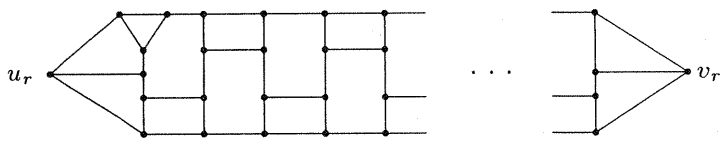

Fig. 1. H_r, r **non-involutory**

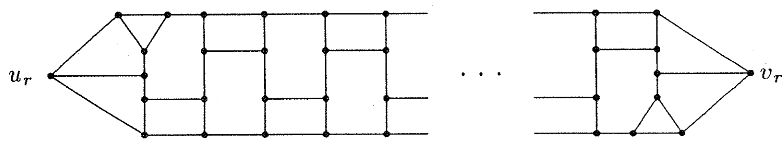

Fig. 2. H_r, r **involutory**

We proceed by constructing a new map M' from M, embedded in the same oriented surface S: Replace a small neighbourhood of each edge of colour $r \in \Omega'$ (i.e., a strip along every edge of the form (g, gr), $g \in G$) by a copy of the map H_r in such way that u_r is identified with g and v_r with gr, and so that the orientations of H_r and S agree (Fig. 3). It is easy to see that our construction of M' guarantees that the (orientation-preserving) map automorphisms of M' are in 1-1 correspondence with the colour-preserving map automorphisms of M. The rest is a consequence of Proposition 1. $\square$

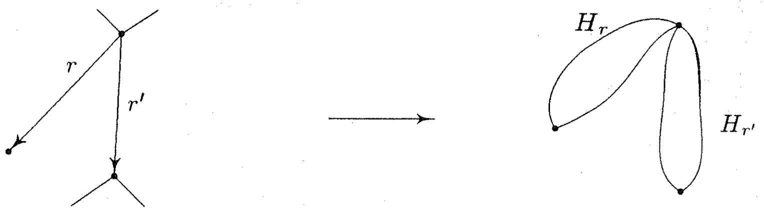

Fig. 3.

The just described method enables us to construct infinite classes of maps with prescribed automorphism group, satisfying several additional requirements. As an example we prove:

Corollary 3. *For every finite group G there exist infinitely many cubic 3-connected orientable maps whose automorphism group is isomorphic to G.*

Proof. Consider the maps M' constructed in the preceding proof, with the H_r's as suggested in Figs. 1 and 2. Now, for each vertex v of M' at which copies of H_r's are attached, do the following: Expand v to a cycle C_v on the supporting surface in such a way that the cyclic order of edges originally incident with v remains the same, see Fig. 4. It is an easy exercise to verify that the automorphism group of the resulting 3-connected cubic map is isomorphic to G. □

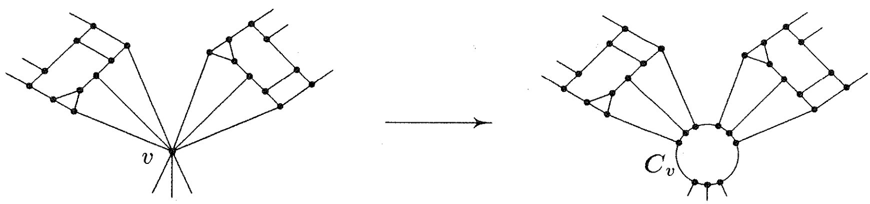

Fig. 4.

3. The non-orientable case

In this section we prove that every finite group G is the automorphism group of some non-orientable map. Before doing this we briefly recall the description of non-orientable 2-cell embeddings by means of generalized embedding schemes [9, 10].

Let K be a connected graph. As before, let P be a rotation of K, i.e., $P = \prod_{v\in V(G)} P_v$ where P_v is a cyclic permutation of arcs emanating from v. In addition, let λ be a voltage assignment on K, with values in the multiplicative group $\{1, -1\}$ subject to the condition that $\lambda(x^{-1}) = \lambda(x)$. The pair (P, λ) is called *generalized embedding scheme* and determines a map $M = (K, P, \lambda)$ of K in a surface S in the *embedding scheme* and determines a map $M = (K, P, \lambda)$ of K in a surface S in the following way. Let x and y be two arcs of K incident with the same vertex and such that $P^{\varepsilon(x)}(x^{-1}) = y$ for some $\varepsilon(x) \in \{1, -1\}$. Then the sequence xy forms a portion of the boundary of some face F of the embedding. The successor z of the arc y on the boundary of F is determined by the formula $z = P^{\varepsilon(y)}(y^{-1})$, where $\varepsilon(y) = \varepsilon(x)\lambda(y)$. Repeatedly applying this rule we eventually compute the boundaries of all faces of the embedding. To be more precise, this procedure yields for every face F two sequences of arcs representing its boundary, namely $(...xyz...)$ and $(...z^{-1}y^{-1}x^{-1}...)$. Geometrically this corresponds to two possible ways of tracing the boundary of F. The fact that every 2-cell embedding in some closed surface admits such a description is proved in [9, 10] or [7]. We remark that the surface supporting the embedding given by the scheme (P, λ) is non-orientable if and only if there exists a cycle in the graph K containing an odd number of edges with voltage -1.

The occurence of face boundaries in pairs has deeper topological reasons. To explain it, we construct for our graph K with an embedding given by (P, λ) an orientable "antipodal" embedding as follows. Let K^λ be a new graph with vertex set $V(K) \times \{1, -1\}$ and edge set $E(K) \times \{1, -1\}$; if x is an arc of K with initial vertex u and terminal vertex v then the arc (x, i), $i \in \{1, -1\}$, has initial vertex (u, i) and terminal vertex $(v, i\lambda(x))$. Now, consider the embedding of K^λ in an orientable surface, given by the rotation

$$P^\lambda(x, i) = (P^i(x), i).$$

It was shown in [10] that the projection $\pi : K^\lambda \to K$ erasing the second coordinate extends to an (unbranched) double covering of the map $M = (K, P, \lambda)$ by the *derived* map $M^\lambda = (K^\lambda, P^\lambda)$. Equivalently we can say that every face $F = (...xyz...)$ of M *lifts* to two oppositely oriented faces in M^λ. If one of them is $\tilde{F} = (...(x, i), (y, j), (z, k), ...)$ then the other (its *mate*) is $m\tilde{F} = (...(z, -k)^{-1}, (y, -j)^{-1}, (x, -i)^{-1}, ...)$. Of course, $m(m\tilde{F}) = \tilde{F}$.

Now we have enough means to construct non-orientable maps with prescribed automorphism groups. Let G be a finite group. If G is trivial or isomorphic to $\mathbb{Z}_2$ then the existence of the required maps is obvious. Thus, assume that the order of G is at least 3. Then we can take a generating set Ω for G which with some two generators s and t also contains their product st. Let p be a cyclic permutation of Ω. Define the rotation P for the Cayley graph $K = C(G, \Omega)$ by setting $P(g, r) = (g, p(r))$ and the voltage assignment λ as follows: $\lambda(g, r) = -1$ if and only if $r = s$ or $r = s^{-1}$. Observe that the triangle $(e, s), (s, t), (st, (st)^{-1})$ contains exactly one arc with negative voltage and therefore the map $M = (K, P, \lambda)$ is non-orientable.

Continuing in our construction, define for each $h \in G$ two mappings $\varphi_{h,j}$, $j \in \{1, -1\}$, of the arc-set of the derived map M^λ by setting

$$\varphi_{h,j}((g, r), i) = ((hg, r), ji) \ , \ i \in \{-1, 1\}.$$

Lemma 4. *For each $h \in G$, the mapping $\varphi_{h,1}$ is an automorphism of the map M^λ while $\varphi_{h,-1}$ is a reflection of M^λ.*

Proof. The first statement follows from the identity $\varphi_{h,1}P^\lambda = P^\lambda\varphi_{h,1}$ which can be proved in the same way as done in Proposition 1. The second statement is easily seen to be equivalent with the identity $\varphi_{h,-1}P^\lambda = (P^\lambda)^{-1}\varphi_{h,-1}$ which is proved in the following lines:

$$\varphi_{h,-1}P^\lambda((g,r),i) = \varphi_{h,-1}(P^i(g,r),i) = \varphi_{h,-1}((g,p^i(r)),i) = ((hg,p^i(r)),-i) =$$

$$(P^i(hg,r),-i) = (P^{(-1)(-i)}(hg,r),-i) = (P^\lambda)^{-1}((hg,r),-i) =$$

$$(P^\lambda)^{-1}\varphi_{h,-1}((g,r),i). \quad \square$$

Using the idea of the double covering we easily obtain the following:

Lemma 5. *The group of all colour-preserving automorphisms of the non-orientable map $M = (C(G,\Omega),P,\lambda)$ is isomorphic to G.*

Proof. Let F be a face of M and let $\tilde{F}$ and $m\tilde{F}$ be the lifts of F in M^λ. Routine calculations show that for each $h \in G$ and each $j \in \{-1,1\}$, $\varphi_{h,j}$ maps the set $\{\tilde{F}, m\tilde{F}\}$ onto the set $\{\varphi_{h,j}(\tilde{F}), m\varphi_{h,j}(\tilde{F})\}$. In other words, $\varphi_{h,j}$ preserves the mates. This altogether shows that the automorphism $\varphi_h : (g,r) \mapsto (hg,r)$ of $C(G,\Omega)$ is at the same time an automorphism of the non-orientable map $(C(G,\Omega),P,\lambda)$. Since $C(G,\Omega)$ does not contain any other colour-preserving automorphisms, our lemma follows. $\square$

Analogously as in the previous section we form the subset $\Omega' \subseteq \Omega$ and for each $R \in \Omega'$ we replace every edge of the form (g,gr) by a copy of the map H_r satisfying the properties (i)-(iv) listed in the proof of Theorem 2, see also Figs. 1, 2 and 3. Thus we obtain

Theorem 6. *For every finite group G there exists a non-orientable map whose automorphism group is isomorphic to G. Moreover, the underlying graph can be chosen to be cubic.* $\square$

References

1. G.Behrendt, *Automorphism groups of pictures*, J. Graph Theory **14** (1990), 423-426.
2. N.Biggs and A.T.White, *Permutation Groups and Combinatorial Structures*, London Math. Soc. Lect. Notes, Cambridge Univ. Press, Cambridge, 1979.
3. R.Cori and A.Machi, *Construction of maps with prescribed automorphism group*, Theoret. Comp. Sci. **21** (1982), 91-98.
4. V.Z.Feinberg, *Automorphism groups of trees*, Dokl. Akad. Nauk BSSR **13** (1969), 1065-1067.
5. R.Frucht, *Herstellung von Graphen mit vorgegebener abstrakten Gruppe*, Compositio Math. **6** (1938), 239-250.
6. R.Frucht, *Graphs of degree three with a given abstract group*, Canadian J. Math. **1** (1949), 365-378.

7. J.L.Gross and T.W.Tucker, *Topological Graph Theory*, Wiley, New York, 1987.
8. E.Mendelsohn, *On the group of automorphisms of Steiner triple and quadruple systems*, J. Combin. Theory (A) **25** (1978), 97-104.
9. G.Ringel, *The combinatorial map color theorem*, J. Graph Theory **1** (1977), 141-155.
10. S.Stahl, *Generalized embedding schemes*, J. Graph Theory **2** (1978), 41-52.
11. T.R.S. Walsh, *Hypermaps versus bipartite maps*, J. Combin. Theory (B) **18** (1975), 155-163.
12. A.T.White, *Graphs, Groups and Surfaces (Second Edition)*, North Holland, Amsterdam, 1984.

(Received 30/3/92)