IN SEARCH OF 4 — (12, 6, 4) DESIGNS: PART I

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Abstract

As a first step towards finding all 4–(12, 6, 4) designs which are not 5–(12, 6, 1) designs, it is shown that if such a design has a pair of blocks with five points in common, then there is a unique way of assigning the replicas of the seven points from that pair of blocks to the other blocks of the design.

1. Introduction

A $t-(v,k,\lambda)$ is a collection of subsets, called *blocks*, of a set S with v elements, called *points*, such that every t-subset of S is contained in precisely λ blocks. If s is a whole number such that $0 \leq s \leq t$, then a t–design is also an s–design with, of course, a different λ value. Thus the 5–(12, 6, 1) design, which is unique in structure, is also a 4–(12, 6, 4) design. However the converse is not true; there are 4–(12, 6, 4) designs which are not 5–(12, 6, 1) designs. Nine non-isomorphic such designs are constructed in Breach, Elmes, Sharry and Street [1]. In this series of papers we show, by a different construction, that, together with the 5–(12, 6, 1) design, these are the only 4–(12, 6, 4) designs. The work proceeds in three major steps. In Part I it is shown that if the design is not a 5–design, then seven of the points determine a unique skeleton for a 4–(12, 6, 4) design. In Part II [2] it is shown that 32 blocks of the skeleton can be completed in a unique fashion, and embedded in these blocks are the 30 blocks of a 3–(10, 4, 1) design. In Part III [3] it is shown that the 100 partially completed blocks left at the end of Part II can be completed in 47 ways. Each of the resulting 4–(12, 6, 4) designs belongs to one of nine equivalence classes.

2. Parameters and Block Types for 4 —(12, 6, 4) Designs

For any $t-(v,k,\lambda)$ design let λ_i be the number of times each i-subset of the v points appears in the design. Thus $\lambda_0 = b$ is the number of blocks; $\lambda_1 = r$ is the number of replicas of each point; and $\lambda_t = \lambda$. Then it is well-known that

$$\lambda_i = \frac{(v-i)(v-i-1)\dots(v-t+1)}{(k-i)(k-i-1)\dots(k-t+1)}\lambda; \qquad \text{where } 0 \le i \le t.$$

For a 4–(12, 6, 4) design we have

$$\lambda_0 = b = 132, \quad \lambda_1 = r = 66, \quad \lambda_2 = 30, \quad \lambda_3 = 12, \quad \lambda_4 = 4.$$

Let S be any 6–subset of the 12 points of a 4–(12, 6, 4) design and let b_i be the number of blocks intersecting S in exactly i points. Then by counting successively blocks, point occurences, pair occurences, etc, in two ways, we have for the *block intersection numbers* the equations;

$$\begin{array}{rrrrrrrcr} b_0 + & b_1 + & b_2 + & b_3 + & b_4 + & b_5 + & b_6 & = & 132, \\ & b_1 + & 2b_2 + & 3b_3 + & 4b_4 + & 5b_5 + & 6b_6 & = & 396, \\ & & 2b_2 + & 6b_3 + & 12b_4 + & 20b_5 + & 30b_6 & = & 900, \\ & & & 6b_3 + & 24b_4 + & 60b_5 + & 120b_6 & = & 1440, \\ & & & & 24b_4 + & 120b_5 + & 360b_6 & = & 1440. \end{array}$$

From these $b_0 = 6 - b_5 - 5b_6$. But $b_6 \geq 1$ so $b_6 = 1$ and there are no repeated blocks. Therefore

$$b_0 = 1 - b_5,\ b_1 = 5b_5,\ b_2 = 45 - 10b_5,\ b_3 = 40 + 10b_5, b_4 = 45 - 5b_5,$$
$$b_5 = 0 \text{ or } 1;$$

and only two solution sets are possible. They are :

	b_0	b_1	b_2	b_3	b_4	b_5	b_6
Type I	1	0	45	40	45	0	1
Type II	0	5	35	50	40	1	1 .

The blocks of either type occur in pairs. A block of Type I is disjoint from just one other block. A block of Type II has five points in common with just one other block and intersects all other blocks. Two blocks of Type II with five points in common are said to be *friendly* blocks.

THEOREM: *If all the blocks of a* 4–(12, 6, 4) *design are of Type I then the design is a* 5–(12, 6, 1) *design.*

Proof: No quintuple of points can occur more than once in the design. The 132 blocks of the design each contain $\binom{6}{5} = 6$ distinct quintuples so the design contains 792 distinct quintuples in all. But $\binom{12}{5} = 792$ so each possible quintuple occurs just once and the design must be a 5–(12, 6, 1) design. □

The design is the well-known 5–(12, 6, 1) design associated with the Mathieu group M_{12}.

As a by-product of the work above we have a short proof of a result of Dehon[4] and Oberschelp[5].

THEOREM: *A* 4–(12, 6, 2) *design cannot exist.*

Proof: If such a design did exist then, by the duplication of each block, a 4–(12, 6, 4) design with repeated blocks could be made; but no 4–(12, 6, 4) design can have repeated blocks. □

3. The Distribution of the Seven Distinct Points from a Pair of Type II Blocks.

To investigate 4–(12, 6, 4) designs which are not 5–(12, 6, 1) designs it must be assumed that such designs have at least one pair of Type II blocks. In that case there is at least one quintuple of points which occurs twice in the design. We take the pair of blocks to be **[123456]** and **[123457]**. The two points, **6** and **7**, which do not belong to both of these friendly blocks, play special roles and will be called *prongs*. Thus each pair of Type II blocks has associated with it a *prong pair*, in this case **67**. The intention is to show that that the seven points associated with a pair of Type II blocks are distributed over the other blocks in a unique pattern. This gives a unique skeleton for a 4–(12, 6, 4) design in which the placing of the seven points **1, 2, 3, 4, 5, 6, 7** is determined. The remaining five points, **8, 9, a, b, c** say, then have to be judiciously inserted to complete the design. Towards this end we embark on a series of lemmas.

LEMMA: *Let N_5 be the number of blocks containing a given quintuple of points. Let E_0 be the number of blocks containing none of the quintuple. Then*

$$E_0 = 2 - N_5.$$

Proof: By the principle of inclusion and exclusion we have

$$\begin{aligned} E_0 &= b - \binom{5}{1} r + \binom{5}{2}\lambda_2 - \binom{5}{3}\lambda_3 + \binom{5}{4}\lambda_4 - N_5 \\ &= 132 - 5.66 + 10.30 - 10.12 + 5.4 - N_5 \\ &= 2 - N_5. \ \square \end{aligned}$$

LEMMA: *Let* **[123456]** *and* **[123457]** *be a pair of friendly blocks in a* 4–(12, 6, 4) *design. Then any block intersecting one of the pair in just one point contains the prong of the other.*

Proof: The quintuple **23456** cannot occur elsewhere in the design. By the previous lemma, $E_0 = 2 - N_5 = 1$, so there is one block not containing any of **2, 3, 4, 5, 6**. But the type II block **[123456]** intersects all other blocks. Therefore there must be a block A, **[1** . . . . .], not containing any of **2, 3, 4, 5, 6**. Now, there are two blocks not containing any of **8, 9, a, b, c**, namely the given pair of type II blocks. Therefore, by the previous lemma again, **8, 9, a, b, c** cannot all be in the same block; for otherwise $E_0 \neq 2$. Hence the block A must contain **7** which is the prong of **[123457]**. □

LEMMA: *If a* 4–(12, 6, 4) *design contains the pair of type* II *blocks* **[123456]** *and* **[123457]** *then it contains the ten blocks*
[16 . . . .], **[26** . . . .], **[36** . . . .], **[46** . . . .], **[56** . . . .],
[17 . . . .], **[27** . . . .], **[37** . . . .], **[47** . . . .], **[57** . . . .],
in which the dots represent elements from the set {**8, 9, a, b, c**}.
Proof: This is a direct consequence of the previous lemma since **1,2,3,4,5** are interchangeable and **6** and **7** are interchangeable. □

LEMMA: *If a* 4–(12, 6, 4) *design contains friendly blocks* **[123456]** *and* **[123457]** *then it contains the ten blocks*
[167 . . .], **[267** . . .], **[367** . . .], **[467** . . .], **[567** . . .],
[167 . . .], **[267** . . .], **[367** . . .], **[467** . . .], **[567** . . .],
in which the dots represent elements from the set {**8, 9, a, b, c**}.
Proof: If E_0 is the number of blocks not containing any of the quadruple **2, 3, 4, 5**, then by the principle of inclusion and exclusion,

$$E_0 = 132 - \binom{4}{1}r + \binom{4}{2}\lambda_2 - \binom{4}{3}\lambda_3 + \binom{4}{4}\lambda_4 = 132 - 264 + 180 - 48 + 4 = 4.$$

But all the blocks in the design must contain members of the quintuple **1, 2, 3, 4, 5**. Therefore the four blocks not containing the quadruple **2, 3, 4, 5**, all contain **1**. Two of these four blocks are the blocks **[16** . . . .], **[17** . . . .] of the previous lemma. The remaining two, which must intersect both of **[123456]** and **[123457]** in at least two points, must be **[167** . . .] and **[167** . . .] where the dots represent elements of the set {**8, 9, a, b, c**}. Cycling through **1, 2, 3, 4, 5** produces the ten blocks in the statement of the lemma. □

LEMMA: *If a* 4–(12, 6, 4) *design contains the pair of friendly blocks* **[123456]** *and* **[123457]** *then the prongs,* **6** *and* **7**, *cannot be orphans; that is to say, they cannot appear on a block either individually or together unless accompanied by at least one of the non-prong points* **1, 2, 3, 4, 5.**
Proof: This is a consequence of the two previous lemmas. □

4. The Proto-skeleton

Given the pair of friendly blocks **[123456]** and **[123457]**, it is a matter of simple counting to determine the distribution of the six points **1, 2, 3, 4, 5, 6** over the rest of the blocks in the design. With the help of the previous lemmas some of the **7**'s can also be placed. To facilitate further progress, in Table 1 there is presented a *proto-skeleton* in which the two specified friendly blocks are in section A and the other 130 partially

A
123456
123457

B
1234 . .
1234 . .
1235 . .
1235 . .
1245 . .
1245 . .
1345 . .
1345 . .
2345 . .
2345 . .

C	
1236 . .	1456 . .
1236 . .	1456 . .
1236 . .	1456 . .
123 . . .	145 . . .
123 . . .	145 . . .
123 . . .	145 . . .
1246 . .	2346 . .
1246 . .	2346 . .
1246 . .	2346 . .
124 . . .	234 . . .
124 . . .	234 . . .
124 . . .	234 . . .
1256 . .	2356 . .
1256 . .	2356 . .
1256 . .	2356 . .
125 . . .	235 . . .
125 . . .	235 . . .
125 . . .	235 . . .
1346 . .	2456 . .
1346 . .	2456 . .
1346 . .	2456 . .
134 . . .	245 . . .
134 . . .	245 . . .
134 . . .	245 . . .
1356 . .	3456 . .
1356 . .	3456 . .
1356 . .	3456 . .
135 . . .	345 . . .
135 . . .	345 . . .
135 . . .	345 . . .

D	
126 . . .	246 . . .
126 . . .	246 . . .
12	24 . . . .
12	24 . . . .
136 . . .	256 . . .
136 . . .	256 . . .
13	25 . . . .
13	25 . . . .
146 . . .	346 . . .
146 . . .	346 . . .
14	34 . . . .
14	34 . . . .
156 . . .	356 . . .
156 . . .	356 . . .
15	35 . . . .
15	35 . . . .
236 . . .	456 . . .
236 . . .	456 . . .
23	45 . . . .
23	45 . . . .

E
167 . . .
167 . . .
267 . . .
267 . . .
367 . . .
367 . . .
467 . . .
467 . . .
567 . . .
567 . . .

F
16 . . . .
17 . . . .
26 . . . .
27 . . . .
36 . . . .
37 . . . .
46 . . . .
47 . . . .
56 . . . .
57 . . . .

Table 1 The Proto-Skeleton

completed blocks are grouped into sections B, C, D, E and F. The blocks of B intersect those of A in four points but do not contain **6** or **7**. The blocks of C contain just three of **1**, **2**, **3**, **4**, **5** and may contain **6** or **7** or both. The blocks of D contain just two of **1**, **2**, **3**, **4**, **5** and may contain **6** or **7** or both. The blocks of E and F contain just one of **1**, **2**, **3**, **4**, **5** and are the blocks described in the previous two lemmas. The positions of the **6**'s within sections C and D are all determined. The placings of the **7**'s within these sections are not so easily decided. Within each subsection of section C the point **7** must appear just three times; within each subsection of D the point **7** must appear just twice. These appearances are subject to each quadruple of points appearing just four times in the completed design.

Here we make use of the rule of five (RF for short). This rule, which encapsulates a very basic priciple, is used frequently in the subsequent work.

LEMMA: **The rule of five;** *Given any three blocks of a* 4–(12, 6, 4) *design, at most two of them can have five points in common.*

Proof: The blocks are are of either Type I or of Type II and any one of them can intersect at most one other block of the design in at most five points. □

An immediate application of the RF proves the following lemma.

LEMMA: *Each subsection of section* C *of the proto-skeleton can contain the pair* **67** *at most twice.* □

LEMMA: *In section* C *of the proto-skeleton there are exactly* 10 *blocks containing both* **6** *and* **7**.

Proof: A quadruple from the set {**8, 9, a, b, c**} can be chosen in five ways. Consider the four points **8, 9, a, b**. There are four blocks that contain all of them. The complement of a 4–(12, 6, 4) design is also a 4–(12, 6, 4) design. Therefore there are four blocks that contain none of **8, 9, a, b**. Two of these blocks are [**123456**] and [**123457**]. The other two blocks must be in section C and must contain three of **1, 2, 3, 4, 5** together with **6, 7,** and **c**. □

With these constraints a computer search showed that there are twenty non-isomorphic ways of inserting the **7**'s into sections C and D. The following parity argument ruled out fifteen of these ways.

LEMMA: *Let* **x** *be one of* **1, 2, 3, 4, 5.** *Then the triple* **x67** *cannot occur an odd number of times in section* C *of the proto-skeleton.*

Proof: Each of the five quadruples **167y**, where **y** ∈ {**8, 9, a, b, c**}, occurs four times in the design, making twenty such quadruples in all. Six of these occur in the first two blocks of section E. Any block of section D containing **167** contains just two of **8, 9, a, b, c**. Any block of section C containing **167** contains just one of **8, 9, a, b, c**. Therefore the number of appearances of **167** in section C must be even. □

One of the five surviving ways contains, in section C, the six-block configuration

[**12367** .],	[**12467** .],	[**13467** .],
[**12367** .],	[**12467** .],	[**13467** .].

Of these blocks, those with **12** require four distinct points from {**8, 9, a, b, c**} for completion if the RF is to hold. But then the remaining two blocks cannot be completed without breaking the RF or avoiding repeated blocks. Thus the number of non-isomorphic ways of inserting the **7**'s into sections C and D is reduced to four, as given in the patterns of Table 2. Once the **7**'s have been placed in section C, the placement of the remaining **7**'s, in section D, is forced. Therefore in Table 2 the patterns for section C only are presented. Since section C must contain ten copies of the pair **67** with at most two copies per subsection, and supposing that these copies are in the first two blocks of the subsection, it suffices to give the first two blocks only for each subsection to determine the whole pattern. The four patterns are given in Table 2.

Pattern I	Pattern II	Pattern III	Pattern IV
12367 .	12367 .	12367 .	12367 .
1236 . .	12367 .	12367 .	12367 .
12467 .	12467 .	12467 .	12467 .
1246 . .	12467 .	1246 . .	1246 . .
12567 .	1256 . .	12567 .	1256 . .
1256 . .	1256 . .	1256 . .	1256 . .
13467 .	1346 . .	13467 .	1346 . .
1346 . .	1346 . .	1346 . .	1346 . .
13567 .	13567 .	13567 .	13567 .
1356 . .	13567 .	1356 . .	1356 . .
14567 .	1456 . .	1456 . .	14567 .
1456 . .	1456 . .	1456 . .	14567 .
23467 .	2346 . .	2346 . .	23467 .
2346 . .	2346 . .	2346 . .	2346 . .
23567 .	2356 . .	2356 . .	23567 .
2356 . .	2356 . .	2356 . .	2356 . .
24567 .	24567 .	24567 .	24567 .
2456 . .	24567 .	24567 .	2456 . .
34567 .	34567 .	34567 .	34567 .
3456 . .	34567 .	34567 .	3456 . .

Table 2 The four inequivalent ways of placing the ten 67 pairs in section C. Only the first two blocks of each subsection are given.

5. The Skeleton for a 4–(12, 6, 4) Design

The four patterns can be arrived at "by hand,"at the peril of overlooking a case. This was the way they were first produced. A reasonably short proof that Pattern II is not viable is available although it is not given here. Before doing the computer run we knew that Pattern I can be completed to a 4–(12, 6, 4) design as we already had examples of such designs. None of the known designs seemed to have Patterns II, III and IV embedded in it

A

123456	
123457	

B

1234 . .	
1234 . .	
1235 . .	
1235 . .	
1245 . .	
1245 . .	
1345 . .	
1345 . .	
2345 . .	
2345 . .	

C

12367 .	14567 .
1236 . .	1456 . .
1236 . .	1456 . .
1237 . .	1457 . .
1237 . .	1457 . .
123 . . .	145 . . .
12467 .	23467 .
1246 . .	2346 . .
1246 . .	2346 . .
1247 . .	2347 . .
1247 . .	2347 . .
124 . . .	234 . . .
12567 .	23567 .
1256 . .	2356 . .
1256 . .	2356 . .
1257 . .	2357 . .
1257 . .	2357 . .
125 . . .	235 . . .
13467 .	24567 .
1346 . .	2456 . .
1346 . .	2456 . .
1347 . .	2457 . .
1347 . .	2457 . .
134 . . .	245 . . .
13567 .	34567 .
1356 . .	3456 . .
1356 . .	3456 . .
1357 . .	3457 . .
1357 . .	3457 . .
135 . . .	345 . . .

D

1267 . .	2467 . .
126 . . .	246 . . .
127 . . .	247 . . .
12	24 . . . .
1367 . .	2567 . .
136 . . .	256 . . .
137 . . .	257 . . .
13	25 . . . .
1467 . .	3467 . .
146 . . .	346 . . .
147 . . .	347 . . .
14	34 . . . .
1567 . .	3567 . .
156 . . .	356 . . .
157 . . .	357 . . .
15	35 . . . .
2367 . .	4567 . .
236 . . .	456 . . .
237 . . .	457 . . .
23	45 . . . .

E

167 . . .	
167 . . .	
267 . . .	
267 . . .	
367 . . .	
367 . . .	
467 . . .	
467 . . .	
567 . . .	
567 . . .	

F

16	
17	
26	
27	
36	
37	
46	
47	
56	
57	

Table 3 The skeleton for a 4–(12, 6, 4) design. This is also the paradigm for the distribution of the points from a friendly pair of blocks.

and endeavours to make designs having these patterns seemed to go nowhere. Therefore an attempt was made to prove that they cannot lead to completed 4–(12, 6, 4) designs. One of us (DRB) has a very long proof (about fifteeen pages) that the Patterns II, III and IV are not viable, a proof which is of the into-the-corner-painting type, and which, since it does not involve novel techniques, will not be given here. However, it was fairly easy to show that Pattern II is not viable; the somewhat asymmetrical Patterns III and

IV are the difficult cases. A computer search eliminated all but Pattern I. The program was tested on Pattern II where the result was known beforehand. Then a short run on Patterns III and IV confirmed that no 4–designs can be completed from them.

The combination of Pattern I and the proto-skeleton gives Table 3, the skeleton of a 4–(12, 6, 4) design with a friendly pair of blocks. Note that the seven distinct points from any pair of friendly blocks will generate the same skeleton, which is therefore the paradigm for the distribution of points from any friendly pair of blocks in the design even though they may not be in section B. The blocks in section F must all be of Type II since they intersect other blocks, namely those of section A, in just one point. Also, in section C there are the two blocks [**1236** . .], [**1236** . .]. There is only *one* other block in the design that could possibly be disjoint from one of these, and that is in section D, namely, the block [**457** . . .]. Therefore section C contains at least twenty Type II blocks and the design contains at least 32 such blocks.

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