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Characterisation of graphs with exclusive sum labelling

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In memory of Mirka Miller

Abstract

A *sum graph* G is a graph with an injective mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. In an *exclusive sum graph* the integers of S that are the sum of two other integers of S form a set of integers that label a collection of isolated vertices associated with the graph G. A graph bears a k-*exclusive sum labelling* (abbreviated k-ESL), if the set of isolated vertices

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is of cardinality k, an *optimal exclusive sum labelling* if k is as small as possible, and Δ-optimal if k equals the maximum degree of the graph.

In this paper, observing that the property of having a k-ESL is hereditary, we provide a characterisation of graphs that have a k-exclusive sum labelling, for any $k \geq 1$, in terms of describing a universal graph for the property.

1 Introduction

All graphs considered here are simple and undirected unless otherwise stated. All graphs are also connected except for the isolated vertices necessary to maintain the labelling. We will define terms specific to this article; for all other terms used the reader is referred to [3].

1.1 Sum Graphs

A *sum graph* G is a graph with an injective mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. More formally, for a *sum labelling* $L : V(G) \to S$, we have $u, v \in V(G), uv \in E(G)$, if and only if there is a $w \in V(G)$ such that $L(u) + L(v) = L(w)$. In this case the vertex w is said to be a *working vertex* whose work is to *witness* the edge uv.

Sum graphs were introduced by Harary in [5] as a terse way of storing and communicating graphs. One of the first properties noticed of sum graphs was that they must be disconnected. The vertex with the largest label must be an isolate. Any graph can be sum labelled by including sufficiently many isolated vertices with the graph. The *sum number* of a graph G, $\sigma(G)$ is the smallest cardinality of a set of isolates that must be included with G in order for it to have a sum labelling.

A sum graph with all working vertices being confined to the set of isolates was postulated in [8] and given the name *exclusive sum graph*. More precisely, for a given positive integer k, a *k-exclusive sum labelling* (abbreviated k-ESL) of a graph G is a sum labelling L of the graph $G \cup \overline{K_k}$ such that, for $u, v \in V(G \cup \overline{K_k})$, we have $uv \in E(G \cup \overline{K_k})$ if and only if $L(u) + L(v) = L(w)$ for some $w \in \overline{K_k}$ (and, similarly as above, we say that the isolate w *witnesses* the edge uv). Note that there is a slight formal difference here: unlike in sum graphs, when saying that a graph G is an exclusive sum graph, we do not consider the isolates to be vertices of G, and, consequently, an exclusive sum graph does not have to be disconnected. We will use $\mathcal{E}_k$ to represent the class of all graphs having a k-ESL.

Thus, a (given) k-ESL assigns to every edge of G an isolate by which it is witnessed. This assignment can be also thought of as an edge colouring of G, in which the colour of an edge equals the label of the isolate by which it is witnessed. Since all labels of vertices have to be distinct, no two edges adjacent to the same vertex can have the same colour and, consequently, this assignment determines a proper edge colouring of G. Moreover, also conversely, once the assignment of labels to the edges of G (i.e., the edge colouring of G) is given, then the labelling L of the vertices of G is uniquely determined, up to an additive constant (provided G is connected; otherwise this is true in each component of G).

However, note that not every proper k-edge-colouring of G determines a k-ESL of G: for example, the graph $K_{2,2,2}$ (see Fig. 7) is 4-edge-colourable while it can be shown [6] that any of its exclusive sum labellings requires at least 7 isolates.

Obviously, if G has a k-ESL, then G has a k'-ESL for every $k' \geq k$ (i.e., adding extra isolates does not change the k-ESL property). The *exclusive sum number* of a graph $G, \epsilon(G)$ is the smallest k for which G has a k-ESL, i.e., the cardinality of the smallest set of isolates that must be included with G in order for it to have an exclusive sum labelling. Clearly $\sigma(G) \leq \epsilon(G)$ and, by the above observations on edge-colourings, $\chi'(G) \leq \epsilon(G)$, where $\chi'(G)$ is the edge chromatic number (also called the chromatic index) of G.

Exclusive sum numbers are known for various families of graphs such as: complete graphs, $\epsilon(K_n) = 2n - 3, n > 3$ [2]; cocktail party graphs, $\epsilon(H_{2,n}) = 2n - 5$ [6]; and odd wheels, $\epsilon(W_n) = n, n$ odd [7].

Since, for an exclusive sum graph G, labels of the isolates determine a proper edge-colouring of G, the fewest number of isolates required for a graph G to bear an exclusive sum labelling is $\chi'(G)$ and, consequently, by Vizing's theorem, the maximum degree of $G, \Delta(G)$. Exclusive sum graphs with Δ isolates are referred to as Δ-*optimal exclusive sum graphs*. Such graphs include caterpillars, shrubs (trees with diameter 4), stars and double stars [10]. By the above observations on edge-colourings, every such graph must satisfy $\chi'(G) = \Delta(G)$, i.e., must be of chromatic class 1. Ryan [9] has produced a survey of exclusive graph labellings while Gallian [4] devotes a section to exclusive sum labellings in his well-known dynamic survey.

The problem with applying an exclusive sum labelling (and indeed a sum labelling) is twofold. First the labelling must witness every edge but also no two non-adjacent vertices should have labels that sum to another label. To do so would have the effect of inducing an edge in the graph that belongs in the complement of the graph. For example, let us label C_4 with 2, 4, 5, 7 (cyclic). Then the isolates required would be 6, 9, 12. However the label 7 induces an edge between vertices labelled 2 and 5 resulting in a graph that is no longer isomorphic to C_4. We call such edges *false edges*.

We say that a graph property $\mathcal{P}$ is *hereditary* if, whenever a graph G has $\mathcal{P}$, so does its every induced subgraph. Similarly, a class $\mathcal{C}$ of graphs is *hereditary* if, when $G \in \mathcal{C}$, all induced subgraphs of G are also in $\mathcal{C}$. (For example, every induced subgraph of a line graph is also a line graph, hence the class of all line graphs is hereditary). Note that if $\mathcal{F}$ is a given (finite or infinite) family of graphs, then the class of all $\mathcal{F}$-free graphs (i.e., graphs that do not contain an induced subgraph isomorphic to any graph from $\mathcal{F}$), is a hereditary class.

Now, it is immediate to observe that, for a given k, if L is a k-ESL of a graph G and G' is an induced subgraph of G, then the restriction of L to $V(G') \cup \overline{K_k}$ is a k-ESL of G'. Thus, the property of "having a k-ESL" is a hereditary property, and the class $\mathcal{E}_k$ of all graphs having a k-ESL is a hereditary class.

There are two ways of characterising hereditary classes of graphs.

- It is a well-known fact that for any hereditary class $\mathcal{C}$ there is a family $\mathcal{F}$ of graphs (called "forbidden induced subgraphs") such that $G \in \mathcal{C}$ if and only if G is $\mathcal{F}$-free. Note that such $\mathcal{F}$ always exists (the graphs in $\mathcal{F}$ are just those elements of $\overline{\mathcal{C}}$ that are minimal under the partial order defined by the relation of being an induced subgraph). A well-known example is the Beineke's characterisation of line graphs in terms of 9 forbidden induced subgraphs [1].

- Sometimes, it is possible to characterise $\mathcal{C}$ in terms of a *universal graph*, i.e., a graph $\mathcal{G}$ such that $G \in \mathcal{C}$ if and only if G is an induced subgraph of $\mathcal{G}$. Note that, unlike with the forbidden subgraphs, there are hereditary classes for which a universal graph does not exist.

In this paper, we will address the question of characterising the class $\mathcal{E}_k$ of all graphs having a k-ESL in either of the above ways. While a forbidden subgraph characterisation seems to be complicated (note that even for $k = 2$, the family $\mathcal{F}$ for $\mathcal{E}_2$ contains all cycles and the claw ($K_{1,3}$) since neither of these graphs has a 2-ESL, hence $\mathcal{F}$ is infinite), we will succeed in finding a universal graph for a generalised version of the problem. Describing the families of forbidden subgraphs for $\mathcal{E}_k, k \geq 3$, remains an open problem.

1.2 Hyperdiamond

A *hyperdiamond* is a generalisation of the honeycomb grid and is defined by the following construction.

1. H_1 is one edge (i.e., K_2).

2. Take a doubly infinite sequence of copies of H_i:
   $\ldots, H_i^{-2}, H_i^{-1}, H_i^0, H_i^1, H_i^2, \ldots$

3. Colour the vertices with 2 colours (black and white) so that corresponding vertices in H_i^j and H_i^{j+1} have different colours.

4. For every j, join every black vertex in H_i^j to its corresponding (white) vertex in H_i^{j+1} with a copy of H_1, and denote the resulting graph as H_{i+1}.

So H_1 is a single edge, H_2 is an infinite path, H_3 is the infinite honeycomb grid, H_4 is the infinite diamond (sometimes also called the "diamond structure"). Figure 1 shows the infinite honeycomb grid H_3 being constructed from copies of the path H_2, and the infinite diamond structure being constructed from copies of H_3.

By the above construction of H_k, we immediately observe that, for $i = 1$, the sequence given in Step 2 is an infinite matching, denoted M_1, and for each $i = 2, \ldots, k$, in Step 4, a perfect matching, denoted M_i, is added to join the copies of H_{i-1}. Thus, for any $k \geq 2$, the matchings $M_1, \ldots, M_k$ define a decomposition of $E(H_k)$ into k perfect matchings. This decomposition will be called the *canonical decomposition* of H_k. Obviously, removing any one of the matchings $M_1, \ldots, M_k$ from the H_k will leave an infinite number of copies of H_{k-1}.

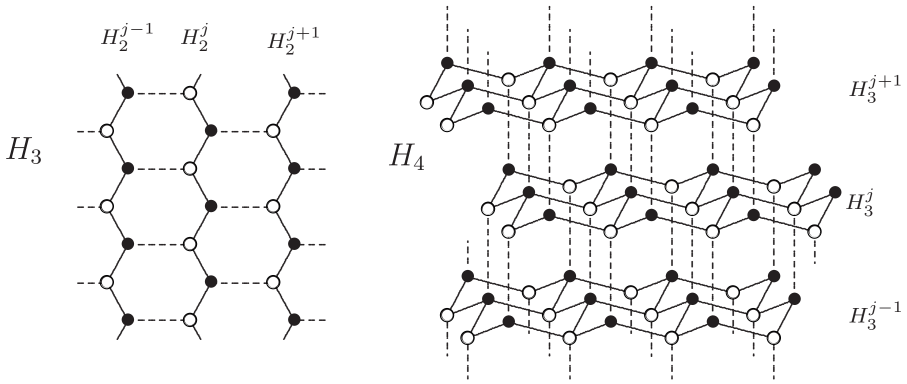

Figure 1: The honeycomb grid H_3 and the diamond structure H_4

Note that, from a purely geometrical point of view, H_3 is 2-dimensional (being in the plane), and H_4 is 3-dimensional (being a crystallographic structure). However, for our purposes, we will consider k (i.e., the number of perfect matchings in a canonical decomposition) to be the *dimension of* H_k.

All graphs considered herein, except the hyperdiamonds, will be finite.

In this article we provide a characterisation of graphs having a k-ESL. In Section 2, we investigate exclusive sum graphs for $k \leq 3$, while in Section 3 we extend this result to all $k \geq 1$. A surprising feature of these results is the central role of a universal graph played by the hyperdiamond structure.

2 Graphs having a 3-ESL

First we consider the (easy) cases of graphs having 1-ESL and 2-ESL. Since the maximum degree of the graph sets the lower bound for the exclusive sum number, the only graph with a 1-ESL is K_2, and a graph has a 2-ESL if and only if it is a path of length at least two [8]. Thus, the first nontrivial case is that of having a 3-ESL.

Let u_e and v_e represent end vertices of an edge e of G, witnessed by an isolate w_e. Define the function f on the edges of G as the sum of labels of end points of an edge minus the edge colour (i.e., the label of the witnessing isolate), formally $f(e) = L(u_e) + L(v_e) - L(w_e) = L(u_e) + L(v_e) - \chi(e)$. In an exclusively labelled sum graph,

$$f(e) = 0, \quad \text{for every } e \in E(G). \tag{1}$$

To simplify the notation, for the rest of the paper, if no confusion can arise, we will identify a vertex with its label, i.e., we will simply write $f(e) = u_e + v_e - w_e$.

Sum labellings and exclusive sum labellings are not necessarily unique. For clarity we will employ the following definitions.

Definition 2.1 *A* particular labelling *is an exclusive sum labelling in which all labels are distinct positive integers.*

Figure 2a) shows an example of a particular labelling.

Definition 2.2 *A* general labelling *is an exclusive sum labelling in which the vertices are labelled with parameters indicating a relationship between labels such that equation (1) holds for all edges.*

Figure 2b) gives an example of a general labelling for the graph of Figure 2a). Setting $x = 1, a = 6, b = 10, c = 14$ gives the particular labelling as shown in Figure 2a). Another particular labelling can result from setting $x = 3, a = 11, b = 22, c = 30$. Other particular labellings can be obtained from appropriate settings of any three of x, a, b, c and solving $(a - x) + (b - x) - c = 0$.

Definition 2.3 *A* generic labelling *is a general labelling such that equation (1) is satisfied for all choices of parameters $x, a, b, \ldots$. In the case where the labelling requires k isolates, we may use the term* generic k-labelling.

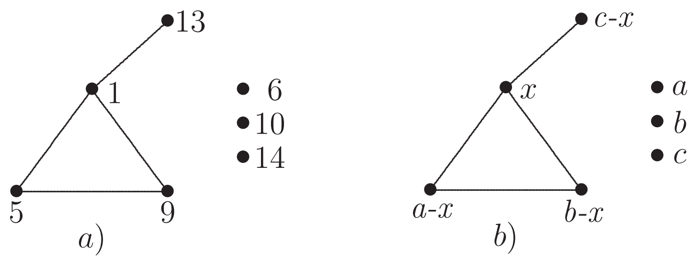

Figure 2: Particular and general labellings

Figure 3a) gives an example of a generic labelling. Apart from x (which is arbitrarily chosen) each of the vertices is chosen so that the incident edge weight satisfies equation (1). First place $a - x, b - x, c - x$, then $c - b + x$ and $c - a + x$; finally, $b - c + a - x$ is chosen so that the right edge has weight b. The final edge must now satisfy equation (1), i.e., $(c - b + x) + (b - c + a - x) - a = 0$, which is identically true. For a generic labelling, the only restriction on the choice of parameters x, a, b, c is that the vertex labels must be distinct, positive integers.

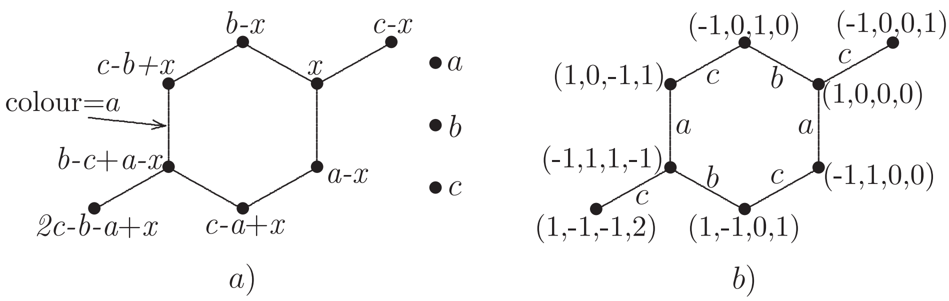

Figure 3: A generic labelling and the corresponding canonical labelling

As another example, we show that C_4, the cycle of length 4, has no generic 3-labelling. Thus, suppose the opposite, and let $C_4 = uvwzu$. By symmetry, we can choose $L(u) = x$, $\chi(u,v) = a$ and $\chi(u,z) = b$. This gives $L(v) = -x + a$ and $L(z) = -x + b$. Now, since necessarily $\chi(v,w) \neq \chi(u,v)$, for $\chi(v,w)$ we have either $\chi(v,w) = b$ or $\chi(v,w) = c$. In the first case we get $L(w) = x - a + b$, implying $f(wz) = L(w) + L(z) - \chi(wz) = -a + 2b - \chi(wz)$ and, since $\chi(wz) \in \{a,c\}$, none of these possibilities gives $f(wz) = 0$ for all values of x, a, b, c. In the second case similarly $L(w) = x - a + c$, and, since $\chi(wz) = a$, we have $f(wz) = L(w) + L(z) - \chi(wz) = -2a + b + c \neq 0$. Thus, C_4 has no generic 3-labelling.

The following concept will be needed throughout the rest of the paper and therefore we define it for arbitrary $k \geq 1$.

Definition 2.4 *We define a specific type of generic labelling, ϕ on H_k, called a* canonical labelling. *Let $M_1, M_2, \ldots, M_k$ be the canonical decomposition of H_k into k perfect matchings. We define a labelling ϕ on $V(H_k)$ by the following construction:*

(i) *select (arbitrarily) an origin and label it $L(u) := x$,*

(ii) *label the isolates with $a_1, a_2, \ldots, a_k$, respectively,*

(iii) *colour every edge $e \in E(M_i)$ with colour $\chi(e) = a_i$, $i = 1, 2, \ldots, k$, (or, equivalently, assign to the edges of M_i the isolate labelled a_i as a witness, $i = 1, 2, \ldots, k$),*

(iv) *for every edge $uv \in E(H_k)$ such that $L(u)$ is already defined while $L(v)$ is not, set $L(v) := \chi(uv) - L(u)$.*

Then each vertex $u \in V(H_k)$ is labelled with an expression

$$\psi * x + \alpha_1 * a_1 + \alpha_2 * a_2 + \ldots + \alpha_k * a_k,$$

where $\psi \in \{-1, 1\}$, and we set $\phi(u) = (\psi, \alpha_1, \alpha_2, \ldots, \alpha_k)$. When we need to specify the dimension, we may speak of a canonical k-labelling *so that, when dealing with graphs in $\mathcal{E}_3$ we may refer to a* canonical 3-labelling.

In terms of the canonical labelling ϕ on H_3, the origin and its three adjacent vertices are labelled $(1,0,0,0), (-1,1,0,0), (-1,0,1,0), (-1,0,0,1)$. Of course, the canonical labelling of H_3 can be restricted to any finite induced subgraph G of H_3, and it immediately gives a generic 3-labelling of G, see Fig. 3b).

If G is a graph embedded in H_3 and $u, v \in V(G)$, then define the *grid distance* $d_H(u,v)$ as the distance between u and v in H_3. Note that $d_H(u,v)$ can be different from $d_G(u,v)$ and depends on the embedding. In Figure 4 $d_G(u,v) = 12$ while $d_H(u,v) = 2$.

The following fact is straightforward.

Observation 2.5 *Consider G embedded in H_3 with origin u (i.e., $\phi(u) = (1,0,0,0)$). Then for $v \in V(G)$ with $\phi(v) = (\pm 1, \alpha, \beta, \gamma)$ we have $d_H(u,v) = |\alpha| + |\beta| + |\gamma|$.*

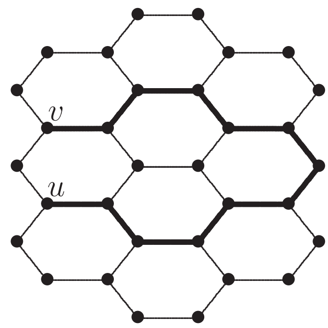

Figure 4: The path P_{13} embedded in H_3

For example, in the graph in Fig. 3, for the left-bottom vertex v with $\phi(v) = (1, -1, -1, 2)$, we have $\alpha = -1$, $\beta = -1$ and $\gamma = 2$, hence $d_H(u, v) = 1 + 1 + 2 = 4$.

The main result of this section is the following theorem.

Theorem 2.6 *A graph G has a generic 3-labelling if and only if G is an induced subgraph of H_3.*

Proof

($\Leftarrow$) A generic 3-labelling of G is obtained as a restriction of a canonical labelling of the H_3 to $V(G)$.

($\Rightarrow$) We prove the following slightly stronger statement.

If G has a generic 3-labelling φ, then G can be embedded in H_3 (as an induced subgraph) in such a way that φ is a restriction of a canonical labelling of H_3.

To anchor the induction, observe that it is easily seen that the theorem is true e.g. for graphs of order at most 3.

Assume that the statement does not hold and let G be the smallest order exclusive sum graph for which the theorem does not hold. That is, G is a smallest graph that has a generic 3-labelling but which is not embeddable in H_3 as an induced subgraph. Then for any vertex $z \in V(G)$, $G - z$ is embeddable in H_3. Choose z to be of degree 1 or 2. This is always possible since we are considering finite graphs (if G was 3-regular, then $G - z$ would be an induced subgraph of H_3 with 3 vertices of degree 2 and no vertex of degree 1, which is not possible).

For $d(z) = 1$, remove z and the graph $G - z$ is embeddable. If we replace z then, since G has a generic exclusive sum labelling, z cannot induce any false edges and so G must have been embeddable.

For $d(z) = 2$, let u, v be the neighbours of z. By assumption, $G - z$ is embeddable in such a way that φ is a restriction of a canonical labelling of H_3. Suppose, without loss of generality that edge zu has colour a and zv has colour b (in the generic labelling of G) and that the canonical labelling of H_3 has u as origin ($\phi(u) = (1, 0, 0, 0)$). Then we have $\phi(z) = (-1, 1, 0, 0)$, that is, $a - x$, and $\phi(v) = (1, -1, 1, 0)$, that is, $b - a + x$. But in $G - z$, ϕ is a canonical labelling so, by Observation 2.5, $d_H(u, v) = 2$.

Let y be the common neighbour of u, v in H_3. Then $y \neq z$ for otherwise G is already embedded in H_3. If $y \in V(G)$, then $y \in V(G-z)$ and since $G-z$ is induced in H_3, both $yu, yv \in E(G-z)$. In this case $C = zvyuz$ is a C_4 in G which is impossible since C_4 has no generic 3-labelling. Hence $y \notin V(G)$ and so $y = z$ and G is embedded in H_3.

It remains to show that, after the embedding, φ is a restriction of a canonical labelling of H_3. This is true for φ on $G-z$, so let ϕ be the canonical labelling of H_3 such that $\phi(w) = \varphi(w)$, for every $w \in G-z$. We need to show that $\varphi(z) = \phi(y)$. Recall that we have $\varphi(u) = \phi(u)$ $(= x)$ and $\varphi(v) = \phi(v)$. Let a_k and b_k be the colours (in ϕ) of the edges uy and vy, respectively. Then from the path uzv we have $\varphi(v) = b - a + x$ while from the path uyv we have $\varphi_k(v) = b_k - a_k + x$. Since both φ and ϕ are generic, they must be the same function, from which $b = b_k$ and $a = a_k$, so $\varphi(z) = b - x = b_k - x = \phi(z)$. □

For a graph to have an exclusive sum labelling, the vertices must be labelled with distinct positive integers. Therefore any generic (or even general) labelling must have a solution in the positive integers. To this end we will employ the following theorem from [9].

Theorem 2.7 *If L is an exclusive sum graph labelling of a graph H in $G = H \cup \overline{K_r}$ then so is the labelling $L'(u) = k_1L(u) + k_2$ for $u \in H$ and $L'(u) = k_1L(u) + 2k_2$ for $u \in \overline{K_r}$, where k_2 is any integer which results only in positive values in L' and k_1 is any positive integer that does not divide $6k_2$.*

Since the generic labelling may involve negative coefficients of a, b, c, the above theorem allows us, by judicious choice of k_1 and k_2, to make sure all vertices have positive integer labels.

Theorem 2.8 *Any graph bearing a generic 3-labelling has also a 3-ESL, i.e., is an exclusive sum graph with 3 isolates.*

Proof For proof it is sufficient to show that in a generic 3-labelling there exists a choice of exclusive sum isolates, a, b, c, such that all vertices are labelled with distinct positive integers.

First we make sure that all labels are distinct. To each vertex in a graph G we assign its generic label, which is a linear function of isolates a, b, c. If φ is the generic labelling, then we can describe the label of a vertex u as $\varphi_u(a, b, c)$. In order to ensure that all labels are distinct, we need to avoid the following:

$$\varphi_u(a, b, c) = \varphi_v(a, b, c), \qquad u, v \in V(G). \tag{2}$$

Each equation (2) gives a linear equation in a, b, c which can be interpreted geometrically as a plane in Euclidean 3-space with orthogonal axes, a, b, c. For a graph on n vertices, this gives $\binom{n}{2}$ equations, i.e., $\binom{n}{2}$ planes in Euclidean 3-space. Additionally to this, the conditions $a \neq b$, $a \neq c$ and $b \neq c$, that have to be also satisfied, give additional three planes in E_3.

Thus we have finitely many linear equations ($O(n^2)$ equations in three variables for a graph on n vertices embeddable in H_3) and we need to find a, b, c so that none of these equations is satisfied. Equivalently, we have a finite number of planes in E_3 and we must find a point (a, b, c) in the first octant with integer coordinates that is not incident with any of them. Of course this is always possible. Therefore we have distinct, positive integers a, b, c and distinct integer vertex labels.

We now need to ensure that the labels of all vertices are positive. Theorem 2.7 allows us to choose k_1 and k_2 in a linear transformation so that the labels of all vertices are increased sufficiently that all labels are positive. We now have an exclusive sum labelling with 3 isolates. □

3 Graphs having a k-ESL

In Section 2, we considered graphs in $\mathcal{E}_3$. We now extend these results to $\mathcal{E}_k$.

The main result in this section is given in the following theorem.

Theorem 3.1 *A graph G has a generic k-labelling if and only if G is an induced subgraph of H_k.*

Proof By construction, the infinite hyperdiamond H_k has k isolates and each vertex in the hyperdiamond is labelled by a function ϕ which is a $k+1$-tuple (the k isolates plus the origin). Call these isolates $a_1, \ldots, a_k$.

($\Leftarrow$) The generic labelling of G is given as a restriction of a canonical labelling of the H_k.

($\Rightarrow$) Proof will be by induction on k (and therefore, the dimension of the hyperdiamond). Assume, by the inductive step, that the theorem is true for all hyperdiamonds up to dimension k. Consider a graph bearing a generic labelling that requires $k+1$ isolates $a_1, a_2, \ldots, a_{k+1}$, $a_i \neq a_j$, to support an exclusive sum labelling but that cannot be embedded in H_{k+1}, and choose such a graph with minimal number of vertices. This means that, by the minimality assumption, embedding the graph induces one or more false edges, that is, edges that are not elements of the hyperdiamond structure. Take one of these unwanted edges and, if this edge is witnessed by an isolate a_i, then remove from the hyperdiamond all edges witnessed by some isolate $a_j \neq a_i$ (recall that these edges form a matching). Then, if the false edge remains, we have a hyperdiamond in k dimensions and a graph with a generic exclusive sum labelling requiring a_k isolates that cannot be embedded, contradicting the induction hypothesis. However a_j is chosen arbitrarily, so removing any matching associated with an isolate must remove the false edge associated with a_i. This is impossible, so graphs with a generic labelling requiring $k+1$ isolates can be embedded in H_{k+1}. □

Recall that for a graph to have an exclusive sum labelling, the vertices must be labelled with distinct positive integers. Therefore any generic (or even general) labelling must have a solution in the positive integers.

Theorem 3.2 *Any graph bearing a generic k-labelling has also a k-ESL, i.e., is an exclusive sum graph with k isolates.*

Proof As in the case of H_3, for proof it is sufficient to show that in a generic k-labelling there exists a choice of labels for isolates $a_1, a_2, \ldots, a_k$ such that all vertices are labelled with distinct positive integers.

The proof follows the same reasoning as the case for H_3 (Theorem 2.8). Here vertices with non-distinct labels can be represented as a linear equation in the k labels of the isolates. This can be viewed as $(k-1)$-dimensional hyperplanes in k-dimensional space. As before, we are considering only finitely many ($O(n^2)$) linear equations so visually only finitely many hyperplanes. We can easily choose a point in the positive sector (2^k-dant), so that it is not coincident with any hyperplane. Now we have positive, distinct, integer isolates and distinct vertex labels and we can again employ Theorem 2.7, if necessary, to ensure that all vertex labels are positive. Thus we have an exclusive sum labelling. □

4 Non-embeddable exclusive sum graphs

In Sections 2 and 3, we have described, for $k \geq 1$, all graphs having a generic k-labelling, and we have shown that

- these graphs are exactly all induced subgraphs of the H_k, and
- each of these graphs also has a (particular) k-ESL.

However, there still remain graphs that have a (particular) k-ESL but not a generic k-labelling. These, of course, do not embed into H_k. For example, Miller *et al.* proved that all cycles can be labelled exclusively with no more than 3 isolates [8], but clearly not all cycles can be embedded in the honeycomb grid H_3.

However, all graphs that have a k-ESL have also a spanning subgraph which has a generic k-labelling and hence can be embedded into H_k, and from which the remaining edges can be determined by solving the corresponding linear equations. See Figure 5 for an example of a chorded 5-cycle with a pendant edge.

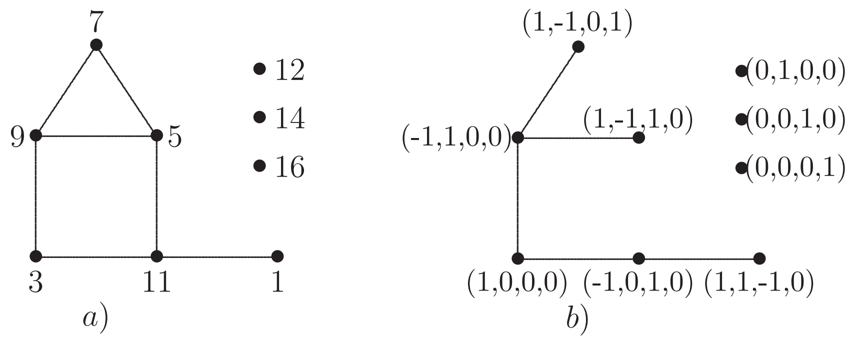

Figure 5: Non-embeddable graph and its spanning subgraph that lies in H_3

This example motivates our next result which shows that there are no graphs with a k-ESL but without a generic k-labelling among trees.

Proposition 4.1 *Let T be a tree and $k \in \mathbb{N}$. Then T has a k-ESL if and only if T has a generic k-labelling.*

Proof
($\Leftarrow$) Assume that T has a k-ESL. Each edge in T may be considered as coloured by the label of its isolate. Choose one vertex in the tree to be the origin, O, and label it (1,0,...,0), where the 1 is the coefficient of some yet to be determined constant and the k 0's are the initial coefficients of the k isolates. For each remaining vertex $u \in T$ assign a generic label by following the colours of the edges from O, that is, follow the path to u from O adding 1 to each coordinate associated with an edge that is transversed on the path. The generic labels are unique since if a vertex bears two different labels that would indicate that it was reached from O in two different ways, resulting in a cycle, which is impossible in a tree.

The generic labelling cannot induce false edges in T since any induced edge must be associated with an isolate and so be witnessed in the specific labelling.

($\Rightarrow$) This is a direct application of Theorem 3.2. □

Since all connected graphs have a spanning tree, Proposition 4.1 implies that any graph G with a particular k-ESL has a spanning subgraph F with a generic k-labelling. This spanning subgraph may not necessarily be a tree and may not require all k isolates for labelling. However, as the next theorem demonstrates, the k-ESL of G can always be constructed by embedding F in H_k and solving the restriction of the canonical labelling for the remaining edges in G, which appear as false edges in H_k.

Consequently, our last result will show that graphs with k-ESL are exactly those which can be obtained by taking an induced subgraph of H_k with the corresponding restriction of the canonical labelling, and substituting appropriate specific values for the parameters.

Theorem 4.2 *Let G be a graph and L a k-ESL of G. Then there is a spanning subgraph $F \subset G$ having a generic k-labelling and such that F can be embedded in H_k in such a way that $L = \phi(x, a_1, a_2, \ldots, a_k)$ for some values of $x, a_1, a_2, \ldots, a_k$, where ϕ is the canonical labelling of H_k.*

Proof
Consider a graph G with a k-ESL L, let $a_1, \ldots, a_k$ be the isolates witnessing the edges of G, and let T be a spanning tree of G. The labelling L obviously determines an edge-colouring of G, hence also of T (where the colour of an edge is the witnessing isolate a_i). Choose an origin x and let φ be the generic labelling defined on the edges of T by x and by the colours of the edges. Since T is not necessarily an induced subgraph of G, it is possible that for some $u, v \in V(G)$ with $uv \notin E(T)$ we have $\varphi(u) + \varphi(v) = a_i$ for some isolate a_i; however, since φ is constructed from a k-ESL L of G, in such case necessarily $uv \in E(G)$. Let F be the graph obtained from T by adding all such edges uv. Then F is a spanning subgraph of G and φ is a generic k-labelling of F. Hence, by Theorem 3.1, F can be embedded in H_k such that φ is a

restriction of a canonical labelling ϕ of H_k. Now, substituting in ϕ the specific values $a_1, \ldots, a_k$ and $L(x)$, we obtain that $L = \phi(x, a_1, a_2, \ldots, a_k)$. This can be done since we know the graph is an element of $\mathcal{E}_k$. □

For example, in Figure 5b) we need to solve for the 2 edges that are missing from Figure 5a). The top edge must be coloured $(0, 1, 0, 0)$ since the end vertices are incident with each of the remaining colours while the bottom edge must be coloured $(0, 0, 0, 1)$ for the same reason. So we have (reverting to the x, a, b, c notation),

$$\begin{aligned}(x - a + c) + (x - a + b) - a &= 0\\ 2x - 3a + b + c &= 0\\ \text{and}&\\ (x - a + b) + (-x + b) - c &= 0\\ -a + 2b - c &= 0\end{aligned}$$

The positive integer values given in Figure 5a) provide solutions for x, a, b, c and thus a labelling for the (non-embeddable) graph.

Note A proper k-edge-colouring of a graph G that is not an induced subgraph of H_k does not imply that G has a k-ESL. Figure 6 shows a graph that is 3-edge-colourable but no arrangement of colours a, b, c will result in equations that can be solved to provide distinct positive integer labels on the vertices. In this arrangement colouring one of the edges coloured c with a new colour (i.e., introducing a new isolate) d allows for a 4-ESL, a minimal (in terms of number of isolates) labelling. This graph is also critical in the sense that removal of any vertex results in a graph with a 3-ESL.

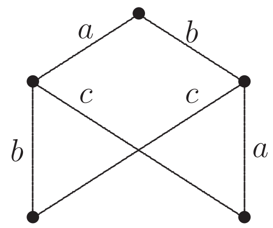

Figure 6: Critical graph with 4-ESL

5 Conclusion

However, even having a maximal spanning subgraph embeddable in H_k is not enough to ensure that the graph has a particular k-labelling. Each of the graphs in Figure 7, although of differing orders, requires a minimum of 7 isolates to support an exclusive sum labelling while they all possess spanning subgraphs that can be embedded in H_3.

Possible labellings are

$V(K_5)$ - 1, 5, 9, 13, 17 with isolates 6, 10, 14, 18, 22, 26, 30

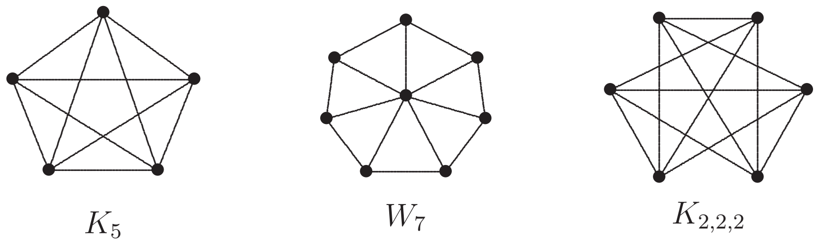

Figure 7: Three graphs, each requiring 7 isolates for an exclusive sum labelling

$V(W_7)$ - centre 16 and (cyclic) 28, 25, 19, 22, 1, 37, 7 with isolates 17, 23, 35, 38, 41, 44, 53
$V(K_{2,2,2})$ - (1, 5), (9, 21), (13, 17) with isolates 10, 14, 18, 22, 26, 34, 38.

The fact that the embedding dimension of a maximal spanning subgraph of a graph G gives no information about the number of isolates required for G to support an exclusive sum labelling then begs the following open question.

Open Question 1. *How difficult is it to determine the exclusive sum number of a graph without actually providing an exclusive sum labelling?*

As mentioned at the conclusion of Subsection 1.2, a forbidden subgraph characterisation for $\mathcal{E}_k$ appears to be difficult. We noted that, even for $k = 2$ the family of forbidden subgraphs for $\mathcal{E}_2$ is infinite containing, as it does, all cycles as well as the claw $K_{1,3}$. While we suspect the family of forbidden subgraphs for $\mathcal{E}_k$ is infinite for all k, we pose a perhaps more approachable problem.

Open Question 2. *Describe the family of forbidden subgraphs for $\mathcal{E}_3$.*

Acknowledgements

We dedicate this paper to the memory of Mirka Miller who passed away during preparation of the final version of the manuscript. We miss her greatly. J.R. and Z.R.

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