

A note on Grundy colorings of central graphs*

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Abstract

A *Grundy coloring* of a graph G is a proper vertex coloring of G where any vertex x , colored with $c(x)$, has a neighbor of any color $1, 2, \dots, c(x) - 1$. A central graph G^c is obtained from G by adding an edge between any two non-adjacent vertices in G and subdividing any edge of G once. In this note we focus on Grundy colorings of central graphs. We present some bounds related to parameters of G and a Nordhaus-Gaddum inequality. We also determine exact values for the Grundy coloring of some central classical graphs.

1 Introduction

We consider connected graphs without loops or multiple edges. In a graph $G = (V, E)$, the set $V(G)$ is the set of vertices ($|V(G)| = n$ is the order of G) and $E(G)$ is the set of edges ($|E(G)| = m$ is the size of G). The number of neighbors of a vertex x in G is its *degree*, denoted $d_G(x)$, and the maximum (resp. minimum) degree of the graph is denoted $\Delta(G)$ (resp. $\delta(G)$). The set of neighbors of x in G is denoted $N_G(x)$.

For a graph $G = (V, E)$, a *k-coloring* of G is defined as a function c on $V(G)$ into a set of colors $\mathcal{C} = \{1, 2, \dots, k\}$ such that for each vertex x_i , $1 \leq i \leq |V(G)|$, we have $c(x_i) \in \mathcal{C}$. A *proper k-coloring* is a k -coloring satisfying $c(x_i) \neq c(x_j)$ for every pair of adjacent vertices $x_i, x_j \in V(G)$. For a given proper k -coloring, a vertex x_i adjacent to a vertex of every color q , $q < c(x_i)$, is called a *Grundy vertex* for the color $c(x_i)$. A *Grundy k-coloring* is a proper k -coloring where any vertex is a Grundy vertex. In 1979 Christen and Selkow [4] introduced the *Grundy number*, denoted by $\Gamma(G)$, as the maximum number of colors among all Grundy colorings of G . It is obvious that $\Gamma(G) \leq \Delta(G) + 1$ but the determination of the Grundy number is NP-complete in general. This parameter was then studied for different classes of graphs (graph

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products [3, 5], power paths and cycles [9], regular graphs [8], fat-extended P_4 -laden graphs [2], etc.).

In this note we consider the family of central graphs. A *central graph* of a graph G , denoted by G^c , is obtained by joining all the non-adjacent vertices in G and subdividing each edge of G exactly once. The set of vertices of G^c is then given by $V(G^c) = V_1(G^c) \cup V_2(G^c)$ where $V_1(G^c)$ is the set of vertices inherited from G and $V_2(G^c)$ contains the added vertices (note that $|V_2(G^c)| = m$). Thus $V_1(G^c) = V(G) = \{x_1, x_2, \dots, x_n\}$ and each vertex of $V_2(G^c)$ subdividing an edge (x_i, x_j) of G is denoted $x_{i,j}$. If there is no ambiguity, sets $V_1(G^c)$ and $V_2(G^c)$ could be denoted V_1 and V_2 respectively.

In Section 2 we present some general bounds on the Grundy number of G^c related to its independence number, the Grundy number of the complement graph \overline{G} and the minimum degree of G . These results allow us to bound the Nordhaus-Gaddum inequality $\Gamma(G^c) + \Gamma(\overline{G^c})$. In Section 3 we focus on the Grundy colorings of some central classical graphs (complete graphs, complete bipartite graphs, complete k -ary trees, paths and cycles) for which the determined exact values reach the lower and upper bounds given in Section 2. Finally we conclude in Section 4 with an extension to a relaxed Grundy coloring, the *partial Grundy coloring*.

2 Nordhaus-Gaddum inequality type for the Grundy number of central graphs

In this section we present an upper bound for the Nordhaus-Gaddum inequality $\Gamma(G^c) + \Gamma(\overline{G^c})$. Different studies consider Nordhaus-Gaddum inequalities' type for the Grundy number (see the survey [1] where such inequalities are presented for several parameters). Zaker [11] proposed some results in 2006 and he conjectured that $\Gamma(G) + \Gamma(\overline{G}) \leq n + 2$ for any graph G . This conjecture was confirmed by Füredi et al. [7] for general graphs with $n \leq 8$ and disproved for $n \geq 9$ and they give the following result.

Theorem 2.1. ([7]) *Let G be a graph of order n ; then*

$$\Gamma(G) + \Gamma(\overline{G}) \leq \begin{cases} n + 2 & \text{if } n \leq 8, \\ 12 & \text{if } n = 9, \\ \lfloor \frac{5n+2}{4} \rfloor & \text{otherwise.} \end{cases}$$

We start our study with bounds for the Grundy numbers of G^c and its complement.

Proposition 2.2. *Let G be a graph of order n . Then $1 + \Gamma(\overline{G}) \leq \Gamma(G^c) \leq \min\{n, n + 2 - \delta(G)\}$ if G is connected and $\Gamma(\overline{G}) \leq \Gamma(G^c) \leq \min\{n, n + 2 - \delta(G)\}$ otherwise.*

Proof. The bound $\Gamma(G^c) \geq \Gamma(\overline{G})$ is obvious since $\overline{G} \subseteq G^c$. Consider a connected graph G and a Grundy coloring \mathcal{C} of \overline{G} . In G^c put the coloring \mathcal{C} on $V_1(G^c)$ shifted

by 1 and put the color 1 on all the vertices of $V_2(G^c)$. Then $\Gamma(G^c) \geq \Gamma(\overline{G}) + 1$. Next, since $\Delta(G^c) \leq n - 1$ we have $\Gamma(G^c) \leq \Delta(G^c) + 1 \leq n$. Moreover every vertex x_i of V_1 has degree $d_{G^c}(x_i) = n - 1$ and it follows that the maximum number of colors in $N_{G^c}(x_i)$ is $(n - 1) - d_G(x_i) + 2$ because $d_G(x_i)$ neighbors of x_i in G^c are vertices of V_2 which can be colored only by 1 or 2. Therefore $\Gamma(G^c) \leq (n - 1) - \delta(G) + 2 + 1$ and the result holds. \square

Corollary 2.3. *Let G be a graph of stability number $\alpha(G)$. Then $\Gamma(G^c) \geq \alpha(G) + 1$.*

Proof. Let S be a stable set in G of size $\alpha(G)$ (S is a clique in \overline{G}). By Proposition 2.2 we have $\Gamma(G^c) \geq \Gamma(\overline{G}) + 1 \geq \alpha(G) + 1$. \square

Proposition 2.4. *Let G be a graph of order n and size m . Then $m \leq \Gamma(\overline{G^c}) \leq m + n - 2$.*

Proof. By construction, vertices of $V_2(G^c)$ form a clique in $\overline{G^c}$. Thus $\Gamma(\overline{G^c}) \geq |V_2(G^c)| = m$. Moreover in $\overline{G^c}$, each vertex of $V_2(G^c)$ has the maximum degree $\Delta(\overline{G^c}) = (m - 1) + (n - 2) = n + m - 3$ and $\Gamma(\overline{G^c}) \leq \Delta(\overline{G^c}) + 1 = n + m - 2$. \square

We next present an upper bound for $\Gamma(G^c) + \Gamma(\overline{G^c})$ which improves the bound of Theorem 2.1 for central graphs when $n \geq 10$ and m is large enough.

Theorem 2.5. *Let G be a graph of order n and size m . Then $\Gamma(G^c) + \Gamma(\overline{G^c}) \leq \min\{m + 2n - 2, m + 2n - \delta(G)\}$.*

Proof. Deduced from Propositions 2.2 and 2.4. \square

Remark. Note that G^c is of order $n' = n + m$. Consider $\delta(G) \geq 2$. Thus from Theorem 2.5, if $n \geq 10$ and $m \geq 3n + 5 - 4\delta(G)$ then

$$\begin{aligned}
 m &\geq 3n + 5 - 4\delta(G), \\
 m + (4m + 5n - 1) &\geq 3n + 5 - 4\delta(G) + (4m + 5n - 1), \\
 5n' - 1 &\geq 4m + 8n + 4 - 4\delta(G), \\
 \left\lfloor \frac{5n' + 2}{4} \right\rfloor &\geq \frac{5n' - 1}{4} \geq m + 2n + 1 - \delta(G) > m + 2n - \delta(G),
 \end{aligned}$$

and the upper bound of Theorem 2.1 is improved for central graphs. By the same way if we consider $\delta(G) = 1$, $n \geq 10$ and $m \geq 3n - 3$, we have a similar relation $\left\lfloor \frac{5n'+2}{4} \right\rfloor > m + 2n - 2$.

3 The Grundy number of some central graphs

In the above section we showed that $\alpha(G) + 1 \leq \Gamma(G^c) \leq \min\{n, n + 2 - \delta(G)\}$. Now we present exact values for the Grundy number of central graphs for some classical graphs where the both bounds are reached. In particular we consider the central graphs of complete graphs, complete bipartite graphs, complete k -ary trees, paths and cycles.

Before studying these classes of graphs we can do some remarks on the Grundy number of a central graph G^c versus the Grundy number of \overline{G} . Indeed we have $\overline{G} \subseteq G^c$ which gives the following bounds.

Proposition 3.1. *Let G be a graph. Then, $\Gamma(\overline{G}) + 1 \leq \Gamma(G^c) \leq \Gamma(\overline{G}) + 2$ if G is connected and $\Gamma(\overline{G}) \leq \Gamma(G^c) \leq \Gamma(\overline{G}) + 2$ otherwise.*

Proof. The lower bounds are given by Proposition 2.2. And if we suppose a Grundy coloring of G^c with $k \geq \Gamma(\overline{G}) + 3$ colors, then a vertex x colored by $c(x) \geq \Gamma(\overline{G}) + 3$ has a neighbor in $V_2(G^c)$ with color $c' \geq 3$ which cannot be satisfied, a contradiction. Therefore $\Gamma(G^c) \leq \Gamma(\overline{G}) + 2$ □

From this relation we see that it is possible to determine the Grundy number of a central graph G^c from the Grundy number of the complement graph \overline{G} in polynomial time.

Proposition 3.2. *Let \overline{G} be the complement graph of a connected graph G with a Grundy coloring of $\Gamma(\overline{G})$ colors. Then a coloring of G^c can be determined in polynomial time.*

Proof. By Proposition 3.1 we see that $\Gamma(G^c)$ can have two possible values. For the lower value, a polynomial coloring algorithm is given in Proposition 2.2. Then note that Zaker [11] introduced the concept of t -atom which is a sufficient and necessary condition to have a Grundy coloring of t colors (i.e. a minimal induced subgraph of G , Grundy colorable with t colors). Moreover in [11], Zaker shows how to construct the set of t -atoms (from $(t - 1)$ -atoms) and he proves (Theorem 1 and its corollary) that there exists a polynomial time algorithm to determine whether $\Gamma(G) \geq t$ for any given graph G and integer t by determining if one of the t -atoms (generated above) exists in G . If exists, the coloring of such a t -atom can be extended in a polynomial time algorithm to the remaining of G by a greedy algorithm and $\Gamma(G) \geq t$. Therefore in a polynomial time algorithm we can determine if $\Gamma(G^c) \geq \Gamma(\overline{G}) + 2$ (and by Proposition 3.1, $\Gamma(G^c) = \Gamma(\overline{G}) + 2$). □

Despite the relation between central and complement graphs, for general central graph the problem of determining the Grundy number stays a NP-complete problem. This is why we focus on particular classes of central graphs. Firstly we consider complete and bipartite graphs.

Theorem 3.3. *Let K_n be a complete graph of order $n \geq 4$. Then $\Gamma(K_n^c) = 3$.*

Proof. In K_n^c there exists an induced subpath P_4 colorable with colors $1 - 2 - 3 - 1$ (a Grundy coloring). Then remaining vertices are colorable in a proper coloring and $\Gamma(K_n^c) \geq 3$. Moreover $\delta(K_n) = n - 1$ and Proposition 2.2 gives $\Gamma(K_n^c) \leq n + 2 - \delta(K_n) = 3$. \square

Theorem 3.4. *Let $K_{p,q}$ be a complete bipartite graph of order $p + q$, with $p \geq q \geq 2$. Then $\Gamma(K_{p,q}^c) = p + 2$.*

Proof. Denote by $S_p = \{x_1, x_2, \dots, x_p\}$ and $S_q = \{x_{p+1}, x_{p+2}, \dots, x_{p+q}\}$ the partitions of $K_{p,q}$. Vertices of S_p form a clique in $K_{p,q}^c$. Color these vertices with colors $3, 4, \dots, p + 2$. Then put $c(x_{p+1}) = 1$ and color the vertices of $V_2(K_{p,q}^c)$ as follows: $c(x_{i,p+1}) = 2$ for any $1 \leq i \leq p$ and the others with color 1. Thus each vertex of S_p is adjacent to all the colors (except its own color). Moreover vertices $x_{i,p+1}$ ($1 \leq i \leq p$) are also satisfied because $c(x_{p+1}) = 1$. The remaining vertices can be properly colored and the coloring is a Grundy coloring. Thus $\Gamma(K_{p,q}^c) \geq p + 2$.

Furthermore we have $\delta(K_{p,q}) = q$ and Proposition 2.2 gives $\Gamma(K_{p,q}^c) \leq (p + q) + 2 - \delta(K_{p,q}) = p + 2$. \square

Secondly we consider complete rooted trees. In a rooted tree of height h , the root is on the level 0 while the leaves are on level h . Moreover for a vertex x on level i ($0 \leq i < h$), a son of x is a neighbor of x on level $i + 1$.

Theorem 3.5. *Let $T_{k,h}$ be a complete k -ary tree of height $h \geq 4$, with $k \geq 2$. Let $L(i)$ be the number of vertices on the level i ($L(i) = k^i$). Then $\Gamma(T_{k,h}^c) = 1 + \sum_{i \in I} L(i)$ where $I = \{0, 2, \dots, h\}$ if h is even and $I = \{1, 3, \dots, h\}$ if h is odd.*

Proof. Let $q = \sum_{i \in I} L(i)$ with $I = \{0, 2, \dots, h\}$ if h is even and $I = \{1, 3, \dots, h\}$ if h is odd. Note that q corresponds to the number of vertices in the even levels if h is even (resp. odd levels if h is odd) of the tree $T_{k,h}$. In $T_{k,h}$ we have $\alpha(T_{k,h}) \geq q$ and by Corollary 2.3 we deduce $\Gamma(T_{k,h}^c) \geq 1 + q$.

The upper bound can be proved by contradiction. Suppose there exists a Grundy coloring of $T_{k,h}^c$ with q' colors such that $q' \geq 2 + q$. We see that $\overline{T_{k,h}}$ can be decomposed into two cliques K_o and K_e (due to respectively odd and even levels) where there exists an edge between each vertex of a set and at most $k + 1$ vertices of the other set (corresponding to its neighbor in the previous level, if exists, and its k neighbors in the next level in $T_{k,h}$, if exist). The graph $T_{k,h}^c$ is the same one where each edge between cliques is splitted by a vertex of V_2 . Note that since $h \geq 4$ and $k \geq 2$ we have $q > \max\{4, k + 1\}$.

Thus we have $q = |K_o| > |K_e|$ if h is odd while $q = |K_e| > |K_o|$ otherwise. Without loss of generality, consider h odd and $q = |K_o| > |K_e|$. We see that color 2 cannot be on a vertex of K_o , otherwise any vertex of K_o with color $c > 3$ must have a neighbor in V_2 colored by 3, a contradiction. Then K_o contains colors 3 to $q + 2$ and these vertices need colors 1 and 2 on V_2 . Thus at least q vertices of V_2 have color 2. Since each vertex of K_e has at most $k + 1$ neighbors of V_2 and $q > k + 1$, it follows that at least two vertices of K_e are adjacent to vertices of V_2 colored by 2

and must be colored with color 1. This implies a non-proper coloring because K_e is a clique, a contradiction. Therefore $\Gamma(T_{k,h}^c) \leq 1 + q$. \square

Finally we focus on central graphs of paths and cycles. Paths and cycles are classical graphs. Several graphs based on them have been studied. For instance the Grundy number of power paths and power cycles was studied in [9]. Power graphs G^p are obtained from G by adding edges between vertices at distance at most p in G . A central graph is then a power graph (where p is the diameter of G) with new added vertices. Moreover recently central paths and central cycles were considered for another maximal coloring called the b -coloring [10].

We start with some results on the number of colors used on Grundy colorings.

Lemma 3.6. *Let G be either a path P_n or a cycle C_n of order $n \geq 6$. In G^c , denote by \mathcal{C}_i the set of colors used on V_i , with $i \in \{1, 2\}$. In any Grundy coloring of G^c only the color $c = 1$ can satisfy $c \in \mathcal{C}_2$ and $c \notin \mathcal{C}_1$.*

Proof. Suppose that two distinct colors c_1 and c_2 appear only on V_2 , i.e. $c_1, c_2 \in \mathcal{C}_2$ and $c_1, c_2 \notin \mathcal{C}_1$ (assume $c_1 > c_2$). To satisfy a vertex colored by c_1 the color c_2 must be repeated on V_1 . This is a contradiction and a unique color c can appear only on V_2 . Suppose that $c > 1$. To satisfy the vertices of V_2 and since $n \geq 6$ (so $|V_2| \geq 4$), there exist two vertices of V_1 colored by 1 such that the distance between them is 2 or more (i.e. they are adjacent in G^c). This contradicts the property of the coloring. Hence all vertices of V_2 are colored with color 1. \square

Lemma 3.7. *Let P_n and C_n be respectively a path and a cycle of order $n = 3q+r \geq 6$, with $0 \leq r \leq 2$. Then,*

- a) *any Grundy coloring of P_n^c or C_n^c using $k \geq 2 \lfloor \frac{n}{3} \rfloor + 1$ colors has at least $\lfloor \frac{k-1}{2} \rfloor$ repeated colors.*
- b) *any Grundy coloring of C_n^c using $k > 2 \lfloor \frac{n}{3} \rfloor + 1$ colors has at least $\lfloor \frac{k-1}{2} \rfloor + 1$ repeated colors, if $r \in \{1, 2\}$.*

Proof. Colors 2 to k are on vertices of V_1 (by Lemma 3.6). Let n_u and n_r be the numbers of respectively *unique* and *repeated* colors on V_1 (note that $n_u + n_r = k - 1$). Moreover two vertices with unique colors must be adjacent to satisfy the largest one (i.e. they cannot be on consecutive vertices of V_1), thus $n_r \geq n_u$.

a) Suppose that $n_r < \lfloor \frac{k-1}{2} \rfloor$. Then we have $\lfloor \frac{k-1}{2} \rfloor > n_r \geq n_u = k - 1 - n_r \geq \lfloor \frac{k-1}{2} \rfloor$, a contradiction. Thus at least $\lfloor \frac{k-1}{2} \rfloor$ vertices of V_1 have repeated colors.

b) Since we have at least $\lfloor \frac{k-1}{2} \rfloor$ repeated colors, suppose the coloring has exactly $n_r = \lfloor \frac{k-1}{2} \rfloor$ repeated colors. If k is even then $n_r = \frac{k-2}{2}$ and $\frac{k-2}{2} = n_r \geq n_u = k - 1 - n_r = \frac{k}{2}$, a contradiction. If k is odd (thus $k \geq 2q + 3$) then we have $n_r = \frac{k-1}{2}$

and $n_u = k - 1 - n_r = \frac{k-1}{2}$. Since the colors of n_u (put once) and n_r (put twice) allow to color V_1 , we have $n \geq n_u + 2n_r \geq \frac{k-1}{2} + k - 1 \geq 3q + 3$, a contradiction. Therefore we have $n_r > \lfloor \frac{k-1}{2} \rfloor$. □

We next propose the following exact value for the Grundy number of central paths.

Theorem 3.8. *Let P_n be a path of order $n = 3q + r \geq 6$, with $0 \leq r \leq 2$. The Grundy number of P_n^c is given by*

$$\Gamma(P_n^c) = \begin{cases} 2\frac{n}{3} + 1 & \text{if } r = 0, \\ 2\lfloor \frac{n}{3} \rfloor + 2 & \text{otherwise.} \end{cases}$$

Proof. By construction only two consecutive vertices on V_1 can have the same color in a proper coloring of P_n^c . Suppose there exists a Grundy coloring of P_n^c on k colors where $k > 2\frac{n}{3} + 1$ if $r = 0$ and $k > 2\lfloor \frac{n}{3} \rfloor + 2$ otherwise. Lemma 3.7 shows that at least $N \geq \lfloor \frac{k-1}{2} \rfloor$ colors are repeated on the vertices of V_1 (remaining colors are then put once). Since a color is repeated at most twice, the n vertices of V_1 are colored by at least $(k - 1) + N$ colors (color 1 on V_2 , see Lemma 3.6). We distinguish two cases:

- case $r = 0$ ($n = 3q$): we have $k > 2\frac{n}{3} + 1 = 2q + 1$, $N \geq \lfloor \frac{k-1}{2} \rfloor \geq q$ and $n \geq k + N - 1 > (2q + 1) + q - 1 = 3q$,
- case $r = \{1, 2\}$ ($n = 3q + r$): we have $k \geq 2\lfloor \frac{n}{3} \rfloor + 3 = 2q + 3$, $N \geq \lfloor \frac{k-1}{2} \rfloor > q$ and $n \geq k + N - 1 > (2q + 3) + q - 1 = 3q + 2$.

We obtain a contradiction for each case. Therefore $\Gamma(P_n^c) \leq 2\frac{n}{3} + 1$ if $r = 0$ and $\Gamma(P_n^c) \leq 2\lfloor \frac{n}{3} \rfloor + 2$ otherwise.

The lower bounds are given by construction. Let $k = 2\lfloor \frac{n}{3} \rfloor + 2$. We color all the vertices of V_2 with the color 1. Then we color $c(x_{3i-1}) = c(x_{3i}) = i + 1$ for $1 \leq i \leq \lfloor \frac{n}{3} \rfloor$. Next we put $c(x_{3i+1}) = \lfloor \frac{n}{3} \rfloor + 2 + i$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$. Thus if $r = 0$ the graph is completely colored. The colors used on V_1 are 2 to $2\lfloor \frac{n}{3} \rfloor + 1 = k - 1$. Then if $r = 1$ it remains to color $c(x_n) = c(x_{n-3}) + 1 = k$. Finally if $r = 2$ we color $c(x_{n-1}) = c(x_n) = c(x_{n-4}) + 1 = k$. The coloring is then a Grundy coloring. Indeed first note that each vertex of V_1 is adjacent to the color 1 since it is put on all the vertices of V_2 . Then since every color c of the interval $[2, \lfloor \frac{n}{3} \rfloor + 1]$ is put twice, every vertex of V_1 can reach the color c (except the vertices colored by c). Moreover, since the distance in P_n between two vertices with colors from $[\lfloor \frac{n}{3} \rfloor + 2; k]$ (resp. $[\lfloor \frac{n}{3} \rfloor + 2; k - 1]$ if $r = 0$) is at least two, they are adjacent in P_n^c . Therefore each vertex of V_1 is a Grundy vertex and $\Gamma(P_n^c) \geq 2\frac{n}{3} + 1$ if $r = 0$ and $\Gamma(P_n^c) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ otherwise. □

Finally we determine the exact value of the Grundy number of central cycles.

Theorem 3.9. *Let C_n be a cycle of order n , with $n = 3q + r \geq 6$ and $0 \leq r \leq 2$. Then the Grundy number of C_n^c is given by*

$$\Gamma(C_n^c) = \begin{cases} 2\frac{n}{3} + 1 & \text{if } r = 0, \\ 2\left\lfloor \frac{n}{3} \right\rfloor + r & \text{otherwise.} \end{cases}$$

Proof. As in Theorem 3.8 we can prove the upper bound by contradiction. Suppose there exists a Grundy coloring of C_n^c with k colors where $k > 2\frac{n}{3} + 1$ if $r = 0$ and $k > 2\left\lfloor \frac{n}{3} \right\rfloor + r$ otherwise. Lemma 3.7 shows that at least N colors are repeated on the vertices of V_1 (remaining colors are then put once), with $N \geq \left\lfloor \frac{k-1}{2} \right\rfloor$ if $r = 0$ and $N \geq 1 + \left\lfloor \frac{k-1}{2} \right\rfloor$ otherwise. Thus the n vertices of V_1 must be colored by at least $(k - 1) + N$ colors (color 1 is on V_2 , by Lemma 3.6). We distinguish three cases:

- case $r = 0$ ($n = 3q$) then $k > 2\frac{n}{3} + 1 = 2q + 1$. By Lemma 3.7.a we have $N \geq \left\lfloor \frac{k-1}{2} \right\rfloor \geq q$. Thus $n \geq k + N - 1 > (2q + 1) + q - 1 = 3q$,
- case $r = 1$ ($n = 3q + 1$) then $k \geq 2\left\lfloor \frac{n}{3} \right\rfloor + 2 = 2q + 2$. By Lemma 3.7.b we have $N > \left\lfloor \frac{k-1}{2} \right\rfloor \geq q$. Thus $n \geq k + N - 1 > (2q + 2) + q - 1 = 3q + 1$,
- case $r = 2$ ($n = 3q + 2$) then $k \geq 2\left\lfloor \frac{n}{3} \right\rfloor + 3 = 2q + 3$. By Lemma 3.7.a we have $N \geq \left\lfloor \frac{k-1}{2} \right\rfloor > q$ and $n \geq k + N - 1 > (2q + 3) + q - 1 = 3q + 2$.

These contradictions imply $\Gamma(C_n^c) \leq 2\frac{n}{3} + 1$ if $r = 0$ and $\Gamma(C_n^c) \leq 2\left\lfloor \frac{n}{3} \right\rfloor + r$ otherwise.

The lower bounds are deduced from the constructions given in Theorem 3.8 for the central graphs of paths. Let G be the induced subgraph of C_n^c defined by $V(G) = V(C_n^c) \setminus \{x_{n,1}\}$. Remark that $P_n^c \equiv G \cup \{(x_1, x_n)\}$. We distinguish three cases. Consider $r = 0$. Put the coloring of Theorem 3.8 (case $r = 0$) on G . Since $c(x_n) = c(x_{n-1})$ the vertex x_1 is satisfied in both G and C_n^c . We complete the coloring of C_n^c by $c(x_{n,1}) = 1$. Therefore $\Gamma(C_n^c) = \Gamma(C(P_n)) = 2\frac{n}{3} + 1$. Next consider $r = 1$. By the same way, put the coloring of Theorem 3.8 (case $r = 1$) on G . Note that since x_1 is not adjacent to x_n (which is the unique vertex of its color) in C_n^c , we recolor x_n such that $c(x_n) = c(x_1)$. Moreover since $c(x_{n-1}) = c(x_{n-2})$ then x_n is a Grundy vertex. We complete the coloring of C_n^c by $c(x_{n,1}) = 1$. Therefore $\Gamma(C_n^c) = \Gamma(C(P_n)) - 1 = 2\left\lfloor \frac{n}{3} \right\rfloor + 1$. Finally consider $r = 2$. Put the coloring of Theorem 3.8 (case $r = 2$) on G and recolor x_n such that $c(x_n) = c(x_1)$ (x_n is a Grundy vertex since x_n and x_{n-1} are not adjacent in C_n^c and $c(x_n) < c(x_{n-1})$). Note that x_{n-1} remains the last vertex of its color but it stays a Grundy vertex. We complete the coloring of C_n^c by $c(x_{n,1}) = 1$ and we have $\Gamma(C_n^c) = \Gamma(C(P_n)) = 2\left\lfloor \frac{n}{3} \right\rfloor + 2$. \square

4 Conclusion

We have presented some results on the Grundy colorings of central graphs. In particular we have given exact values for the central graphs of some classical graphs and

have shown that lower and upper bounds given in Proposition 2.2 and Corollary 2.3 are reached.

A relaxed approach of the Grundy coloring, called the *partial Grundy coloring*, was introduced by Erdős et al. [6] where every color has at least one Grundy vertex. The *partial Grundy number*, denoted $\delta\Gamma(G)$, is then defined as the maximum integer k such that G admits a partial Grundy k -coloring. The determination of the partial Grundy number is also NP-complete in general but it is obvious that $\delta\Gamma(G) \geq \Gamma(G)$. The results presented above on Grundy colorings give first bounds for the partial Grundy number of central graphs.

Corollary 4.1. *Let G be a graph of order n and size m . Then,*

- a) $1 + \Gamma(\overline{G}) \leq \delta\Gamma(G^c) \leq n$,
- b) $\delta\Gamma(G^c) \geq \alpha(G) + 1$,
- c) $m \leq \delta\Gamma(\overline{G^c}) \leq m + n - 2$,
- d) $\delta\Gamma(G^c) + \delta\Gamma(\overline{G^c}) \leq m + 2n - 2$.

Proof. This is deduced from the fact that $\Gamma(G) \leq \delta\Gamma(G) \leq \Delta(G) + 1$ for any graph G . For case c, see Proposition 2.4. □

However even for such a coloring we can find classes of central graphs for which the upper bound $\delta\Gamma(G^c) \leq n$ is reached (for instance central chordal graphs) while it is not for some others (for instance central even paths). A graph is a *chordal graph* if each induced cycle of length 4 or more has a chord (an edge connecting two non-adjacent vertices). Chordal graphs are sometimes called *triangulated graphs*.

Theorem 4.2. *Let G be a chordal graph of order $n \geq 3$. Then $\delta\Gamma(G^c) = n$.*

Proof. By Corollary 4.1 we have $\delta\Gamma(G^c) \leq n$. The lower bound is given by construction. Color the vertices of V_1 with colors $1, 2, \dots, n$. Then in G^c , for every triangle $\{x_i, x_j, x_k\}$, with $c(x_i) > c(x_j) > c(x_k)$, vertex x_i does not reach colors $c(x_j)$ and $c(x_k)$ (and vertex x_j does not reach color $c(x_k)$). Then put $c(x_{i,j}) = c(x_k)$ and $c(x_{i,k}) = c(x_j)$ to make x_i and x_j Grundy vertices. Thus all the vertices of V_1 are Grundy vertices. If it remains non colored vertices of V_2 they can be properly colored with color 1, 2 or 3 since they have degree 2. Therefore $\delta\Gamma(G^c) \geq n$. □

Theorem 4.3. *Let P_n be a path of order $n \geq 7$. Then,*

$$\delta\Gamma(P_n^c) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n - 1 & \text{otherwise.} \end{cases}$$

Proof. Consider n odd. Color $c(x_i) = i$ for odd i and $c(x_i) = i + 2$ for even i , with $1 \leq i \leq n - 2$. Then put $c(x_{n-1}) = n$ and $c(x_n) = 2$. Vertices x_i , with odd i , are already Grundy vertices. Finally color $c(x_{i,i-1}) = c(x_{i+1})$ and $c(x_{i,i+1}) = c(x_{i-1})$ for even i with $1 \leq i \leq n$. Thus all the vertices are colored and each vertex of V_1 is a Grundy vertex ($\delta\Gamma(P_n^c) \geq n$). And by Corollary 4.1 we have $\delta\Gamma(P_n^c) \leq n$.

Consider n even. The lower bound is given by construction. Put $c(x_i) = i$ for any $1 \leq i \leq n-1$, and $c(x_{j,j+1}) = c(x_{j-1})$ for any $2 \leq j \leq n-1$. Thus every colored vertex of V_1 is a Grundy vertex. Remaining vertices can be properly colored and $\delta\Gamma(P_n^c) \geq n-1$. Next suppose there exists a partial Grundy coloring of P_n^c using $q \geq n$ colors. Note that $c(x_1) \leq c(x_2)$ and $c(x_n) \leq c(x_{n-1})$ to have a proper coloring of $c(x_{1,2})$ and $c(x_{n-1,n})$. Moreover we also observe that either $c(x_1) < c(x_2)$ or $c(x_n) < c(x_{n-1})$ (or both), otherwise the $q-1 \geq n-1$ colors (only one color can appear only on V_2) are on the $n-2$ remaining vertices of V_1 , a contradiction. Then there exist four consecutive vertices $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ such that $c(x_i) < c(x_{i+1}) < c(x_{i+2})$ and $c(x_{i+2}) > c(x_{i+3})$ (or $c(x_{i+1}) > c(x_{i+2}) > c(x_{i+3})$ and $c(x_i) < c(x_{i+1})$). In both cases vertices x_{i+1} and x_{i+2} are Grundy vertices in a proper coloring of P_n^c which implies $c(x_{i+1,i+2}) = c(x_i)$ (for x_{i+1}) and $c(x_{i+1,i+2}) = c(x_{i+3})$ (for x_{i+2}), a contradiction. Therefore $\delta\Gamma(P_n^c) \leq n-1$.

□

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