AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **68(2)** (2017), Pages 186–201

On completing partial Latin squares with two filled rows and at least two filled columns

Jaromy Kuhl   Donald McGinn

Department of Mathematics and Statistics
University of West Florida
Pensacola, FL 32514
United States of America
jkuhl@uwf.edu   dmcginn@uwf.edu

Abstract

In this paper we give an alternate proof that it is always possible to complete partial Latin squares with two filled rows and two filled columns, except for a few small counterexamples. The proof here is significantly shorter than the most recent proof by Adams, Bryant, and Buchanan. Additionally, we find sufficient conditions under which a partial Latin square with two filled rows and at least three filled columns can be completed.

1 Introduction

Let n be a positive integer and S be a finite symbol set. If P is an $n \times n$ array in which each nonempty cell contains exactly one symbol of S, then we call P an *array of order n over S*. The symbol of S appearing in a nonempty cell (i, j) of P is denoted $P(i, j)$. Unless stated otherwise, we assume that $S = [n] = \{1, 2, \ldots, n\}$. We will often treat P as a subset of $[n] \times [n] \times [n]$, where $(r, c, s) \in P$ if and only if $s = P(r, c)$.

If symbols occur at most once in a row (or column) of P, then that row (or column) is called *Latin*. The array P is a *partial Latin square* (PLS) if each row and column of P is Latin. If P is a PLS and each cell is nonempty, then P is called a *Latin square*. A Latin square of order 2 that occurs as a subset of a PLS is called an *intercalate*. We use $I_{j,k}$ to denote an intercalate over symbols $\{j, k\}$.

Let PLS(n) denote the set of all partial Latin squares of order n and let LS(n) denote the set of all Latin squares of order n. Furthermore, let PLS$(a, b; n)$ be the subset of PLS(n) in which the first a rows and first b columns are filled, and the remaining cells empty.

We say that $P \in \mathrm{PLS}(n)$ is *completable* if there exists $L \in \mathrm{LS}(n)$ such that $P \subset L$. The problem of determining if a partial Latin square is completable is NP-complete [4]. However, there are many known families of completable partial Latin squares (see e.g. [5], [6], [8], [9]). Of particular interest is the result that all elements of $\mathrm{PLS}(2,2;n)$ for $n \geqslant 6$ are completable, first proven in Buchanan's PhD thesis [3]. A slightly shortened version appears in [1], which is still over 25 pages long and relies on a computer search to check the values $n \in \{6,7,8\}$. In the present paper, we provide a significantly shorter proof without computer aid. We also give sufficient conditions under which elements of $\mathrm{PLS}(2,b;n)$ can be completed for $b \geqslant 3$. Critical to these proofs are the notions of isotopisms and conjugates, which we define below.

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$. For $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$, we use $\theta(P) \in \mathrm{PLS}(n)$ to denote the array in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively. The mapping θ is called an *isotopism*, and P and $\theta(P)$ are said to be *isotopic*. It is well known that P can be completed if and only if $\theta(P)$ can be completed for any $\theta \in S_n \times S_n \times S_n$. Thus, when completing an element of $\mathrm{PLS}(2,b;n)$, we may assume that the subarray induced by cells $[2] \times [b]$ is over a subset of $[2b]$ and that the symbols in the first row appear in natural order.

A *conjugate* of P is an array in which the coordinates of each triple of P are uniformly permuted. There are six, not necessarily distinct, conjugates of P. Let $\lambda \in \{\epsilon, (rc), (rs), (cs), (rcs), (rsc)\}$. We use λ to permute coordinates, where r, c, and s correspond to the row, column, and symbol coordinates, respectively. The conjugate of P induced by λ is denoted by P^λ. When $\lambda = (rc)$, we use standard notation for the transpose of a matrix, P^T. As with isotopisms, P can be completed if and only if any conjugate of P can be completed.

In Section 2, we develop an operation that reduces elements of $\mathrm{PLS}(2,b;n)$ to elements of $\mathrm{PLS}(2,b;n-1)$ for $b \geqslant 2$. In Section 3, we describe an inductive technique for completing partial Latin squares invented by Smetaniuk, as well as a generalization. In Section 4, we prove necessary and sufficient conditions for when it is always possible to complete the elements of $\mathrm{PLS}(2,2;n)$. In Section 5, we find sufficient conditions under which $P \in \mathrm{PLS}(2,b;n)$ can be completed for $b \geqslant 3$.

2 Reducing Elements of $\mathrm{PLS}(2,2;n)$

Let $a,b,j,k \in [n]$ and let $X \in \mathrm{PLS}(a,b;n)$. We use C_j and R_k to denote column j and row k as subarrays of X. Furthermore, we shall often treat column j of X as the set of triples $C_j = \{(i,j,s) \mid i \in [n], (i,j,s) \in X\}$ and row k of X as $R_k = \{(k,i,s) \mid i \in [n], (k,i,s) \in X\}$.

Definition 2.1. Let $X \in \mathrm{PLS}(a,b;n), l \in [a]$, and $m \in [b]$. For columns C_j and C_k

of X and rows R_j and R_k of X, define $C_j \circ_l C_k$ and $R_j \circ_m R_k$ as:

$$C_j \circ_l C_k = (C_j \backslash (l, j, X(l, j))) \cup (l, j, X(l, k))$$
$$R_j \circ_m R_k = (R_j \backslash (j, m, X(j, m))) \cup (j, m, X(k, m))$$

Example 2.2. Consider the following element of PLS$(2, 2; 6)$:

1	2	3	4	5	6
2	3	1	5	6	4
6	5				
3	4				
4	1				
5	6				

Then $C_3 \circ_1 C_6 = [6\ 1\ \cdot\ \cdot\ \cdot\ \cdot]^T$ and $C_6 \circ_1 C_3 = [3\ 4\ \cdot\ \cdot\ \cdot\ \cdot]^T$, where $\cdot$ denotes an empty cell. Additionally, $R_3 \circ_2 R_4 = [6\ 4\ \cdot\ \cdot\ \cdot\ \cdot]$ and $R_4 \circ_2 R_3 = [3\ 5\ \cdot\ \cdot\ \cdot\ \cdot]$.

Lemma 2.3. *Let $j, k \in [n]$ and $X \in$ PLS$(a, b; n)$.*

(i) Let $l \in [a]$ and suppose that $j, k > b$. Then $C_j \circ_l C_k$ is Latin if and only if $X(l, k) \neq X(i, j)$ for each $i \in [a]$.

(ii) Let $l \in [b]$ and suppose that $j, k > a$. Then $R_j \circ_l R_k$ is Latin if and only if $X(k, l) \neq X(j, i)$ for each $i \in [b]$.

Definition 2.4. Let $X \in$ PLS$(a, b; n)$. Let $\beta \in [n]$ such that $(i, m_i, \beta) \in X$ and $m_i > b$ for each $i \in [a]$. Let $j \in [n] \backslash [b]$. If C_j is a column such that each column of $\{C_{m_1} \circ_1 C_j, \ldots, C_{m_a} \circ_a C_j\}$ is Latin, then β is called a *column-replaceable symbol*, and we say that C_j *replaces* β. If $C_j = C_{m_i}$ for some i, we also say that C_j *replaces itself*. We define a symbol γ to be *row-replaceable* if it is column-replaceable in X^T. We similarly define when R_j *replaces itself* for some $j \in [n] \backslash [a]$. If a symbol is both column-replaceable and row-replaceable, we simply say it is *replaceable*.

Let $\alpha \in [n]$ be a replaceable symbol in X and suppose that C_j and R_k replace α. Furthermore, let $(i, m_i, \alpha) \in X$ for each $i \in [a]$ and $(p_i, i, \alpha) \in X$ for each $i \in [b]$. Let $C = C_{m_1} \cup \ldots \cup C_{m_a}$, $R = R_{p_1} \cup \ldots \cup R_{p_b}$, and $D = C \cup R$. Arrays

$$\begin{aligned} R(X; R_k, s_\alpha) &= ((X \backslash R) \cup (R_{p_1} \circ_1 R_k) \cup \ldots \cup (R_{p_b} \circ_b R_k)) \backslash R_k \\ R(X; C_j, s_\alpha) &= ((X \backslash C) \cup (C_{m_1} \circ_1 C_j) \cup \ldots \cup (C_{m_a} \circ_a C_j)) \backslash C_j \\ R(X; R_k, C_j, s_\alpha) &= ((X \backslash D) \cup (C_{m_1} \circ_1 C_j) \cup \ldots \cup (C_{m_a} \circ_a C_j) \cup (R_{p_1} \circ_1 R_k) \cup \ldots \\ &\quad \cup (R_{p_b} \circ_b R_k)) \backslash (C_j \cup R_k) \end{aligned}$$

are called a *row-reduction*, *column-reduction*, and *reduction* of X, respectively.

Lemma 2.5. *Let $X \in$ PLS$(a, b; n)$. Suppose that α is replaceable in X by C_j and R_k. Then the following hold:*

(i) $R(X; R_k, s_\alpha)$ *is an* $(n-1) \times n$ *array over* $[n]$ *in which each row and column is Latin,*

(ii) $R(X; C_j, s_\alpha)$ *is an* $n \times (n-1)$ *array over* $[n]$ *in which each row and column is Latin, and*

(iii) $R(X; R_k, C_j, s_\alpha)$ *is an* $(n-1) \times (n-1)$ *array over* $[n]\backslash\{\alpha\}$ *in which each row and column is Latin.*

Henceforth, we consider $R(X; R_k, C_j, s_\alpha)$ as an element of $\mathrm{PLS}(a, b; n-1)$. In Sections 4 and 5 we use reductions to complete elements of $\mathrm{PLS}(2, b; n)$ for $b \geqslant 2$.

Example 2.6. Let $X \in \mathrm{PLS}(2, 2; 6)$ from Example 2.2. The arrays immediately below are $X, R(X; R_3, C_3, s_4)$, and $R(X; R_5, C_4, s_4)$ respectively:

1	2	3	4	5	6
2	3	1	5	6	4
6	5				
3	4				
4	1				
5	6				

1	2	5	6	3
2	3	6	1	5
5	6			
3	5			
6	1			

1	2	3	5	6
2	3	1	6	5
6	5			
5	6			
3	1			

Observe that in the third array of Example 2.6, R_5 and C_4 replace themselves. Also observe that both arrays are elements of $\mathrm{PLS}(2, 2; 5)$ over $\{1, 2, 3, 5, 6\}$.

Let $X \in \mathrm{PLS}(2, b; n)$. By Lemma 2.3, a column of X replaces itself if and only if it is not a column of an intercalate. Reducing X with columns that replace themselves plays a crucial role in proving the main results of this paper.

The next lemma gives a sufficient condition for when we are guaranteed to find a row that replaces a symbol.

Lemma 2.7. *Let* $b \geqslant 2$ *and* $n \geqslant b^2 - b + 3$*. For every* $X \in \mathrm{PLS}(2, b; n)$ *and any symbol* α *not occurring in cells* $[2] \times [b]$ *of* X*, there exists a row that replaces* α*.*

Proof. Let $X \in \mathrm{PLS}(2, b; n)$ and $n \geqslant b^2 - b + 3$. Suppose that symbol α does not occur in cells $[2] \times [b]$ of X. Without loss of generality, assume that symbol α occurs in the last b rows of X. There are at most $b - 1$ rows that cannot replace α in R_n since there are exactly $b - 1$ values of i for which $X(i, 1) = X(n, \beta)$ for some $\beta \in ([b]\backslash\{1\})$. Similarly, there are at most $b - 1$ rows that cannot replace α in R_k, where $n - b + 1 \leqslant k \leqslant n$. Therefore, there are at most $b(b - 1)$ rows that cannot replace α. Since $n > b(b - 1) + 2$, there exists a row replacing α. $\square$

3 A Variation on Smetaniuk's Method

Reducing an element of PLS$(2, 2; n)$ is one of the two main ideas in Section 4. The second is Smetaniuk's inductive argument confirming the famous Evans Conjecture [9]. We briefly outline his argument, and then describe a simple generalization.

For a $k \times k$ array, let D be the set of back diagonal cells; that is, $D = \{(1, k), (2, k-1), \ldots, (k, 1)\}$. We say that cell (i, j) is *above* D if $(i, m) \in D$ for some $m > j$ and *below* D if $(i, m) \in D$ for some $m < j$.

Let $X \in \mathrm{LS}(n)$. The array $T(X) \in \mathrm{PLS}(n + 1)$ is formed from X in the following way:

(i) $T(X)(i, j) = X(i, j)$ if cell (i, j) is above D of $T(X)$,

(ii) $T(X)(i, j) = n + 1$ if cell $(i, j) \in D$, and

(iii) the cells of $T(X)$ below D are empty.

Smetaniuk's main result in [9] says that $T(X)$ can be completed.

Theorem 3.1. *For each* $X \in \mathrm{LS}(n)$, $T(X) \in \mathrm{PLS}(n + 1)$ *is completable.*

Example 3.2. Let $X \in \mathrm{LS}(5)$. The arrays below are X, $T(X) \in \mathrm{PLS}(6)$, and a completion of $T(X)$ respectively. The completion was found in accordance with the proof of Theorem 3.1, which we describe below.

1	2	3	4	5
3	4	1	5	2
5	3	2	1	4
2	5	4	3	1
4	1	5	2	3

1	2	3	4	5	6
3	4	1	5	6	
5	3	2	6		
2	5	6			
4	6				
6					

1	2	3	4	5	6
3	4	1	5	6	2
5	3	2	6	4	1
2	5	6	3	1	4
4	6	5	1	2	3
6	1	4	2	3	5

The argument used to complete $T(X)$ in Theorem 3.1 is inductive. We describe the inductive step and call the entire inductive procedure *the Smetaniuk Method.* See either [9] or [2] for a full proof. Induction is performed on the parameter k, where $0 \leqslant k \leqslant n - 1$.

- To fill the empty cells of R_{n-k}, set $T(X)(n-k, j) = X(n-k, j)$ for $k+3 \leqslant j \leqslant n$ and $T(X)(n - k, n + 1) = X(n - k, k + 2)$.
- Every row and column of $T(X)$ is Latin, except possibly C_{n+1}.
- Suppose that $T(X)(n-k, n+1) = T(X)(m_1, n+1)$ for some $n-k+1 \leqslant m_1 \leqslant n$.
- In this case, switch symbols in cells $(m_1, k + 2)$ and $(m_1, n + 1)$.
- If all rows and columns are Latin, fill $R_{n-(k+1)}$ next.

- Otherwise, there exists an $n-k+1 \leqslant m_2 \leqslant n$ such that $T(X)(m_1, n+1) = T(X)(m_2, n+1)$.
- Switch symbols in cells $(m_2, k+2)$ and $(m_2, n+1)$.
- The process of switching symbols in cells $(m_i, k+2)$ and $(m_i, n+1)$ terminates in a finite number of steps with every row and column of $T(X)$ being Latin.
- R_{n+1} is guaranteed to have a unique completion.

Whenever we complete $T(X)$ using the Smetaniuk Method, we call the result the *Smetaniuk completion*. The following observation will be important.

Observation 3.3. *Let $X \in \mathrm{LS}(n)$ and L be the Smetaniuk completion of $T(X)$.*

(i) $L(i,j) = X(i,j)$ *if cell (i,j) is above D of $T(X)$.*

(ii) $L(i,j) = n+1$ *if cell $(i,j) \in D$ of $T(X)$.*

(iii) If there is a $k \in [n-1]$ such that $\{X(n,2), X(n-1,3), \ldots, X(n-k+1,k+1)\}$ are k distinct symbols, then $X(n-i+1, i+1) = L(n+1, i+1)$ for each $i \in [k]$. Furthermore, $X(i,j) = L(i,j)$ for all cells (i,j) lying below the back diagonal of L and such that $i \leqslant n$ and $2 \leqslant j \leqslant k+1$.

(iv) $L(2, n+1) = X(2, n)$.

We now describe a completion technique that generalizes the Smetaniuk Method. Let $X \in \mathrm{LS}(n)$. If $n+2 = 2k+1$, then D^2 is the 2×2 back diagonal of $[n-1]\times([n+2]\setminus[3])$ along with $(([n+2]\setminus[n])\times[1])\cup(\{n\}\times\{2,3\})$ (see cells containing boldfaced symbols in Example 3.4(b)). If $n+2 = 2k$ for some positive integer k, then D^2 is the set of cells constituting the 2×2 back diagonal of $[n+2]\times[n+2]$ (see cells containing boldfaced symbols in Example 3.4(e)). The PLS $T^2(X)$ of order $n+2$ is formed from X in the following way:

(i) $T^2(X)(i,j) = X(i,j)$ if cell (i,j) is above D^2 of $T^2(X)$,

(ii) $T^2(X)(i,j) \in \{n+1, n+2\}$ if cell $(i,j) \in D^2$, and

(iii) the cells of $T^2(X)$ below D^2 are empty.

Example 3.4. Let $X \in \mathrm{LS}(3)$ and $Y \in \mathrm{LS}(4)$. The arrays below are X, $T^2(X) \in \mathrm{PLS}(5)$, a completion of $T^2(X)$, Y, $T^2(Y) \in \mathrm{PLS}(6)$, and a completion of $T^2(Y)$ respectively. The completions were found in accordance with the generalization of the proof of Theorem 3.1 described below.

2	3	1
3	1	2
1	2	3

2	3	1	**4**	**5**
3	1	2	**5**	**4**
1	**4**	**5**		
4				
5				

2	3	1	**4**	**5**
3	1	2	**5**	**4**
1	**4**	**5**	2	3
4	5	3	1	2
5	2	4	3	1

1	2	4	3
3	4	1	2
2	1	3	4
4	3	2	1

1	2	4	3	**5**	**6**
3	4	1	2	**6**	**5**
2	1	**5**	**6**		
4	3	**6**	**5**		
5	**6**				
6	**6**				

1	2	4	3	**5**	**6**
3	4	1	2	**6**	**5**
2	1	**5**	**6**	3	4
4	3	**6**	**5**	2	1
5	**6**	3	1	4	2
6	**6**	2	4	1	3

Below is the inductive procedure for the Smetaniuk completion of $T^2(X) \in \mathrm{PLS}(n+2)$ when n is odd, where the induction is performed on the parameter k for $0 \leqslant k \leqslant n-1$.

- To fill the empty cells of R_{n-k} when k is even, set $T^2(X)(n-k, j) = X(n-k, j)$ for $k+4 \leqslant j \leqslant n$ and $T^2(X)(n-k, n+1) = X(n-k, k+2)$ and $T^2(X)(n-k, n+2) = X(n-k, k+3)$; to fill the empty cells of R_{n-k} when k is odd, set $T^2(X)(n-k, j) = X(n-k, j)$ for $k+5 \leqslant j \leqslant n$ and $T^2(X)(n-k, n+1) = X(n-k, k+3)$ and $T^2(X)(n-k, n+2) = X(n-k, k+4)$.
- Every row and column of $T^2(X)$ is Latin, except possibly C_{n+1} and C_{n+2}.
- Suppose that $T^2(X)(n-k, n+1) = T^2(X)(m_1, n+1)$ for some $n-k+1 \leqslant m_1 \leqslant n$, or $T^2(X)(n-k, n+2) = T^2(X)(l_1, n+2)$ for some $n-k+1 \leqslant l_1 \leqslant n$ (or both).
- When k is even, switch symbols in cells $(m_1, k+2)$ and $(m_1, n+1)$, or switch symbols in cells $(l_1, k+3)$ and $(l_1, n+2)$ (or both); when k is odd, switch symbols in cells $(m_1, k+3)$ and $(m_1, n+1)$, or switch symbols in cells $(l_1, k+4)$ and $(l_1, n+2)$ (or both).
- If all rows and columns of $T^2(X)$ are Latin, fill $R_{n-(k+1)}$ next.
- Otherwise, there exists an $n-k+1 \leqslant m_2, l_2 \leqslant n$ such that $T^2(X)(m_1, n+1) = T^2(X)(m_2, n+1)$, or $T^2(X)(l_1, n+2) = T^2(X)(l_2, n+2)$ (or both).
- When k is even, switch symbols in cells $(m_2, k+2)$ and $(m_2, n+1)$, or switch symbols in cells $(l_2, k+3)$ and $(l_2, n+2)$ (or both); when k is odd, switch symbols in cells $(m_2, k+3)$ and $(m_2, n+1)$, or switch symbols in cells $(l_2, k+4)$ and $(l_2, n+2)$ (or both).
- The process of switching symbols in these cells terminates in a finite number of steps with every row and column of $T^2(X)$ being Latin.
- R_{n+1} and R_{n+2} are guaranteed to have a completion by Hall's Theorem [6].

If n is even, the inductive procedure is the same, except that the parity of k changes. This accounts for the fact that the procedure (which starts on R_n) begins with a 2×2 block instead of a 1×2 block on the back diagonal.

Theorem 3.5. *For each $X \in \mathrm{LS}(n), T^2(X) \in \mathrm{PLS}(n+2)$ is completable.*

The following variation of Observation 3.3 will be important.

Observation 3.6. *Let $X \in \mathrm{LS}(n)$ and L be the Smetaniuk completion of $T^2(X)$.*

(i) $L(i,j) = X(i,j)$ *if cell* (i,j) *is above* D^2 *of* $T^2(X)$.

(ii) $L(i,j) \in \{n+1, n+2\}$ *if cell* $(i,j) \in D^2$ *of* $T^2(X)$.

(iii) For odd n, if the symbols in cells $(n,2),(n,3)$ *of* X *are distinct from the symbols in cells* $(n-1,4),(n-1,5),(n-2,4),(n-2,5)$ *of* X, *then* $L(n,4) = X(n,4)$ *and* $L(n,5) = X(n,5)$.

When constructing $T^2(X)$, there is more than one way to fill the cells of D^2. However, any appropriate filling of D^2 results in a completed PLS by the Smetaniuk method. Regardless of whether we are completing $T(X)$ or $T^2(X)$, we refer to the completed array as the *Smetaniuk completion of* X.

4 Completing Arrays in $\mathrm{PLS}(2,2;n)$

It is clear that the following two arrays, denoted Y and Z respectively, cannot be completed:

1	2	3	4
3	4	2	1
2	3		
4	1		

1	2	3	4	5
3	1	2	5	4
2	3			
4	5			
5	4			

Let Γ denote the set of all isotopisms of Y and Z.

Theorem 4.1. *Let $n \geqslant 1$ and $A \in \mathrm{PLS}(2,2;n+1)$. The partial Latin square A can be completed if and only if $A \notin \Gamma$.*

Proof. Before we prove the theorem by induction on the order $n+1$ of A, we state four completion methods for reducing and then completing A. We assume that the reduction of A is not an element of Γ, the reduction of A is completable, and the order of A is at least 5. The first three completion methods use the classical Smetaniuk's method, as well as the reductions defined in Section 2. The fourth completion method uses the variation of Smetaniuk's method for $T^2(A)$ and a reduction of A that simply removes two rows and two columns.

Completion Method 1: A contains a replaceable symbol, and a row and a column that replaces that symbol and themselves.

Without loss of generality, assume that the row and column replacing themselves are C_{n+1} and R_{n+1}, $n+1$ is the replaceable symbol, and $n+1$ is located in cells $(1,n+1)$, $(2,n)$, $(n,2)$, and $(n+1,1)$. Let $(2,n+1,\alpha),(n+1,2,\beta) \in A$ for some $\alpha,\beta \in [n]$. By Lemma 2.5, A can be reduced to $R(A; R_{n+1}, C_{n+1}, n+1) \in \mathrm{PLS}(2,2;n)$. Let C

be the completion of $R(A; R_{n+1}, C_{n+1}, n+1)$. The Smetaniuk completion of $T(C)$ is a completion of A by Observation 3.3; (i) guarantees that every symbol above the back diagonal of A remains in their cells, (ii) guarantees that $n+1$ returns to cells $(1, n+1), (2, n), (n, 2)$, and $(n+1, 1)$, (iii) guarantees that β returns to cell $(n+1, 2)$, and (iv) guarantees that α returns to cell $(2, n+1)$.

Completion Method 2: A or A^T, but not both, contains a column that replaces itself.

Without loss of generality, assume that A contains a column that replaces itself, the column replacing itself is C_{n+1}, $n+1$ is the replaceable symbol, and $n+1$ is located in cells $(1, n+1)$ and $(2, n)$. Let $(2, n+1, \alpha) \in A$ for some $\alpha \in [n]$. By Lemma 2.7, there is a row replacing $n+1$ and we assume that it is R_{n+1} (by assumption R_{n+1} does not replace itself). This means that symbol $n+1$ appears in an intercalate and we may assume $(n, 1, n+1), (n-1, 2, n+1) \in A$. Reduce A to $R(A; R_{n+1}, C_{n+1}, n+1) \in \mathrm{PLS}(2, 2; n)$, and let C be the completion of $R(A; R_{n+1}, C_{n+1}, n+1)$. Let $\theta = ((1), (1\ 2\ 3), (1)) \in S_n \times S_n \times S_n$ and set $C' = \theta(C)$ (i.e., in C, move C_1 to C_2, C_2 to C_3, C_3 to C_1, and leave the rows and symbols unchanged). As in Completion Method 1, the Smetaniuk completion of $T(C')$ guarantees a completion of $\theta(A)$ by Observation 3.3, which guarantees a completion of A.

Completion Method 3: Neither A nor A^T has a column that replaces itself and there is at least one intercalate in R_1 and R_2 of A that does not occur in C_1 and C_2 of A.

Let S be the set of symbols appearing in cells $[2] \times [2]$ of A. Symbols from $[n+1] \backslash S$ must occur as intercalates in A. Without loss of generality, assume that intercalate $I_{n,n+1}$ occurs in cells $[2] \times \{n, n+1\}$, but does not occur in C_1 and C_2. For some $\alpha \in [n-1] \backslash S$, assume that intercalate $I_{\alpha,n+1}$ occurs in cells $\{n, n+1\} \times [2]$, specifically with α in cells $(n, 1)$ and $(n+1, 2)$. Let $(1, j, \alpha) \in A$, where $3 \leqslant j \leqslant n-1$. Reduce A to $R(A; R_{n-1}, C_j, n+1) \in \mathrm{PLS}(2, 2; n)$, and let C be the completion of $R(A; R_{n-1}, C_j, n+1)$.

Note that there exists k such that $(k, n-1, \alpha) \in C$. Since α occurs in cells $(n-1, 1)$ and $(n, 2)$ of C, it must be that $k \leqslant n-2$. Let $\theta = ((1\ 2\ \ldots\ k), (1\ 2\ 3), (1)) \in S_n \times S_n \times S_n$. Permute rows and columns of $\theta(C)$, keeping rows and columns 2, 3, $n-1$, and n stationary, until α appears in cells $(n, 3), (n-1, 2), (n-2, 1), (2, n), (1, n-1)$, and cells of the form (i, j) where $i + j = n+1$ (note that α is already in cells $(n, 3), (n-1, 2), (2, n), (1, n-1)$ by θ). Call this new LS (isotopic to $\theta(C)$) C'.

Let L be the Smetaniuk completion of $T(C')$. Since the symbols in cells $(n, 2)$ and $(n-1, 3)$ of C' are distinct, α appears in cells $(n, 3)$ and $(n-1, 2)$ in both arrays C' and L. Furthermore, by Observation 3.3 (iii), $L(n+1, 2) = C'(n, 2) = A(n-1, 1)$ and $L(n+1, 3) = C'(n-1, 3) = A(n-1, 2)$. By Observation 3.3 (iv), $L(2, n+1) = C'(2, n) = A(1, j)$. Since each occurrence of α is above the back diagonal of $T(C')$, except in cells $(n, 3)$ and $(2, n)$, $L(3, n+1) = C'(3, n-1) = A(2, j)$. Thus, the Smetaniuk completion of $T(C')$ guarantees a completion of A.

Completion Method 4: Neither A nor A^T has a column that replaces itself and the intercalates in R_1 and R_2 of A are the same as the intercalates in C_1 and C_2 of A.

Assume that $I_{n,n+1}$ occurs in cells $[2] \times \{n, n+1\}$ and in cells $\{n, n+1\} \times [2]$. Let

$A' \in \mathrm{PLS}(2,2;n-1)$ be the PLS formed from A by removing rows and columns n and $n+1$. Let C be the completion of A'.

If $n-1$ is even, a Smetaniuk completion of $T^2(C)$ guarantees a completion of A by Observation 3.6 (i) and (ii). Suppose that $n-1$ is odd. Let $\theta = ((1),(1\ 4)(2\ 5),(1))$. In $\theta(C)$, we may assume that intercalate $I_{\gamma,\delta}$ occurs in cells $\{n-3,n-2\} \times \{4,5\}$ for some $\gamma,\delta \in [n-1]$. Furthermore, $n+1 \geqslant 9$, since otherwise $A \in \Gamma$ or $A' \in \Gamma$. We may assume that $\theta(C)(n-1,2), \theta(C)(n-1,3) \notin \{\gamma,\delta\}$. Let L be the Smetaniuk completion of $T^2(\theta(C))$. By Observation 3.6 (ii) L contains intercalate $I_{n,n+1}$ on the 2×2 back diagonal. Furthermore, L contains intercalate $I_{\gamma,\delta}$ in cells $\{n,n+1\}\times\{4,5\}$. By Observation 3.6 (iii), $L(n-1,4) = \theta(C)(n-1,4) = A(n-1,1)$ and $L(n-1,5) = \theta(C)(n-1,5) = A(n-1,2)$. Thus, a Smetaniuk completion of $T^2(\theta(C))$ guarantees a completion of $\theta(A)$, which guarantees a completion of A.

We are now ready to begin the induction proof, starting with small cases for $n = 2,3,4,5$, and 6. The result clearly holds for $n \in \{2,3\}$ (for $n=3$ there is only one empty cell and a completion follows using Hall's Theorem). Additionally, all elements of $\mathrm{PLS}(2,2;4)\backslash\Gamma$ are isotopic to one of the PLSs in Figure 1 (in bold face) and each has a completion.

Suppose that $A \in \mathrm{PLS}(2,2;5)$. If A has a row or column (or both) that replaces itself, say R_5 or C_5 (or both), then for $R(A;R_5,C_5,5) \notin \Gamma$, use either Completion Method 1 or 2 to complete A. Otherwise, $R(A;R_5,C_5,5)$ is isotopic to Y. In this case, reduce A to $R(A;R_3,C_5,5)$ or $R(A;R_4,C_5,5)$, since at least one of these arrays does not belong to Γ, and then use either Completion Method 1 or 2. For example, in Figure 2, the first array is $A \in \mathrm{PLS}(2,2;5)$, the second array is $R(A;R_5,C_5,5)$ (which is isotopic to Y), the third array is $R(A;R_4,C_5,5)$, and the fourth array is the completion of a PLS isotopic to A found in accordance with Completion Method 2. If A does not have a row or column that can replace itself, then A must be isotopic to Z.

Suppose that $A \in \mathrm{PLS}(2,2;6)$. If A has a row or column (or both) that replaces itself, say R_6 or C_6 (or both), then for $R(A;R_6,C_6,6) \notin \Gamma$, use either Completion Method 1 or 2 to complete A. Otherwise, $R(A;R_6,C_6,6)$ is isotopic to Z. In this case, reduce A to $R(A;R_3,C_6,6), R(A;R_4,C_6,6)$, or $R(A;R_5,C_6,6)$, since at least two of these arrays do not belong to Γ, and then use either Completion Method 1 or 2. For example, in Figure 3, the first array is $A \in \mathrm{PLS}(2,2;6)$, the second array is $R(A;R_6,C_6,6)$ (which is isotopic to Z), the third array is $R(A;R_3,C_6,6)$, and the fourth array is the completion of a PLS isotopic to A found in accordance with Completion Method 2.

If A does not have a row or column that replaces itself and the intercalates in the first two rows are not identical to the intercalates of the first two columns, use Completion Method 3 to complete A. Otherwise, assume that $I_{5,6}$ occurs in cells $[2] \times \{5,6\}$ and cells $\{5,6\} \times [2]$. Let $A' \in \mathrm{PLS}(2,2;4)$ be the PLS formed from A by removing R_5, R_6, C_5, and C_6. If A' is not isotopic to Y, use Completion Method 4. Otherwise, A' is isotopic to the first PLS (in bold face) in Figure 4, which has a completion.

Now we begin the inductive step. Let $n \geqslant 6$. Suppose that all elements of

$\mathrm{PLS}(2,2;n)\backslash\Gamma$ can be completed and let $A \in \mathrm{PLS}(2,2;n+1)$.

If A has a row or column (or both) that replaces itself, say R_{n+1} or C_{n+1} (or both), then $R(A; R_{n+1}, C_{n+1}, n+1) \notin \Gamma$, and use either Completion Method 1 or 2. If A does not have a row or column that replaces itself and the intercalates in the first two rows are not identical to the intercalates of the first two columns, use Completion Method 3. Otherwise, assume that $I_{n,n+1}$ is in cells $[2] \times \{n, n+1\}$ and cells $\{n, n+1\} \times [2]$. Let $A' \in \mathrm{PLS}(2,2;n-1)$ be the array formed from A by removing R_n, R_{n+1}, C_n, and C_{n+1}. If A' is not isotopic to Z, use Completion Method 4. Otherwise, A is isotopic to the second PLS in Figure 4, which has a completion. $\square$

1	**2**	**3**	**4**
2	**1**	**4**	**3**
3	**4**	2	1
4	**3**	1	2

1	**2**	**3**	**4**
3	**1**	**4**	**2**
4	**3**	2	1
2	**4**	1	3

1	**2**	**3**	**4**
3	**4**	**1**	**2**
2	**1**	4	3
4	**3**	2	1

1	**2**	**3**	**4**
3	**4**	**1**	**2**
2	**3**	4	1
4	**1**	2	3

1	**2**	**3**	**4**
3	**4**	**2**	**1**
2	**1**	4	3
4	**3**	1	2

Figure 1: Completions for all elements of $\mathrm{PLS}(2,2;4)\backslash\Gamma$, up to isotopisms.

1	2	3	4	5
3	4	5	1	2
2	5			
4	1			
5	3			

1	2	4	3
3	4	1	2
4	1		
2	3		

1	2	4	3
3	4	1	2
2	1		
4	3		

1	2	3	4	5
3	4	5	1	2
2	5	4	3	1
5	3	1	2	4
4	1	2	5	3

Figure 2: Completing $A \in \mathrm{PLS}(2,2;5)$ when $R(A; R_5, C_5, 5)$ is isotopic to Y.

1	2	3	4	5	6
3	1	6	5	4	2
2	3				
4	6				
5	4				
6	5				

1	2	4	5	3
3	1	5	4	2
2	3			
5	4			
4	5			

1	2	4	5	3
3	1	5	4	2
5	4			
4	3			
2	5			

1	2	3	4	5	6
3	1	6	5	4	2
5	4	1	2	6	3
4	6	2	1	3	5
6	5	4	3	2	1
2	3	5	6	1	4

Figure 3: Completing $A \in \mathrm{PLS}(2,2;6)$ when $R(A; R_6, C_6, 6)$ is isotopic to Z.

5 Completing Arrays in $\mathrm{PLS}(2,b;n)$ for $b \geqslant 3$

First, we define arrays in $\mathrm{PLS}(2,b;n)$ that are incompletable, and conjecture that these characterize all incompletable arrays of $\mathrm{PLS}(2,b;n)$ for $b \geqslant 2$. Then we provide sufficient conditions under which arrays in $\mathrm{PLS}(2,b;n)$ for $b \geqslant 3$ are always completable.

Definition 5.1. Let $A \in \mathrm{PLS}(2,b;n)$, where $b+2 \leqslant n \leqslant 2b+1$. Let i and j be positive integers such that $n = b+1+j$ and $j \leqslant i \leqslant b$. We say that A is a *bad array* if A satisfies the following:

1	**2**	**3**	**4**	**5**	**6**
3	**4**	**2**	**1**	**6**	**5**
4	**1**	5	6	3	2
2	**3**	6	5	1	4
5	**6**	1	2	4	3
6	**5**	4	3	2	1

1	**2**	**3**	**4**	**5**	**6**	**7**
3	**1**	**2**	**5**	**4**	**7**	**6**
2	**3**	1	6	7	5	4
4	**5**	7	1	6	2	3
5	**4**	6	7	1	3	2
6	**7**	4	2	3	1	5
7	**6**	5	3	2	4	1

Figure 4: Completions of elements of PLS$(2, 2; 6)$ and PLS$(2, 2; 7)$.

(i) there exists a subset $T \subset [n]$ of i symbols such that each of the bottom k rows of X (with $b + j - i \leqslant k \leqslant b$) contains T, and

(ii) the last column of X contains 2 symbols from $[n] \backslash T$.

Let Γ_b be the set of isotopisms of bad arrays in PLS$(2, b; n)$.

Observe that the arrays in Γ_b can not be completed. To see this, let $A \in \Gamma_b$. In order to complete A, cells $(n - k + 1, n), \ldots, (n, n)$ must contain symbols outside of T. Therefore, $i + k + 2 \geqslant b + j + 2$ symbols are needed to complete A. Hence, A can not be completed.

Example 5.2. The following is an example of a bad array in PLS$(2, 3; 5)$ and (an isotopism of) a bad array in PLS$(2, 4; 9)$, respectively:

1	2	3	4	5
4	1	5	2	3
3	5	2		
5	3	4		
2	4	1		

1	2	3	4	5	6	7	8	9
3	1	5	2	4	9	6	7	8
2	5	4	1					
4	3	2	5					
5	4	1	3					
9	8	7	6					
6	9	8	7					
7	6	9	8					
8	7	6	9					

Conjecture 5.3. *Let $A \in$ PLS$(2, b; n)$. Then A cannot be completed if and only if $A \in \Gamma_b$.*

Note that Conjecture 5.3 implies that $A \in$ PLS$(2, b; n)$ is completable when $n \geqslant 2b+2$. In Section 4, Conjecture 5.3 was confirmed for $b = 2$. When $b = 2$, there exists (up to isotopisms) exactly two bad arrays, one of order 4 and one of order 5.

Let $b \geqslant 3$ be a positive integer and $A \in$ PLS$(2, b; n)$. By Lemma 2.3, C_k of X for $k > b$ replaces itself if and only if it is not a column of an intercalate. As in the $b = 2$ case, finding a column that replaces itself means that A can be reduced, allowing the use of Smetaniuk's Method.

Proposition 5.4. *Let $A \in \mathrm{PLS}(2, b; n)$ for $b \geqslant 3$. Assume that n is a replaceable symbol, R_n and C_n replace n, and C_n replaces itself. If $R(A; R_n, C_n, n)$ can be completed, then A can be completed.*

Proof. By assumption, $R(A; R_n, C_n, n)$ can be completed, and we denote the completion C. Since C_n replaces itself, Completion Method 1 or 2 (from the proof of Theorem 4.1) guarantees a completion of A. $\square$

The obvious set back to using Proposition 5.4 to complete elements of $\mathrm{PLS}(2, b; n)$ is verifying the completability of $R(A; R_n, C_n, n)$, especially if there exists elements of $\mathrm{PLS}(2, b; n-1)$ that cannot be completed. Furthermore, if no column replaces itself, we can no longer work with A^T since there is more than one nonempty cell below the back diagonal of A^T. Additionally, if we reduce A twice by removing an intercalate in the first two rows, Hall's Theorem will not fix symbols in cells of the last two rows. Given these set backs, in what follows, we instead reduce the conjugate $A^{(rs)}$. The main advantage in reducing $A^{(rs)}$ is that the reduction of $A^{(rs)}$ has a guaranteed completion by Hall's Theorem (this is explained in the proof of Theorem 5.11).

Example 5.5. The PLS below is $A \in \mathrm{PLS}(2, 3; 10)$, followed by $A^{(rs)}$:

1	2	3	4	5	6	7	8	9	10
3	1	2	5	4	7	8	6	10	9
2	4	1							
4	5	6							
5	6	7							
6	7	9							
7	9	5							
8	3	10							
9	10	8							
10	8	4							

1	2	3							
3	1	2							
2	8	1							
4	3	10	1	2					
5	4	7	2	1					
6	5	4			1		2		
7	6	5			2	1			
8	10	9				2	1		
9	7	6						1	2
10	9	8						2	1

Let $A \in \mathrm{PLS}(2, b; n)$. Note that $A^{(rs)}$ has the property that symbols 1 and 2 occur in every row and column, while each of the symbols in $\{3, ..., n\}$ occur in exactly b rows and b columns. Additionally, the first b columns in $A^{(rs)}$ are filled. If cells $[2] \times [b]$ consist only of symbols from $[b]$, then all occurrences of symbols 1 and 2 in the first b columns are in rows $1, \ldots, b$ in $A^{(rs)}$.

With R_1 and R_2 of $A \in \mathrm{PLS}(2, b; n)$, define the permutation $\sigma_A : [n] \to [n]$ such that $\sigma_A(i) = j$ if and only if $(2, i, j) \in A$. For example, $A \in \mathrm{PLS}(2, 3; 10)$ from Example 5.5 has permutation $\sigma_A = (1\ 3\ 2)(4\ 5)(6\ 7\ 8)(9\ 10)$. Let $I \subseteq [n]$. We use $\sigma_A(I)$ to denote the restriction of the domain of σ_A to I.

Suppose that σ_A consists of t cycles of lengths $l_1, \ldots, l_t$. Let $k_i = l_1 + \cdots + l_i$ for $1 \leqslant i \leqslant t$. The rows and columns of $A^{(rs)}$ can be permuted so that symbol 1 occurs on the back diagonal, symbol 2 either occurs immediately above the back diagonal or in cells $(n, k_1), (n-k_1, k_2), (n-k_2, k_3) \ldots, (n-k_{t-1}, k_t)$, and the completely filled columns are still the first b columns. Such an isotopism of $A^{(rs)}$ is called a *standard cycle-form*.

Example 5.6. Let $A^{(rs)}$ be as in Example 5.5. The following arrays are $A^{(rs)}$ and a standard cycle-form of $A^{(rs)}$:

1	2	3							
3	1	2							
2	8	1							
4	3	10	1	2					
5	4	7	2	1					
6	5	4			1		2		
7	6	5			2	1			
8	10	9				2	1		
9	7	6						1	2
10	9	8						2	1

9	10	8						2	1
5	6	7					2	1	
4	5	6					1		2
8	9	10			2	1			
6	7	9			1	2			
7	4	5	2	1					
10	3	4	1	2					
3	2	1							
2	1	3							
1	8	2							

The following lemmas and definition are used to prove the next main result (see [7] for a proof of the first lemma).

Lemma 5.7. *Let G be a balanced bipartite graph on $2n$ vertices with $\delta(G) = k$. Let $S \subseteq E(G)$ be a set of s independent edges ($s \geqslant 1$), and let $T \subseteq E(G) \setminus S$ be a set of t edges. If $k - s \geqslant \frac{(n-1)}{2}$ and $t \leqslant k - s - 1$, then G contains a 1-factor F in which $S \subseteq F$ and $F \cap T = \emptyset$.*

We use the following definition to prove our main result.

Definition 5.8. Let $A \in \mathrm{PLS}(2, b; n)$ and $\theta \in S_n \times S_n \times S_n$. Let $\alpha \in [n]$ such that $(m_i, i, \alpha) \in \theta(A^{(rs)})$ for each $i \in [b]$. Let $j \in [n]$. If R_j is a row in $\theta(A^{(rs)})$ such that each row of $\{R_{m_1} \circ_1 R_j, \ldots, R_{m_b} \circ_b R_j\}$ is Latin, then R_j *replaces* α in $\theta(A^{(rs)})$. Let $R = R_{m_1} \cup \ldots \cup R_{m_b}$. The array formed from replacing α with R_j in $\theta(A^{(rs)})$ is

$$((\theta(A^{(rs)}) \backslash R) \cup (R_{m_1} \circ_1 R_j) \cup \ldots \cup (R_{m_b} \circ_b R_j)) \backslash R_j.$$

Lemma 5.9. *Let $b \geqslant 2, A \in \mathrm{PLS}(2, b; n)$, and $B = A^{(rs)}$ in standard cycle-form. Pick any α not occurring in cells $[2] \times [b]$ of B. Furthermore, let C be a $k \times n$ subrectangle of B. If $k \geqslant b^2 - b + 3$, then there is a row of C replacing symbol α in B.*

Proof. Let $A \in \mathrm{PLS}(2, b; n), B = A^{(rs)}$ in standard cycle-form, and C be a $k \times n$ subrectangle of B. Pick any α not occurring in cells $[2] \times [b]$ of B. Similar to the proof of Lemma 2.7, if R_j of B contains α then there are at most $b - 1$ rows of B that cannot replace α in R_j of B. Therefore, there are at most $b(b - 1)$ rows of B that cannot replace α in B. Since $k \geqslant b^2 - b + 3$, there must be a row of C replacing α in B. □

Lemma 5.10. *Let $A \in \mathrm{PLS}(2, b; n)$ and $k \in \mathbb{Z}$. Suppose that R_n replaces symbol 1 in $A^{(rs)}$ and $b + k \leqslant \frac{n-3}{2}$. Then there exists an array $B \in \mathrm{PLS}(n)$ containing $A^{(rs)}$ such that the first $b + k$ columns of B are filled and R_n of B replaces symbol 1.*

Proof. Let $k \in \mathbb{Z}$ and suppose that $b+k \leqslant \frac{n-3}{2}$. Suppose inductively that there exists an array B' containing $A^{(rs)}$ such that the first $b+k'-1$ columns of B' are filled for some $k' \in [k]$ and R_n of B' replaces symbol 1. Build a bipartite graph $G = (V_1, V_2; E)$ in which $V_1 = \{R_1, \ldots, R_n\}$, $V_2 = \{s_1, \ldots, s_n\}$, and for $j > 2$, $R_i s_j \in E$ if and only if symbol j does not occur in R_i of B'. If $j \in \{1, 2\}$, then s_j is adjacent to each vertex of V_1. It follows that $d(R_i) \geqslant n-b-k'+1$ for each $i \in [n]$ and $d(s_j) \geqslant n-b-k'+1$ for each $j \in [n]$.

Consider $C_{b+k'}$ of B'. Suppose that $(j_1, b+k', 1), (j_2, b+k', 2) \in B'$, and suppose that symbol $\alpha \in [n]$ does not occur in R_{j_1} and R_n of B'. Such a symbol exists since $2(b+k-1) < n$. Consider independent edges $R_{j_1}s_1$, $R_{j_2}s_2$, and $R_n s_\alpha$ in G. By Lemma 5.7, these three edges can be extended to a 1-factor of G if $n-b-k'-2 \geqslant \frac{n-1}{2}$, or $b+k' \leqslant \frac{n-3}{2}$. This holds since $b+k \leqslant \frac{n-3}{2}$.

Let F be a 1-factor of G which includes $R_{j_1}s_1$, $R_{j_2}s_2$, and $R_n s_\alpha$. Place symbol β in cell $(i, b+k')$ of B' if and only if $R_i\beta \in F$. Since α occurs in cell $(n, b+k')$ and does not occur in R_{j_1}, R_n replaces symbol 1 in this new array. This process continues until the first $b+k$ columns are completed, resulting in the desired array B containing $A^{(rs)}$ such that R_n replaces symbol 1 in B. □

Theorem 5.11. *Let $A \in \mathrm{PLS}(2, b; n)$. Suppose that cells $[2] \times [b]$ consist only of symbols from $[b]$. If $n \geqslant 2b^2 - 2b + 5$ and $\sigma_A([n] \backslash [b])$ contains a cycle of length at least $\frac{n+5}{2}$, then A is completable.*

Proof. Let $A \in \mathrm{PLS}(2, b; n)$ and suppose that cells $[2] \times [b]$ consist only of symbols from $[b]$. Suppose that $n \geqslant 2b^2 - 2b + 5$ and $\sigma'_A = \sigma_A([n] \backslash [b])$ contains a cycle of length $l \geqslant \frac{n+5}{2}$. Without loss of generality, assume that $A^{(rs)}$ is in a standard cycle-form in which the first b columns are filled and the longest cycle of σ'_A occurs in cells $[l] \times ([n] \backslash [n-l])$.

Since $n \geqslant 2b^2 - 2b + 5$ and $l \geqslant \frac{n+5}{2}$, by Lemma 5.9 there is a row among the first l rows of $A^{(rs)}$ that replaces symbol 1. We may assume that R_l replaces symbol 1. Set $\theta = ((\ n\ n-1\ \ldots\ l), (1\ 2\ \ldots\ n-l+1), (1)) \in S_n \times S_n \times S_n$. Observe that in $\theta(A^{(rs)})$, R_n replaces symbol 1. Since $A^{(rs)}$ is in a standard cycle-form, symbol 1 appears on the back diagonal cells (i.e., cells $(n,1), (n-1,2), ..., (1,n)$) of $\theta(A^{(rs)})$. By Lemma 5.10, since $b + (n-l+1-b) \leqslant \frac{n-3}{2}$, $C_1, C_{b+2}, C_{b+3}, \ldots, C_{n-l+1}$ of $\theta(A^{(rs)})$ can be completed. Let $B \in \mathrm{PLS}(n)$ denote the array containing $\theta(A^{(rs)})$ with the first $n-l+1$ columns completed. Furthermore, from Lemma 5.10, we may assume that R_n replaces symbol 1 in B. Observe that symbol 2 only appears below symbol 1 once in the last l columns of B, specifically in cell (n,n).

In the first $n-l+1$ columns of B, replace symbol 1 with R_n, remove symbol 1 in the last $l-1$ columns, and remove R_n and C_n. Call the reduced array B' and observe that B' is the (rs)-conjugate of an element of $\mathrm{PLS}(1, n-l+1; n-1)$. Since each array in $\mathrm{PLS}(1, n-l+1; n-1)$ can be completed using Hall's Theorem, let C be a completion of B'. Furthermore, let L be the Smetaniuk completion of $T(C)$. Since the symbols in cells $(n-1, 2), (n-2, 3), \ldots, (l, n-l+1)$ of C are all distinct, by Observation 3.3 (iii), $L(i,j) = C(i,j)$ if (i,j) is below the back diagonal cells of

L and such that $i \leqslant n-1$ and $2 \leqslant j \leqslant n-l+1$. Also, by Observation 3.3 (iii), $L(n, i+1) = C(n-i, i+1) = B(n, i+1)$ for $i \in [n-1]$. Furthermore, since symbol 2 is above symbol 1 in columns $n-l+2, ..., n-1$, it follows that $(n, n, 2) \in L$. Thus, the Smetaniuk completion of $T(C)$ guarantees a completion of B. Since B contains $\theta(A^{(rs)})$, a completion of A exists. □

References

[1] P. Adams, D. Bryant and M. Buchanan, Completing partial Latin squares with two filled rows and two filled columns, *Electron. J. Combin.* 15 (2008), #R56.

[2] A. Asratian, T. M. J. Denleyx, and R. Häggkvist, *Bipartite Graphs and Their Applications*, Cambridge University Press, 1998.

[3] M. Buchanan, *Embedding, existence and completion problems for Latin squares*, PhD thesis, University of Queensland, 2007.

[4] C. J. Colbourn, The complexity of completing partial Latin squares, *Discrete Appl. Math.* 8 (1984), 25–30.

[5] T. Evans, Embedding incomplete Latin squares, *Amer. Math. Monthly* 67 (1960), 958–961.

[6] M. Hall, An existence theorem for Latin squares, *Bull. Amer. Math. Soc.* 51 (1945), 387–388.

[7] T. Denley and J. Kuhl, Completing partial Latin squares: a conjecture of Häggkvist, Proc. Thirty-Fifth Southeastern Int. Conf. Combinatorics, Graph Theory and Computing, *Congr. Numer.* 171 (2004), 123–128.

[8] H. J. Ryser, A combinatorial theorem with an application to Latin rectangles, *Proc. Amer. Math. Soc.* 2 (1951), 550–552.

[9] B. Smetaniuk, A new construction of Latin squares I. A proof of the Evans conjecture, *Ars Combin.* 11 (1981), 155–172.

(Received 10 May 2016; revised 10 Oct 2016, 21 Jan 2017)