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Packing vector spaces into vector spaces

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Abstract

A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces with trivial intersection such that each non-zero vector is covered at most once. How many t-dimensional subspaces can be packed into $\mathbb{F}_q^n$, i.e., what is the maximum cardinality of a partial t-spread? An upper bound, given by Drake and Freeman, survived more than forty years without any improvement. At the end of 2015, the upper bounds started to crumble. Here, we give self-contained elementary proofs of the current (unpublished) state of the art—inviting the reader to pursue the newly opened path to improved upper bounds for partial t-spreads.

1 Introduction

Let $q > 1$ be a prime power and n a positive integer. A *vector space partition* $\mathcal{P}$ of $\mathbb{F}_q^n$ is a collection of subspaces with the property that every non-zero vector is contained in a unique member of $\mathcal{P}$. If $\mathcal{P}$ contains m_d subspaces of dimension d, then $\mathcal{P}$ is of type $k^{m_k} \dots 1^{m_1}$, where the dimensions are written in decreasing order. We may leave out some of the cases with $m_d = 0$. Subspaces of dimension 1 are called *holes*. If there is at least one non-hole, then $\mathcal{P}$ is called non-trivial.

A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces such that the non-zero vectors are covered at most once, i.e., a vector space partition of type $t^{m_t}1^{m_1}$. By $A_q(n, 2t; t)$ we denote the maximum value of m_t.

The more general notation $A_q(n, 2t-2w; t)$ denotes the maximum cardinality of a collection of t-dimensional subspaces, whose pairwise intersections have a dimension of at most w. Those objects are called *constant dimension codes*, see e.g. [6]. For known bounds, we refer to [10] containing also the generalization to *subspace codes* of mixed dimension.

Writing $n = kt + r$, with $k, r \in \mathbb{N}_0$ and $r \leq t - 1$, we can state that for $r \leq 1$ or $n \leq 2t$ the exact value of $A_q(n, 2t; t)$ was known for more than forty years [1]. Via a computer search the cases $A_2(3k+2, 6; 3)$ were settled in 2010 by El-Zanati et al. [5].

In 2015 the case $q = r = 2$ was resolved by continuing the original approach of Beutelspacher [13], i.e., by *considering* the set of holes in $(n-2)$-dimensional subspaces and some averaging arguments. Very recently, Năstase and Sissokho found a very clear generalized averaging method for the number of holes in $(n-j)$-dimensional subspaces, where $j \le t-2$, and general q, see [14]. Their Theorem 5 determines the exact values of $A_q(kt+r, 2t; t)$ in all cases where $t > \begin{bmatrix} r \\ 1 \end{bmatrix}_q := \frac{q^r-1}{q-1}$. Here, we streamline and generalize their approach leading to improved upper bounds on $A_q(n, 2t; t)$, cf. [15]. All previously known upper bounds are condensed in Theorems 2.9 and 2.10 at the end of the paper. The very few cases, known to us, where the corresponding upper bound can be further lowered, are listed in the Appendix.

2 Subspaces with the minimum number of holes

Definition 2.1 A vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ has *hole-type* (t, s, m_1), if it is of type $t^{m_t} \dots s^{m_s} 1^{m_1}$, for some integers $n > t \ge s \ge 2$, $m_i \in \mathbb{N}_0$ for $1 \le i \le t$, $m_i = 0$ for $2 \le i \le s-1$, and $\mathcal{P}$ is non-trivial.

Lemma 2.2 *(Cf. [14, Proof of Lemma 9].) Let $\mathcal{P}$ be a non-trivial vector space partition of $\mathbb{F}_q^n$ of type $t^{m_t} \dots s^{m_s} 1^{m_1}$ and $l, x \in \mathbb{N}_0$ with $\sum_{i=s}^{t} m_i = lq^s + x$. $\mathcal{P}_H = \{U \cap H \,:\, U \in \mathcal{P}\}$ is a vector space partition of type $t^{m'_t} \dots (s-1)^{m'_{s-1}} 1^{m'_1}$, for a hyperplane H containing $\widehat{m}_1$ holes of $\mathcal{P}$. We have $\mathbb{Z} \ni \frac{m_1+x-1}{q} \equiv \widehat{m}_1 \equiv m_1 \pmod{q^{s-1}}$. If $s > 2$, then $\mathcal{P}_H$ is non-trivial and $m'_1 = \widehat{m}_1$.*

Proof. If $U \in \mathcal{P}$, then $\dim(U) - \dim(U \cap H) \in \{0, 1\}$ for an arbitrary hyperplane H. Since $\mathcal{P}$ is non-trivial, we have $n \ge s$.

For $s > 2$, counting the 1-dimensional subspaces of $\mathbb{F}_q^n$ and H, via $\mathcal{P}$ and $\mathcal{P}_H$, yields $(lq^s + x) \cdot \begin{bmatrix} s \\ 1 \end{bmatrix}_q + aq^s + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and $(lq^s + x) \cdot \begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q + a'q^{s-1} + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ for some $a, a' \in \mathbb{N}_0$. By subtracting we obtain $(lq^s + x) \cdot q^{s-1} + aq^s - a'q^{s-1} + m_1 - \widehat{m}_1 = q^{n-1}$, so that $m_1 \equiv \widehat{m}_1 \pmod{q^{s-1}}$. Since $1 + q \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q = 0$ we conclude $1 + q\widehat{m}_1 - m_1 - x \equiv 0 \pmod{q^s}$. Thus, $\mathbb{Z} \ni \frac{m_1+x-1}{q} \equiv \widehat{m}_1 \pmod{q^{s-1}}$. Since $\mathcal{P}$ is non-trivial there exists an element $U \in \mathcal{P}$ with $\dim(U) \ge 3$, so that $U \cap H \in \mathcal{P}_H$ with $\dim(U \cap H) \ge 2$, i.e., $\mathcal{P}_H$ is non-trivial. Since every $U \in \mathcal{P}$ satisfies $\dim(U) = 1$ or $\dim(U) \ge 3$, we have $\dim(U \cap H) = 1$ if and only if $\dim(U) = 1$ and $U \subseteq H$. Thus, we have $m'_1 = \widehat{m}_1$.

For $s = 2$ we have $(lq^2 + x) \cdot (q+1) + aq^2 + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ and $(lq^2 + x) \cdot 1 + a'q + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ leading to the same conclusion $\mathbb{Z} \ni \frac{m_1+x-1}{q} \equiv \widehat{m}_1 \equiv m_1 \pmod{q^{s-1}}$. □

Lemma 2.3 *(Cf. [14, Proof of Lemma 9].) Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t, s, m_1), $l \in \mathbb{N}_0$, $x \in \mathbb{N}_{\ge 1}$ with $\sum_{i=s}^{t} m_i = lq^s + x$, and $b, c \in \mathbb{Z}$ with $m_1 = bq^s + c \ge 1$, where we do not require $x < q^s$, $c < q^s$, or $b, c \ge 0$. Then, there exists a hyperplane $\widehat{H}$ with $\widehat{m}_1 = \widehat{b}q^{s-1} + \widehat{c}$ holes, where $\widehat{c} := \frac{c+x-1}{q} \in \mathbb{Z}$, $b > \widehat{b} \in \mathbb{Z}$, and $\widehat{m}_1 \equiv \widehat{c} \equiv c \equiv m_1 \pmod{q^{s-1}}$.*

Proof. The average number of holes per hyperplane is given by $m_1 \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, which is strictly smaller than $\frac{m_1}{q}$. Let $\widehat{H}$ be a hyperplane with the minimum number of holes, whose quantity is denoted by $\widehat{m}_1$, hence $\widehat{m}_1 < \frac{m_1}{q}$. Apply Lemma 2.2 and observe $m_1 \equiv c \pmod{q^s}$. Assuming $\widehat{b} \ge b$ yields $q\widehat{m}_1 \ge q \cdot (bq^{s-1} + \widehat{c}) = bq^s + c + x - 1 \ge m_1$, which is a contradiction. $\square$

Corollary 2.4 *Using the notation from Lemma 2.3, let $\mathcal{P}$ be a non-trivial vector space partition with $x \ge 1$ and f be the largest integer such that q^f divides c. For each $0 \le j \le s - \max\{1, f\}$ there exists an $(n-j)$-dimensional subspace U containing $\widehat{m}_1$ holes with $\widehat{m}_1 \equiv \widehat{c} \pmod{q^{s-j}}$ and $\widehat{m}_1 \le (b-j) \cdot q^{s-j} + \widehat{c}$, where $\widehat{c} = \frac{c + \begin{bmatrix} j \\ 1 \end{bmatrix}_q \cdot (x-1)}{q^j} \in \mathbb{Z}$.*

Proof. Observe $\widehat{m}_1 \equiv c \not\equiv 0 \pmod{q^{s-j}}$, i.e., $\widehat{m}_1 \ge 1$, for all $j < s - f$. $\square$

So far, we can guarantee that some subspace contains not *too many* holes. Next, we adjust to the situation of partial spreads before we come up with some non-existence results for vector space partitions with *few* holes.

Lemma 2.5 *Let $\mathcal{P}$ be a non-trivial vector space partition of type $t^{m_t} 1^{m_1}$ of $\mathbb{F}_q^n$ with $m_t = lq^t + x$, where $l = \frac{q^{n-t} - q^r}{q^t - 1}$, $x \ge 2$, $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z + u > r$, $q^f | x-1$, $q^{f+1} \nmid x-1$, and $f, u, z, r, x \in \mathbb{N}_0$. For $\max\{1, f\} \le y \le t$ there exists an $(n-t+y)$-dimensional subspace U with $L \le (z+y-1-u)q^y + w$ holes, where $w = -(x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q$ and $L \equiv w \pmod{q^y}$.*

Proof. Due to

$$m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q - m_t \cdot \begin{bmatrix} t \\ 1 \end{bmatrix}_q = \begin{bmatrix} r \\ 1 \end{bmatrix}_q q^t - \begin{bmatrix} t \\ 1 \end{bmatrix}_q (x-1),$$

we have $m_1 = bq^t + c$ for $b = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $c = -\begin{bmatrix} t \\ 1 \end{bmatrix}_q (x-1)$, where $q^{f'} | x-1$ if and only if $q^{f'} | c$ for $f' \in \mathbb{N}_0$. Setting $s = t$ and $j = t - y$, we observe $0 \le j \le s - \max\{1, f\}$, since $\max\{1, f\} \le y \le t$. With this, we apply Corollary 2.4 and obtain an $(n-t+y)$-dimensional subspace U with

$$\begin{aligned} L &= \widehat{m}_1 \le (b-j) \cdot q^{s-j} + \frac{c + \begin{bmatrix} j \\ 1 \end{bmatrix}_q \cdot (x-1)}{q^j} \\ &= (z+y-1-u) \cdot q^y - (x-1) \cdot \frac{\begin{bmatrix} t \\ 1 \end{bmatrix}_q - \begin{bmatrix} t-y \\ 1 \end{bmatrix}_q}{q^{t-y}} \\ &= (z+y-1-u)q^y - (x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q = (z+y-1-u)q^y + w \end{aligned}$$

holes, so that $L \le (z+y-1-u)q^y + w$ and $L \equiv w \pmod{q^y}$. $\square$

Lemma 2.6 *Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ with $c \geq 1$ holes and let a_i denote the number of hyperplanes containing i holes. Then, $\sum_{i=0}^{c} a_i = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, $\sum_{i=0}^{c} ia_i = c \cdot \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ and $\sum_{i=0}^{c} i(i-1)a_i = c(c-1) \cdot \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$.*

Proof. Double-count the incidences of the tuples (H), (B_1, H), and (B_1, B_2, H), where H is a hyperplane and $B_1 \neq B_2$ are points contained in H. □

Lemma 2.7 *Let $m \in \mathbb{Z}$, and $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t, s, c). Then, $\tau_q(c, q^{s-1}, m) \cdot q^{n-2s} - m(m-1) \geq 0$, where $\tau_q(c, q^{s-1}, m) = m(m-1)q^{2s} - c(2m-1)(q-1)q^s + c(q-1)\Big(c(q-1)+1\Big)$.*

Proof. As an abbreviation we set $\Delta = q^{s-1}$. Consider the three equations from Lemma 2.6. $(c - m\Delta)\Big(c - (m-1)\Delta\Big)$ times the first minus $\Big(2c - (2m-1)\Delta - 1\Big)$ times the second plus the third equation gives

$$\sum_{i=0}^{c} (m\Delta - c + i) \cdot (m\Delta - c + i + \Delta) a_i = \frac{1}{q-1} \cdot \Big(m(m-1)\Delta^2 q^n$$
$$-c(2m-1)(q-1)\Delta q^{n-1} + c(q-1)\Big(c(q-1)+1\Big)q^{n-2} - m(m-1)\Delta^2\Big).$$

Due to Lemma 2.2 we can have $a_i > 0$ only if $i \equiv c \pmod{\Delta}$. So, substituting $i = c - h\Delta$ and dividing the equation by $\Delta^2/(q-1)$, yields

$$(q-1) \cdot \sum_{h=0}^{\lfloor c/\Delta \rfloor} (m-h)(m-h-1) a_{c-h\Delta} = \tau_q(c, \Delta, m) \cdot \frac{q^{n-2}}{\Delta^2} - m(m-1).$$

We observe $a_i \geq 0$ for all $0 \leq i \leq c$ and $(m-h)(m-h-1) \geq 0$ for all $m, h \in \mathbb{Z}$. □

Lemma 2.8 *For integers $n > t \geq s \geq 2$ and $i \leq s-1$, there exists no vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ of hole-type (t, s, c), where $c = i \cdot q^s - \begin{bmatrix} s \\ 1 \end{bmatrix}_q + s - 1$.*

Proof. Since we have $c < 0$ for $i \leq 0$, we can assume $i \geq 1$ in the following. Let, to the contrary, $\mathcal{P}$ be such a vector space partition and apply Lemma 2.7 with $m = i(q-1)$ onto $\mathcal{P}$. We compute $\tau_q(c, q^{s-1}, m) = (m-1-a)\, q^s + a(a+1)$ using $c(q-1) = q^s(m-1) + a$, where $a := 1 + (s-1)(q-1)$. Setting $y = s-1-i$, we have $0 \leq y \leq s-2$ and $\tau_q(c, q^{s-1}, m) = -q^s(y(q-1)+2) + (s-1)^2 q^2 - q(s-1)(2s-5) + (s-2)(s-3)$.

If $q = 2$, then $y \geq 0$ and $s \geq 2$ yields

$$\tau_2(c, 2^{s-1}, m) = -2^s(y+2) + s^2 + s \leq \big(s^2 - s - 2^s\big) + (2s - 2^s) < 0.$$

If $s = 2$, then we have $y = 0$ and $\tau_q(c, q^{s-1}, m) = -q^2 + q < 0$. If $q, s \geq 3$, then we have $q(2s-5) \geq s-3$, so that $\tau_q(c, q^{s-1}, m) \leq -2q^s + (s-1)^2 q^2 \leq -2 \cdot 3^{s-2} q^2 + (s-1)^2 q^2$ due to $y \geq 0$ and $q \geq 3$. Since $2 \cdot 3^{s-2} > (s-1)^2$ for $s \geq 3$, we have $\tau_q(c, q^{s-1}, m) < 0$ in all cases.

Thus, Lemma 2.7 yields a contradiction, since $q^{n-2s} > 0$ and $m(m-1) \geq 0$ for every integer m. □

For more general non-existence results of vector space partitions see e.g. [9, Theorem 1] and the related literature.

After these preparations we are ready to prove two upper bounds on the maximum cardinality $A_q(n,2t;t)$. Suppose $A_q(n,2t;t)=m_t$ for some integer m_t; then there exists a vector space partition of $\mathbb{F}_q^n$ of type $t^{m_t}1^{m_1}$, where $m_1=\begin{bmatrix}n\\1\end{bmatrix}_q-m_t\cdot\begin{bmatrix}t\\1\end{bmatrix}_q$.

Theorem 2.9 *(Cf. [14, Lemma 10], which covers the case $z=1$.) For integers $r\ge 1$, $k\ge 2$, $u\ge 0$, and $0\le z\le \begin{bmatrix}r\\1\end{bmatrix}_q/2$ with $t=\begin{bmatrix}r\\1\end{bmatrix}_q+1-z+u>r$ we have $A_q(n,2t;t)\le lq^t+1+z(q-1)$, where $l=\frac{q^{n-t}-q^r}{q^t-1}$ and $n=kt+r$.*

Proof. Apply Lemma 2.5 with $x=2+z(q-1)\ge 2$ in order to deduce the existence of an $(n-t+y)$-dimensional subspace U with $L\le (z+y-1-u)q^y-(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q$ holes, where $L\equiv -(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q \pmod{q^y}$. Now, we set $y=z+1$. Observe that $q^f|x-1=1+z(q-1)\ge 1$, for $f\in\mathbb{N}_0$, implies $q^f\le 1+z(q-1)$, so that $y>z\ge (q^f-1)/(q-1)\ge 2^f-1\ge f$. Additionally, we have $1\le y=z+1\le \begin{bmatrix}r\\1\end{bmatrix}_q+1-z\le t$. If $z=0$, then $y=1$, $x=2$, and $L\le -uq-1<0$. For $z\ge 1$, we apply Lemma 2.8 onto U with $s=y$, $c=L=$

$$(z+y-1-u)q^y-(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q-jq^y=(y-1-j-u)q^y-\begin{bmatrix}y\\1\end{bmatrix}_q+y-1$$

for some $j\in\mathbb{N}_0$, and $i=y-1-j-u\in\mathbb{Z}$. Thus, $A_q(n,2t;t)\le lq^t+x-1$. □

The known constructions for partial t-spreads give $A_q(kt+r,2t;t)\ge lq^t+1$; see e.g. [1] (or [13] for an interpretation using the more general multilevel construction for subspace codes). Thus Theorem 2.9 is tight for $t\ge \begin{bmatrix}r\\1\end{bmatrix}_q+1$, cf. [14, Theorem 5].

Theorem 2.10 *For integers $r\ge 1$, $k\ge 2$, $y\ge\max\{r,2\}$, $z\ge 0$ with $\lambda=q^y$, $y\le t$, $t=\begin{bmatrix}r\\1\end{bmatrix}_q+1-z>r$, $n=kt+r$, and $l=\frac{q^{n-t}-q^r}{q^t-1}$, we have*

$$A_q(n,2t;t)\le lq^t+\left\lceil \lambda-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda\left(\lambda-(z+y-1)(q-1)-1\right)}\right\rceil.$$

Proof. From Lemma 2.5 we conclude $L\le (z+y-1)q^y-(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q$ and $L\equiv -(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q \pmod{q^y}$ for the number of holes of a certain $(n-t+y)$-dimensional subspace U of $\mathbb{F}_q^n$. $\mathcal{P}_U:=\{P\cap U\mid P\in\mathcal{P}\}$ is of hole-type (t,y,L) if $y\ge 2$. Next, we will show that $\tau_q(c,q^{y-1},m)\le 0$, see Lemma 2.7, where $c=iq^y-(x-1)\begin{bmatrix}y\\1\end{bmatrix}_q$ with $1\le i\le z+y-1$, for suitable integers x and m. Note that, in order to apply Lemma 2.5, we have to satisfy $x\ge 2$ and $y\ge f$ for all integers f with $q^f|x-1$. Applying Lemma 2.7 then gives the desired contradiction, so that $A_q(n,2t;t)\le lq^t+x-1$.

We choose $m=i(q-1)-(x-1)+1$, so that $\tau_q(c,q^{y-1},m)=x^2-(2\lambda+1)x+\lambda(i(q-1)+2)$. Solving $\tau_q(c,q^{y-1},m)=0$ for x gives $x_0=\lambda+\frac{1}{2}\pm\frac{1}{2}\theta(i)$, where

$\theta(i) = \sqrt{1-4i\lambda(q-1)+4\lambda(\lambda-1)}$. We have $\tau_q(c, q^{y-1}, m) \le 0$ for $|2x-2\lambda-1| \le \theta(i)$. We need to find an integer $x \ge 2$ such that this inequality is satisfied for all $1 \le i \le z+y-1$. The strongest restriction is attained for $i = z+y-1$. Since $z+y-1 \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $\lambda = q^y \ge q^r$, we have $\theta(i) \ge \theta(z+y-1) \ge 1$, so that $\tau_q(c, q^{y-1}, m) \le 0$ for $x = \left\lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil$. (Observe $x \le \lambda + \frac{1}{2} + \frac{1}{2}\theta(z+y-1)$ due to $\theta(z+y-1) \ge 1$.) Since $x \le \lambda+1$, we have $x-1 \le \lambda = q^y$, so that $q^f|x-1$ implies $f \le y$ provided $x \ge 2$. The latter is true due to $\theta(z+y-1) \le \sqrt{1-4\lambda(q-1)+4\lambda(\lambda-1)} \le \sqrt{1+4\lambda(\lambda-2)} < 2(\lambda-1)$, which implies $x \ge \left\lceil \frac{3}{2} \right\rceil = 2$.

So far we have constructed a suitable $m \in \mathbb{Z}$ such that $\tau_q(c, q^{y-1}, m) \le 0$ for $x = \left\lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil$. If $\tau_q(c, q^{y-1}, m) < 0$, then Lemma 2.7 gives a contradiction, so that we assume $\tau_q(c, q^{y-1}, m) = 0$ in the following. If $i < z+y-1$ we have $\tau_q(c, q^{y-1}, m) < 0$ due to $\theta(i) > \theta(z+y-1)$, so that we assume $i = z+y-1$. Thus, $\theta(z+y-1) \in \mathbb{N}_0$. However, we can write $\theta(z+y-1)^2 = 1+4\lambda\left(\lambda-(z+y-1)(q-1)-1\right) = (2w-1)^2 = 1+4w(w-1)$ for some integer w. If $w \notin \{0,1\}$, then $\gcd(w, w-1) = 1$, so that either $\lambda = q^y \mid w$ or $\lambda = q^y \mid w-1$. Thus, in any case, $w \ge q^y$, which is impossible since $(z+y-1)(q-1) \ge 1$. Finally, $w \in \{0,1\}$ implies $w(w-1) = 0$, so that $\lambda - (z+y-1)(q-1) - 1 = 0$. Thus, $z+y-1 = \begin{bmatrix} y \\ 1 \end{bmatrix}_q \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ since $y \ge r$. The assumptions $y \le t$ and $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z$ imply $z+y-1 = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $y = r$. This gives $t = r$, which is excluded. □

We remark that our specific choice of m can be motivated as follows. Solving $\frac{\partial \tau_q(c, q^{y-1}, m)}{\partial m} = 0$, i.e., minimizing $\tau_q(c, q^{y-1}, m)$, yields $m = i(q-1)-(x-1)+\frac{1}{2}+\frac{x-1}{q^y}$. For $y \ge r$ we can assume $x-1 < q^y$ due the known constructions for partial spreads, so that up-rounding yields the optimum integer choice. For $y < r$ the interval $\left[\lambda+\frac{1}{2}-\frac{1}{2}\theta(i), \lambda+\frac{1}{2}+\frac{1}{2}\theta(i)\right]$ may contain no integer.

The special case of $y = t$ in Theorem 2.10 is equivalent to [4, Corollary 8] – the mentioned upper bound of Drake and Freeman, which is based on [3, Theorem 1B]. And indeed, our analysis is very similar to the technique used in [3]. Actually, their analysis grounds on [16] and is strongly related to the classical second-order Bonferroni Inequality [2, 7, 8] in Probability Theory, see e.g. [11, Section 2.5] for another application to subspace codes. Compared to [3, 4], the new ingredients essentially are lemmas 2.2 and 2.3, see also [14, Proof of Lemma 9]. A weaker version of Theorem 2.10 was obtained independently and very recently in [15, Theorem 6,7].

Postponing the details and proofs to a more extensive and technical paper [12], we state some further results in the appendix. The rough idea is to show that the set of holes of a partial t-spread in $\mathbb{F}_q^n$ is equivalent to a projective linear code over $\mathbb{F}_q^n$ whose weights of the codewords are divisible by q^{t-1}. So, instead of Lemma 2.6 we can utilize the more general MacWilliams identities. Via the so-called linear programming method the mentioned upper bounds on $A_q(n, 2t; t)$ can be obtained by excluding the existence of certain linear codes. See also the web-page mentioned in footnote 1 for more numerical values and comparisons of the different upper bounds. So, stay tuned and perhaps join the journey to improved upper bounds for partial t-spreads.

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Appendix: Further improved upper bounds

Using the aforementioned linear programming method, the upper bounds of Theoremis 2.9 and 2.10 can be lowered by 1 in the following cases. For $k \geq 2$ we have

- $2^4 l + 1 \leq A_2(4k+3, 8; 4) \leq 2^4 l + 4$, where $l = \frac{2^{4k-1}-2^3}{2^4-1}$;
- $2^6 l + 1 \leq A_2(6k+4, 12; 6) \leq 2^6 l + 8$, where $l = \frac{2^{6k-2}-2^4}{2^6-1}$;
- $2^6 l + 1 \leq A_2(6k+5, 12; 6) \leq 2^6 l + 18$, where $l = \frac{2^{6k-1}-2^5}{2^6-1}$;
- $3^4 l + 1 \leq A_3(4k+3, 8; 4) \leq 3^4 l + 14$, where $l = \frac{3^{4k-1}-3^3}{3^4-1}$;
- $3^5 l + 1 \leq A_3(5k+3, 10; 5) \leq 3^5 l + 13$, where $l = \frac{3^{5k-2}-3^5}{3^3-1}$;
- $3^5 l + 1 \leq A_3(5k+4, 10; 5) \leq 3^5 l + 44$, where $l = \frac{3^{5k-1}-3^4}{3^5-1}$;
- $3^6 l + 1 \leq A_3(6k+4, 12; 6) \leq 3^6 l + 41$, where $l = \frac{3^{6k-2}-3^4}{3^6-1}$;
- $3^6 l + 1 \leq A_3(6k+5, 12; 6) \leq 3^6 l + 133$, where $l = \frac{3^{6k-1}-3^5}{3^6-1}$;
- $3^7 l + 1 \leq A_3(7k+4, 14; 7) \leq 3^7 l + 40$, where $l = \frac{3^{7k-3}-3^4}{3^7-1}$;
- $4^5 l + 1 \leq A_4(5k+3, 10; 5) \leq 4^5 l + 32$, where $l = \frac{4^{5k-2}-4^3}{4^5-1}$;
- $4^6 l + 1 \leq A_4(6k+3, 12; 6) \leq 4^6 l + 30$, where $l = \frac{4^{6k-3}-4^3}{4^6-1}$;
- $4^6 l + 1 \leq A_4(6k+5, 12; 6) \leq 4^6 l + 548$, where $l = \frac{4^{6k-1}-4^5}{4^6-1}$;
- $4^7 l + 1 \leq A_4(7k+4, 14; 7) \leq 4^7 l + 128$, where $l = \frac{4^{7k-3}-4^4}{4^7-1}$;
- $5^5 l + 1 \leq A_5(5k+2, 10; 5) \leq 5^5 l + 7$, where $l = \frac{5^{5k-3}-5^2}{5^5-1}$;
- $5^5 l + 1 \leq A_5(5k+4, 10; 5) \leq 5^5 l + 329$, where $l = \frac{5^{5k-1}-5^4}{5^5-1}$;
- $7^5 l + 1 \leq A_7(5k+4, 10; 5) \leq 7^5 l + 1246$, where $l = \frac{7^{5k-1}-7^2}{7^5-1}$;

- $8^4l + 1 \leq A_8(4k+3, 8; 4) \leq 8^4l + 264$, where $l = \frac{8^{4k-1}-8^3}{8^4-1}$;
- $8^5l + 1 \leq A_8(5k+2, 10; 5) \leq 8^5l + 25$, where $l = \frac{8^{5k-3}-8^2}{8^5-1}$;
- $8^6l + 1 \leq A_8(6k+2, 12; 6) \leq 8^6l + 21$, where $l = \frac{8^{6k-4}-8^2}{8^6-1}$;
- $9^3l + 1 \leq A_9(3k+2, 6; 3) \leq 9^3l + 41$, where $l = \frac{9^{3k-1}-9^2}{9^3-1}$;
- $9^5l + 1 \leq A_9(5k+3, 10; 5) \leq 9^5l + 365$, where $l = \frac{9^{5k-2}-9^3}{9^5-1}$.

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