AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **68(1)** (2017), Pages 94–105

Fans are cycle-antimagic

Ali Ovais   Muhammad Awais Umar

Abdus Salam School of Mathematical Sciences
GC University, Lahore
Pakistan
aligureja_2@hotmail.com   owais054@gmail.com

Martin Bača*   Andrea Semaničová-Feňovčíková†

Department of Applied Mathematics and Informatics
Technical University, Košice
Slovakia
martin.baca@tuke.sk   andrea.fenovcikova@tuke.sk

Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs at least to one subgraph of G isomorphic to a given graph H. Then the graph G admitting an H-covering is (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$, form an arithmetic progression with the initial term a and the common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

This paper is devoted to studying the existence of super (a, d)-H-antimagic labelings for fans when subgraphs H are cycles.

1 Introduction

We consider finite and simple graphs. Let the vertex and edge sets of a graph G be denoted by $V = V(G)$ and $E = E(G)$, respectively. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the

* Also at Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan. Corresponding author.
† Also at Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan.

graph G admits an *H-covering.* Note that in this case all subgraphs of G isomorphic to H must be in the H-covering. A bijective function $f : V \cup E \to \{1, 2, \ldots, |V|+|E|\}$ is an *(a, d)-H-antimagic labeling* of a graph G admitting an H-covering whenever, for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$$

form an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d)-H-antimagic labeling is called *(super) (a, d)-H-antimagic.* For $d = 0$ it is called *H-magic* and *H-supermagic*, respectively.

The notion of H-supermagic graphs was introduced by Gutiérrez and Lladó [8] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [11] and Enomoto, Lladó, Nakamigawa and Ringel [7], respectively. They proved that some classes of connected graphs are H-supermagic, such as the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. More precisely they proved that the cycle C_n is P_h-supermagic for any $2 \leq h \leq n-1$ such that $\gcd(n, h(h-1)) = 1$. Lladó and Moragas [12] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati, Salman, Baskoro, Ryan and Miller [16] and also Salman, Ngurah and Izzati [18] proved that certain families of trees are path-supermagic. Ngurah, Salman and Susilowati [17] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [15] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

The (a, d)-H-antimagic labeling was introduced by Inayah, Salman and Simanjuntak [9]. In [10] the authors investigate the super (a, d)-H-antimagic labelings for some families of connected graphs H. In [19] was proved that wheels W_n, $n \geq 3$, are super (a, d)-C_k-antimagic for every $k = 3, 4, \ldots, n-1, n+1$ and $d = 0, 1, 2$.

The (super) (a, d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1, 1, 0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [13]. Further information on super d-antimagic labelings can be found in [2, 5].

For $H \cong K_2$, (super) (a, d)-H-antimagic labelings are also called (super) (a, d)-edge-antimagic total labelings and have been introduced in [20]. More results on (a, d)-edge-antimagic total labelings, can be found in [4, 14]. The vertex version of these labelings for generalized pyramid graphs is given in [1].

The existence of super (a, d)-H-antimagic labelings for disconnected graphs is studied in [6] and there is proved that if a graph G admits a (super) (a, d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b,d)-H-antimagic labeling as well. In [3] is shown that the disjoint union of multiple copies of a (super) $(a,1)$-tree-antimagic graph is also a (super) $(b,1)$-tree-antimagic. A natural question is whether the similar result holds also for another differences and another H-antimagic graphs.

A *fan* F_n, $n \geq 2$, is a graph obtained by joining all vertices of the path P_n to a further vertex, called the *centre*. The vertices on the path we will call the *path vertices*. The edges adjacent to the central vertex we will call the *spokes* and the remaining edges we will call the *path edges*. Thus F_n contains $n+1$ vertices, say, $v_1, v_2, \dots, v_{n+1}$, and $2n-1$ edges, say, $v_{n+1}v_i$, $1 \leq i \leq n$, and v_iv_{i+1}, $1 \leq i \leq n-1$.

In this paper we investigate the existence of super (a,d)-H-antimagic labelings for fans when subgraphs H are cycles.

2 Super (a,d)-cycle-antimagic labeling of fan

Let C_k be a cycle on k vertices. Every cycle C_k in F_n is of the form $C_k^j = v_jv_{j+1}v_{j+2}\dots v_{j+k-2}v_{n+1}v_j$, where $j = 1, 2, \dots, n-k+2$. It is easy to see that each edge of F_n belongs to at least one cycle C_k^j if $k = 3, 4, \dots, \lfloor \frac{n}{2} \rfloor + 2$.

For the C_k-weight of the cycle C_k^j, $j = 1, 2, \dots, n-k+2$, under a total labeling f we get

$$\begin{aligned} wt_f(C_k^j) &= \sum_{v \in V(C_k^j)} f(v) + \sum_{e \in E(C_k^j)} f(e) \\ &= \sum_{s=0}^{k-3} \Big(f(v_{j+s}) + f(v_{j+s}v_{j+s+1}) \Big) + \big(f(v_{j+k-2}) + f(v_{j+k-2}v_{n+1}) \big) \\ &\quad + f(v_{n+1}) + f(v_jv_{n+1}). \end{aligned} \tag{1}$$

2.1 Differences $d = 1, 3$

The next theorem shows that F_n admits super cycle-antimagic labelings for differences $d = 1$ and $d = 3$.

Theorem 1. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan F_n admits a super (a,d)-C_k-antimagic labeling for $d = 1, 3$.*

Proof. Let us consider the total labelings f_1 and f_2 of F_n defined in the following way

$$\begin{aligned} f_1(v_i) = f_2(v_i) &= i, && \text{for } i = 1, 2, \dots, n+1 \\ f_1(v_iv_{n+1}) &= 2n+2-i, && \text{for } i = 1, 2, \dots, n \\ f_2(v_iv_{n+1}) &= n+1+i, && \text{for } i = 1, 2, \dots, n \\ f_1(v_iv_{i+1}) = f_2(v_iv_{i+1}) &= 3n+1-i, && \text{for } i = 1, 2, \dots, n-1. \end{aligned}$$

It is easy to see that f_1 and f_2 are super labelings as the vertices of F_n are labeled by the labels $1, 2, \dots, n+1$.

Under both labelings the spokes attain the labels $n+2, n+3, \dots, 2n+1$ and the path edges are labeled by the numbers $2n+2, 2n+3, \dots, 3n$.

The sum of the path vertex label and the corresponding incident path edge label is a constant. More precisely, for every $i = 1, 2, \dots, n-1$ and for $m = 1, 2$ we have

$$f_m(v_i) + f_m(v_i v_{i+1}) = i + (3n + 1 - i) = 3n + 1. \tag{2}$$

Under the labeling f_1 the sum of the path vertex label and the incident spoke label is a constant, that is, for every $i = 1, 2, \dots, n$

$$f_1(v_i) + f_1(v_i v_{n+1}) = i + (2n + 2 - i) = 2n + 2. \tag{3}$$

On the other side under the labeling f_2 the sums of the path vertex label and corresponding spoke label form an arithmetic sequence with difference 2, that is, for every $i = 1, 2, \dots, n$

$$f_2(v_i) + f_2(v_i v_{n+1}) = i + (n + 1 + i) = n + 1 + 2i. \tag{4}$$

According to (1), (2) and (3) we obtain

$$\begin{aligned} wt_{f_1}(C_k^j) &= (k-2)(3n+1) + (2n+2) + (n+1) + (2n+2-j) \\ &= (k-2)(3n+1) + 5n + 5 - j \end{aligned}$$

and with respect to (1), (2) and (4) we obtain

$$\begin{aligned} wt_{f_2}(C_k^j) &= (k-2)(3n+1) + (n+1+2(j+k-2)) + (n+1) + (n+1+j) \\ &= (k-1)(3n+3) + 3j. \end{aligned}$$

Thus under the labeling f_1 the set of all the C_k-weights consists of consecutive integers and under the labeling f_2 the C_k-weights form the arithmetic sequence with the difference 3. This concludes the proof. □

2.2 Differences depending on the length of cycle

The following theorem proves the existence of super cycle-antimagic labelings for differences $2k - 5$, $2k - 1$ and $3k - 1$.

Theorem 2. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan F_n admits a super (a, d)-C_k-antimagic labeling for $d = 2k - 5, 2k - 1, 3k - 1$.*

Proof. Let us consider the total labelings f_3, f_4 and f_5 of F_n defined in the following way

$$f_m(v_i) = i, \qquad \text{for } i = 1, 2, \dots, n+1 \text{ and } m = 3, 4, 5$$

$$f_m(v_i v_{n+1}) = \begin{cases} 3n + 1 - i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 3 \\ 2n + i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 4 \\ n + 2i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 5 \end{cases}$$

$$f_m(v_i v_{i+1}) = \begin{cases} n + 1 + i, & \text{for } i = 1, 2, \dots, n-1 \text{ and } m = 3, 4 \\ n + 1 + 2i, & \text{for } i = 1, 2, \dots, n-1 \text{ and } m = 5. \end{cases}$$

It is easy to see that f_m is a super labeling for every $m = 3, 4, 5$. Under the labelings f_3 and f_4 the path edges are labeled with the numbers $n+2, n+3, \dots, 2n$ and under the labeling f_5 they attain the numbers $n+3, n+5, \dots, 3n-1$. The labelings f_3 and f_4 assign to spokes the labels $2n+1, 2n+2, \dots, 3n$ and the labeling f_5 assigns labels $n+2, n+4, \dots, 3n$.

For every $i = 1, 2, \dots, n-1$ we have

$$f_m(v_i) + f_m(v_i v_{i+1}) = i + (n + 1 + i) = n + 1 + 2i, \qquad \text{if } m = 3, 4 \tag{5}$$

$$f_5(v_i) + f_5(v_i v_{i+1}) = i + (n + 1 + 2i) = n + 1 + 3i. \tag{6}$$

For every $i = 1, 2, \dots, n$ we get

$$f_3(v_i) + f_3(v_i v_{n+1}) = i + (3n + 1 - i) = 3n + 1, \tag{7}$$

$$f_4(v_i) + f_4(v_i v_{n+1}) = i + (2n + i) = 2n + 2i, \tag{8}$$

$$f_5(v_i) + f_5(v_i v_{n+1}) = i + (n + 2i) = n + 3i. \tag{9}$$

For C_k-weights from (1), (5) and (7) it follows

$$\begin{aligned} wt_{f_3}(C_k^j) =& \sum_{s=0}^{k-3} \Big(n + 1 + 2(j + s)\Big) + (3n + 1) + (n + 1) + (3n + 1 - j) \\ =& (k - 2)(n + k - 2) + 7n + 3 + j(2k - 5), \end{aligned}$$

by (1), (5) and (8) we obtain

$$\begin{aligned} wt_{f_4}(C_k^j) =& \sum_{s=0}^{k-3} \Big(n + 1 + 2(j + s)\Big) + (2n + 2(j + k - 2)) + (n + 1) + (2n + j) \\ =& (k - 2)(n + k) + 5n + 1 + j(2k - 1), \end{aligned}$$

and by (1), (6) and (9) we get

$$\begin{aligned} wt_{f_5}(C_k^j) =& \sum_{s=0}^{k-3} \Big(n + 1 + 3(j + s)\Big) + (n + 3(j + k - 2)) + (n + 1) + (n + 2j) \\ =& (k + 1)(n + 4) + \frac{3(k - 3)(k - 2)}{2} - 11 + j(3k - 1). \end{aligned}$$

Thus under the labelings f_m, $m = 3, 4, 5$, the C_k-weights form the arithmetic sequence with the differences $2k - 5$, $2k - 1$ and $3k - 1$, respectively. □

The existence of super cycle-antimagic labelings of a fan for differences $3k-9$, $k-7$ and $k+1$ follows from the next theorem.

Note that for some of these values of difference d is negative, which only means that the cycle-weights form decreasing sequence, or alternatively the difference in the corresponding increasing arithmetic sequence is $|d|$. Note that if $d=0$ then the cycle-weights are the same.

Theorem 3. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$. Then the fan F_n is super (a,d)-C_k-antimagic for $d = 3k-9, k-7, k+1$.*

Proof. Define the total labeling f_m, $m = 6,7,8$, of F_n as follows

$$f_m(v_i) = \begin{cases} i, & \text{for } i = 1,2,\dots,n+1 \text{ and } m=6 \\ n+1-i, & \text{for } i = 1,2,\dots,n \text{ and } m = 7,8 \\ n+1, & \text{for } i = n+1 \text{ and } m = 7,8 \end{cases}$$

$$f_m(v_iv_{n+1}) = \begin{cases} 3n+2-2i, & \text{for } i = 1,2,\dots,n \text{ and } m = 6,7 \\ n+2i, & \text{for } i = 1,2,\dots,n \text{ and } m = 8 \end{cases}$$

$$f_m(v_iv_{i+1}) = n+1+2i, \qquad \text{for } i = 1,2,\dots,n-1 \text{ and } m = 6,7,8.$$

Since the labelings f_m, $m = 6,7,8$, assign the smallest possible labels to the vertices of F_n, they are super. For each labeling f_m, $m = 6,7,8$, the path edges attain the labels $n+3, n+5, \dots, 3n-1$ and the spokes are labeled by the labels $n+2, n+4, \dots, 3n$.

For every $i = 1,2,\dots,n-1$ we get

$$f_6(v_i) + f_6(v_iv_{i+1}) = i + (n+1+2i) = n+1+3i, \tag{10}$$

$$f_m(v_i) + f_m(v_iv_{i+1}) = (n+1-i) + (n+1+2i) = 2n+2+i, \text{ if } m = 7,8 \tag{11}$$

For every $i = 1,2,\dots,n$ we have

$$f_6(v_i) + f_6(v_iv_{n+1}) = i + (3n+2-2i) = 3n+2-i, \tag{12}$$

$$f_7(v_i) + f_7(v_iv_{n+1}) = (n+1-i) + (3n+2-2i) = 4n+3-3i, \tag{13}$$

$$f_8(v_i) + f_8(v_iv_{n+1}) = (n+1-i) + (n+2i) = 2n+1+i. \tag{14}$$

According to (1), (10) and (12)

$$\begin{aligned} wt_{f_6}(C_k^j) = &\sum_{s=0}^{k-3} \Big(n+1+3(j+s)\Big) + (3n+2-(j+k-2)) + (n+1) \\ &+ (3n+2-2j) = n(k+5) + \frac{3(k-3)(k-2)}{2} + 5 + j(3k-9), \end{aligned}$$

with respect to (1), (11) and (13)

$$wt_{f_7}(C_k^j) = \sum_{s=0}^{k-3}\Big(2n+2+(j+s)\Big) + (4n+3-3(j+k-2)) + (n+1)$$
$$+ (3n+2-2j) = (2n-1)(k+2) + \frac{(k-3)(k-2)}{2} + 10$$
$$+ j(k-7).$$

and from (1), (11) and (14) it follows

$$wt_{f_8}(C_k^j) = \sum_{s=0}^{k-3}\Big(2n+2+(j+s)\Big) + (2n+1+(j+k-2)) + (n+1)$$
$$+ (n+2j) = (2n+3)k + \frac{(k-3)(k-2)}{2} - 4 + j(k+1).$$

One can see that under the labelings f_m, $m = 6,7,8$, the C_k-weights constitute the arithmetic sequences with the differences $3k-9$, $k-7$ and $k+1$, respectively. □

3 C_3-antimagicness of fans

In [13] Lih proved that F_n is C_3-supermagic for every n except when $n \equiv 2 \pmod 4$. Ngurah, Salman and Susilowati [17] completed this result and they proved that for any integer $n \geq 2$ the fan F_n is C_3-supermagic.

Immediately from Theorems 1 through 3 we obtain that if F_n satisfies the necessary condition for covering by C_3, then there exist super (a,d)-C_k-antimagic labelings of F_n for every $d \in \{0,1,3,4,5,8\}$. Moreover, in the next theorem we are able to prove also that differences $d=2$ and $d=6$ are feasible.

Theorem 4. *The fan F_n, $n \geq 4$, is super (a,d)-C_3-antimagic for $d = 0,1,2,3,4,5,6,8$.*

Proof. The existence of such labelings for $d = 0,1,3,4,5,8$ immediately follows from Theorems 1 through 3. For $d = 2,6$ let us consider the following.

Construct the total labelings g_m, $m = 1,2$, of F_n such that

$$g_m(v_i) = \begin{cases} \frac{i+1}{2}, & \text{for } 1 \leq i \leq n,\ i \equiv 1 \pmod 2 \text{ and } m = 1 \\ \left\lceil \frac{n}{2} \right\rceil + \frac{i}{2}, & \text{for } 2 \leq i \leq n,\ i \equiv 0 \pmod 2 \text{ and } m = 1 \\ n+1, & \text{for } i = n+1 \text{ and } m = 1 \\ i, & \text{for } i = 1,2,\dots,n+1 \text{ and } m = 2 \end{cases}$$

$$g_m(v_iv_{n+1}) = \begin{cases} 2n+i, & \text{for } i = 1,2,\dots,n \text{ and } m = 1 \\ n+1+i, & \text{for } 1 \leq i \leq n,\ i \equiv 1 \pmod 2 \text{ and } m = 2 \\ n+2\left\lceil \frac{n}{2} \right\rceil + i, & \text{for } 2 \leq i \leq n,\ i \equiv 0 \pmod 2 \text{ and } m = 2 \end{cases}$$

$$g_m(v_iv_{i+1}) = \begin{cases} 2n+1-i, & \text{for } i = 1,2,\dots,n-1 \text{ and } m = 1 \\ n+1+2i, & \text{for } i = 1,2,\dots,n-1 \text{ and } m = 2. \end{cases}$$

The labelings g_1 and g_2 are super as the vertices of F_n are labeled with the smallest possible labels. Under the labeling g_1 or g_2 the path edges attain the labels $n+2, n+3, \dots, 2n$ or $n+3, n+5, \dots, 3n-1$, respectively, and the spokes admit the labels $2n+1, 2n+2, \dots, 3n$ or $n+2, n+4, \dots, 3n$, respectively.

For the C_3-weights of the cycle $C_3^j = v_j v_{j+1} v_{n+1} v_j$, $j = 1, 2, \dots, n-1$, we get

$$\begin{aligned} wt_{g_1}(C_3^j) =& g_1(v_j) + g_1(v_j v_{j+1}) + g_1(v_{j+1}) + g_1(v_{j+1} v_{n+1}) + g_1(v_{n+1}) \\ &+ g_1(v_j v_{n+1}) \end{aligned}$$

$$= \begin{cases} \frac{j+1}{2} + (2n+1-j) + \left(\left\lceil \frac{n}{2} \right\rceil + \frac{j+1}{2} \right) + (2n + (j+1)) + (n+1) \\ \qquad +(2n+j) = 7n + \left\lceil \frac{n}{2} \right\rceil + 4 + 2j \\ \qquad\qquad \text{for } 1 \le j \le n-1,\ j \equiv 1 \pmod 2 \\ \left(\left\lceil \frac{n}{2} \right\rceil + \frac{j}{2} \right) + (2n+1-j) + \frac{j+2}{2} + (2n + (j+1)) + (n+1) \\ \qquad +(2n+j) = 7n + \left\lceil \frac{n}{2} \right\rceil + 4 + 2j \\ \qquad\qquad \text{for } 2 \le j \le n-1,\ j \equiv 0 \pmod 2 \end{cases}$$

and

$$\begin{aligned} wt_{g_2}(C_3^j) =& g_2(v_j) + g_2(v_j v_{j+1}) + g_2(v_{j+1}) + g_2(v_{j+1} v_{n+1}) + g_2(v_{n+1}) \\ &+ g_2(v_j v_{n+1}) \\ =& \begin{cases} j + (n+1+2j) + (j+1) + (n + 2\left\lceil \frac{n}{2} \right\rceil + (j+1)) + (n+1) \\ \qquad +(n+1+j) = 4n + 2\left\lceil \frac{n}{2} \right\rceil + 5 + 6j \\ \qquad\qquad \text{for } 1 \le j \le n-1,\ j \equiv 1 \pmod 2 \\ j + (n+1+2j) + (j+1) + (n+1+(j+1)) + (n+1) \\ \qquad +(n + 2\left\lceil \frac{n}{2} \right\rceil + j) = 4n + 2\left\lceil \frac{n}{2} \right\rceil + 5 + 6j \\ \qquad\qquad \text{for } 2 \le j \le n-1,\ j \equiv 0 \pmod 2). \end{cases} \end{aligned}$$

For $j = 1, 2, \dots, n-1$ that is

$$wt_{g_1}(C_3^j) = 7n + \left\lceil \tfrac{n}{2} \right\rceil + 4 + 2j$$

and

$$wt_{g_2}(C_3^j) = 4n + 2\left\lceil \tfrac{n}{2} \right\rceil + 5 + 6j.$$

This means that under the labelings g_1 and g_2 the C_3-weights form the arithmetic sequences with the differences 2 and 6, respectively. □

4 C_4-antimagicness of fans

Every cycle C_4 in F_n is of the form $C_4^j = v_j v_{j+1} v_{j+2} v_{n+1} v_j$, $j = 1, 2, \dots, n-2$, and for $n \ge 4$, each edge of F_n belongs to at least one cycle of C_4^j. For the C_4-weight of the cycle C_4^j, $j = 1, 2, \dots, n-2$, under a total labeling f we have

$$\begin{aligned} wt_f(C_4^j) =& f(v_j) + f(v_j v_{j+1}) + f(v_{j+1}) + f(v_{j+1} v_{j+2}) + f(v_{j+2}) + f(v_{j+2} v_{n+1}) \\ &+ f(v_{n+1}) + f(v_j v_{n+1}). \end{aligned} \tag{15}$$

From Theorems 1 through 3 it follows that F_n, provided necessary condition for the covering by C_4 cycles is met, then there exist super (a, d)-C_4-antimagic labelings for every $d \in \{1, 3, 5, 7, 11\}$. The following theorem shows also that differences $d = 0, 2, 4$ and $d = 6$ are feasible.

Theorem 5. *The fan F_n, $n \geq 4$, is super (a, d)-C_4-antimagic for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$.*

Proof. For $d = 1, 3, 5, 7, 11$ the results follow from Theorems 1 through 3. If $d = 0, 2, 4, 6$ let us consider the following.

For F_n, $n \geq 4$, define the total labelings h_m, $1 \leq t \leq 4$, in the following way

$$h_m(v_i) = \begin{cases} n+1-i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 1, 3 \\ n+1, & \text{for } i = n+1 \text{ and } m = 1, 3 \\ i, & \text{for } i = 1, 2, \dots, n+1 \text{ and } m = 2, 4 \end{cases}$$

$$h_m(v_i v_{n+1}) = \begin{cases} 2n+i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 1, 4 \\ 3n+1-i, & \text{for } i = 1, 2, \dots, n \text{ and } m = 2, 3 \end{cases}$$

$$h_m(v_i v_{i+1}) = \begin{cases} n+1+\frac{i+1}{2}, & \text{for } 1 \leq i \leq n-1,\ i \equiv 1 \pmod 2, \\ & \quad \text{and } m = 1, 2, 3, 4 \\ n+\left\lfloor \frac{n}{2} \right\rfloor + 1 + \frac{i}{2}, & \text{for } 2 \leq i \leq n-1,\ i \equiv 0 \pmod 2, \\ & \quad \text{and } m = 1, 2, 3, 4. \end{cases}$$

It is easy to see that h_m is a super labeling for every $m = 1, 2, 3, 4$. Under all labelings the path edges attain the labels $n+2, n+3, \dots, 2n$ and the spokes are labeled by the labels $2n+1, 2n+2, \dots, 3n$.

According to (15)

$$wt_{h_1}(C_4^j) = \begin{cases} (n+1-j) + \left(n+1+\frac{j+1}{2}\right) + (n+1-(j+1)) \\ \qquad + \left(n + \left\lfloor \frac{n}{2} \right\rfloor + 1 + \frac{j+1}{2}\right) + (n+1-(j+2)) \\ \qquad + (2n+(j+2)) + (n+1) + (2n+j) \\ \qquad = 10n + \left\lfloor \frac{n}{2} \right\rfloor + 6 \\ \qquad\qquad \text{for } 1 \leq j \leq n-2,\ j \equiv 1 \pmod 2 \\ (n+1-j) + \left(n + \left\lfloor \frac{n}{2} \right\rfloor + 1 + \frac{j}{2}\right) + (n+1-(j+1)) \\ \qquad + \left(n+1+\frac{(j+1)+1}{2}\right) + (n+1-(j+2)) \\ \qquad + (2n+(j+2)) + (n+1) + (2n+j) \\ \qquad = 10n + \left\lfloor \frac{n}{2} \right\rfloor + 6 \\ \qquad\qquad \text{for } 2 \leq j \leq n-2,\ j \equiv 0 \pmod 2, \end{cases}$$

$$wt_{h_2}(C_4^j) = \begin{cases} j + \left(n+1+\frac{j+1}{2}\right) + (j+1) \\ \qquad + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j+1}{2}\right) + (j+2) \\ \qquad +(3n+1-(j+2)) + (n+1) + (3n+1-j) \\ \qquad = 9n + \left\lfloor\frac{n}{2}\right\rfloor + 7 + 2j \\ \qquad\qquad \text{for } 1 \le j \le n-2,\ j \equiv 1 \pmod 2 \\ j + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j}{2}\right) + (j+1) \\ \qquad + \left(n+1+\frac{(j+1)+1}{2}\right) + (j+2) \\ \qquad +(3n+1-(j+2)) + (n+1) + (3n+1-j) \\ \qquad = 9n + \left\lfloor\frac{n}{2}\right\rfloor + 7 + 2j \\ \qquad\qquad \text{for } 2 \le j \le n-2,\ j \equiv 0 \pmod 2, \end{cases}$$

$$wt_{h_3}(C_4^j) = \begin{cases} (n+1-j) + \left(n+1+\frac{j+1}{2}\right) + (n+1-(j+1)) \\ \qquad + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j+1}{2}\right) + (n+1-(j+2)) \\ \qquad +(3n+1-(j+2)) + (n+1) + (3n+1-j) \\ \qquad = 12n + \left\lfloor\frac{n}{2}\right\rfloor + 4 - 4j \\ \qquad\qquad \text{for } 1 \le j \le n-2,\ j \equiv 1 \pmod 2 \\ (n+1-j) + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j}{2}\right) + (n+1-(j+1)) \\ \qquad + \left(n+1+\frac{(j+1)+1}{2}\right) + (n+1-(j+2)) \\ \qquad +(3n+1-(j+2)) + (n+1) + (3n+1-j) \\ \qquad = 12n + \left\lfloor\frac{n}{2}\right\rfloor + 4 - 4j \\ \qquad\qquad \text{for } 2 \le j \le n-2,\ j \equiv 0 \pmod 2, \end{cases}$$

and

$$wt_{h_4}(C_4^j) = \begin{cases} j + \left(n+1+\frac{j+1}{2}\right) + (j+1) \\ \qquad + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j+1}{2}\right) + (j+2) \\ \qquad +(2n+(j+2)) + (n+1) + (2n+j) \\ \qquad = 7n + \left\lfloor\frac{n}{2}\right\rfloor + 9 + 6j \\ \qquad\qquad \text{for } 1 \le j \le n-2,\ j \equiv 1 \pmod 2 \\ j + \left(n + \left\lfloor\frac{n}{2}\right\rfloor + 1 + \frac{j}{2}\right) + (j+1) \\ \qquad + \left(n+1+\frac{(j+1)+1}{2}\right) + (j+2) \\ \qquad +(2n+(j+2)) + (n+1) + (2n+j) \\ \qquad = 7n + \left\lfloor\frac{n}{2}\right\rfloor + 9 + 6j \\ \qquad\qquad \text{for } 2 \le j \le n-2,\ j \equiv 0 \pmod 2. \end{cases}$$

This means that under the labelings h_m, $m = 1, 2, 3, 4$, the C_4-weights form the arithmetic sequences with the differences $d = 0, 2, 4$ and 6, respectively. □

5 Conclusion

In this paper we examined the existence of super (a, d)-C_k-antimagic labelings for fans. We proved that the fan F_n, $n \geq 3$, admits a super (a, d)-C_k-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{1, 3, k-7, k+1, 2k-5, 2k-1, 3k-9, 3k-1\}$. We showed that there exists a super (a, d)-C_3-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 8$ and a super (a, d)-C_4-antimagic labeling for $d = 0, 1, 2, 3, 4, 5, 6, 7, 11$ of F_n, $n \geq 4$.

For further investigation we propose the following open problem.

Open Problem 1. *Find a super (a, d)-C_k-antimagic labeling of the fan F_n for $d \neq 1, 3, k-7, k+1, 2k-5, 2k-1, 3k-9, 3k-1$.*

Acknowledgements

The research for this article was supported by APVV-15-0116.

References

[1] S. Arumugam, M. Miller, O. Phanalasy and J. Ryan, Antimagic labeling of generalized pyramid graphs, *Acta Math. Sin. (Engl. Ser.)* **30** (2014), 283–290.

[2] M. Bača, L. Brankovic and A. Semaničová-Feňovčíková, Labelings of plane graphs containing Hamilton path, *Acta Math. Sin. (Engl. Ser.)* **27**(4) (2011), 701–714.

[3] M. Bača, Z. Kimáková, A. Semaničová-Feňovčíková and M. A. Umar, Tree-antimagicness of disconnected graphs, *Math. Probl. Eng.* **2015** (2015), Article ID 504251, 4 pp.

[4] M. Bača and M. Miller, *Super edge-antimagic graphs: A wealth of problems and some solutions*, Brown Walker Press, Boca Raton, Florida, 2008.

[5] M. Bača, M. Miller, O. Phanalasy and A. Semaničová-Feňovčíková, Super d-antimagic labelings of disconnected plane graphs, *Acta Math. Sin. (Engl. Ser.)* **26**(12) (2010), 2283–2294.

[6] M. Bača, M. Miller, J. Ryan and A. Semaničová-Feňovčíková, On H-antimagicness of disconnected graphs, *Bull. Aust. Math. Soc.* **94**(2) (2016), 201–207.

[7] H. Enomoto, A. S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[8] A. Gutiérrez and A. Lladó, Magic coverings, *J. Combin. Math. Combin. Comput.* **55** (2005), 43–56.

[9] N. Inayah, A. N. M. Salman and R. Simanjuntak, On (a, d)-H-antimagic coverings of graphs, *J. Combin. Math. Combin. Comput.* **71** (2009), 273–281.

[10] N. Inayah, R. Simanjuntak, A. N. M. Salman and K. I. A. Syuhada, On (a, d)-H-antimagic total labelings for shackles of a connected graph H, *Australas. J. Combin.* **57** (2013), 127–138.

[11] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[12] A. Lladó and J. Moragas, Cycle-magic graphs, *Discrete Math.* **307** (2007), 2925–2933.

[13] K. W. Lih, On magic and consecutive labelings of plane graphs, *Utilitas Math.* **24** (1983), 165–197.

[14] A. M. Marr and W. D. Wallis, *Magic Graphs*, Birkhäuser, New York, 2013.

[15] T. K. Maryati, A. N. M. Salman and E. T. Baskoro, Supermagic coverings of the disjoint union of graphs and amalgamations, *Discrete Math.* **313** (2013), 397–405.

[16] T. K. Maryati, A. N. M. Salman, E. T. Baskoro, J. Ryan and M. Miller, On H-supermagic labelings for certain shackles and amalgamations of a connected graph, *Utilitas Math.* **83** (2010), 333–342.

[17] A. A. G. Ngurah, A. N. M. Salman and L. Susilowati, H-supermagic labelings of graphs, *Discrete Math.* **310** (2010), 1293–1300.

[18] A. N. M. Salman, A. A. G. Ngurah and N. Izzati, On (super)-edge-magic total labelings of subdivision of stars S_n, *Utilitas Math.* **81** (2010), 275-284.

[19] A. Semaničová–Feňovčíková, M. Bača, M. Lascsáková, M. Miller and J. Ryan, Wheels are cycle-antimagic, *Electron. Notes Discrete Math.* **48** (2015), 11–18.

[20] R. Simanjuntak, M. Miller and F. Bertault, Two new (a, d)-antimagic graph labelings, *Proc. Eleventh Australas. Workshop Combin. Alg. (AWOCA)* (2000), 179-189.

(Received 16 Mar 2016; revised 7 Mar 2017)