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The number of contractible edges in a 4-connected graph having a contractible edge not contained in triangles

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Abstract

Let G be a 4-connected graph, let $\tilde{E}(G)$ denote the set of those edges of G which are not contained in a triangle, and let $E_c(G)$ denote the set of 4-contractible edges of G. We show that if $|\tilde{E}(G)| \geq 1$, then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless G satisfies certain conditions.

1 Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. A subset S of $V(G)$ is called a cutset if $G - S$ is disconnected. A cutset with cardinality i is simply referred to as an i-cutset. For an integer $k \geq 1$, we say that G is k-connected if $|V(G)| \geq k + 1$ and G has no $(k-1)$-cutset. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. For $x \in V(G)$, let $\deg_G(x)$ denote the degree of x. For an integer $i \geq 0$, we let $V_i(G)$ denote the set of vertices x of G with $\deg_G(x) = i$ and let $V_{\geq j}(G) = \bigcup_{i \geq j} V_i(G)$. The complete graph of order n is denoted by K_n. For a graph H, let nH denote the graph with n components, each isomorphic to H. The path of length n is denoted by P_n.

Let G be a 4-connected graph. For $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that e is 4-contractible or 4-noncontractible according as G/e is 4-connected or not. A 4-noncontractible edge $e = ab$ is said to be trivially 4-noncontractible if there exists $z \in V_4(G)$ such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{tn}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively. Note that if $|V(G)| \geq 6$, then $e \in E_n(G)$ if and only if there exists a 4-cutset S

such that $V(e) \subseteq S$. Finally we let $\tilde{E}(G)$ and $E_{\triangle}(G)$ denote the set of those edges of G which are not contained in a triangle or which are contained in a triangle, respectively.

The following result concerning a lower bound on $|E_c(G)|$ in terms of $|\tilde{E}(G)|$ was proved by Ando and Egawa in [3].

Theorem A *Let G be a 4-connected graph, and suppose that $|\tilde{E}(G)| \geq 15$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$.*

The following similar result concerning $|\tilde{E}(G)|$ was proved by Kotani and Nakamura in [7].

Theorem B *Let G be a 4-connected graph, and suppose that $7 \leq |\tilde{E}(G)| \leq 8$ or $10 \leq |\tilde{E}(G)| \leq 14$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$.*

Let $\tilde{V}$ denote the set of those vertices of G which are incident with an edge in $\tilde{E}(G) \cap E_n(G)$, and let $\hat{G}$ denote the induced subgraph of G with edge set $\tilde{E}(G) \cap E_n(G)$; that is to say, $\tilde{V} = \cup_{e \in \tilde{E}(G) \cap E_n(G)} V(e)$ and $\hat{G} = (\tilde{V}, \tilde{E}(G) \cap E_n(G))$. Let $P_1 = x_1x_2$ be the path of order 2, and let y_1, y_2, z_1 and z_2 be four distinct vertices different from x_1, x_2. We let Y^* denote the graph defined by

$$V(Y^*) = V(P_1) \cup \{y_1, y_2, z_1, z_2\}, \quad E(Y^*) = E(P_1) \cup \{x_1y_1, x_1y_2, x_2z_1, x_2z_2\}.$$

Further, the following result concerning $|\tilde{E}(G)|$ was proved in [6].

Theorem C *Let G be a 4-connected graph and suppose that $|\tilde{E}(G)| = 9$ or $2 \leq |\tilde{E}(G)| \leq 6$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 4)/4$. Further we have $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless one of the following holds:*
(1) $|\tilde{E}(G)| = 9$ *and* $\hat{G} \simeq Y^*$,
(2) $|\tilde{E}(G)| = 6$ *and* $\hat{G} \simeq 3K_2$,
(3) $|\tilde{E}(G)| = 5$ *and* $\hat{G} \simeq 2K_2$ *or* P_2,
(4) $|\tilde{E}(G)| = 4$ *and* $\hat{G} \simeq 2K_2$,
(5) $|\tilde{E}(G)| = 3$ *and* $\hat{G} \simeq K_2$,
(6) $|\tilde{E}(G)| = 2$ *and* $\hat{G} \simeq \emptyset$.

It was shown that the lower bound of $|E_c(G)|$ in Theorems A and C are best possible in [3] and [6], respectively. Our main result is the following theorem.

Theorem 1 *Let G be a 4-connected graph and suppose that $|\tilde{E}(G)| = 1$. Then $|E_c(G)| \geq 2$. Furthermore, $|E_c(G)| \geq 3$ unless G satisfies the condition:*
(7) $|\tilde{E}(G)| = 1$ *and* $\hat{G} \simeq K_2$.

From Theorems A, B, C and 1, we obtain the following theorem.

Theorem 2 *Let G be a 4-connected graph and suppose that $|\tilde{E}(G)| \geq 1$. Then $|E_c(G)| \geq (|\tilde{E}(G)|+4)/4$. Furthermore, $|E_c(G)| \geq (|\tilde{E}(G)|+8)/4$, unless one of the conditions (1) to (7) in the statement of Theorems C and 1 holds.*

Theorem 1 is best possible in the sense that there exist infinitely many 4-connected graphs G with $|\tilde{E}(G)| = 1$ and $\hat{G} \simeq K_2$ such that $|E_c(G)| = 2$. To construct such graphs, let $n \geq 8$ and set $m = \lceil n/2 \rceil$. Define a graph G of order n by

$$V(G) = \{x_i \mid 1 \leq i \leq n\},\ E(G) = \{x_i x_{i+1}, x_i x_{i+2} \mid 1 \leq i \leq n\} \cup \{x_1 x_m\},$$

where we take $x_{n+1} = x_1$ and $x_{n+2} = x_2$. Then G is 4-connected, $\hat{G} \simeq K_2$, and $x_2 x_n$ and $x_{m-1} x_{m+1}$ are the only 4-contractible edges of G.

Our notation is standard, and is mostly taken from Diestel [5]. Possible exceptions are as follows. For $x \in V(G)$, let $E_c(x)$ denote the set of 4-contractible edges of G which are incident with x, and let $N_G(x)$ denote the neighborhood of x. For $X \subseteq V(G)$, the subgraph induced by X in G is denoted by $G[X]$.

The organization of this paper is as follows. Section 2 contains the results from [1, 2, 4, 6]. We prove several preliminary results in Section 3, and prove Theorem 1 in Section 4.

2 Assumed results

In this section, we state the assumed results for the proof of Theorem 1. The following lemma appears as Theorem 2 in [1].

Lemma 2.1 *Each 4-connected graph G has at least $|V_{\geq 5}(G)|$ 4-contractible edges.*

Set $\mathcal{L} = \{(S, A) \mid S$ is a 4-cutset, A is the union of the vertex set of some components of $G - S, \emptyset \neq A \neq V(G) - S\}$.
For $(S, A) \in \mathcal{L}$, we let $\bar{A} = V(G) - S - A$. Thus if $(S, A) \in \mathcal{L}$, then $(S, \bar{A}) \in \mathcal{L}$.

The following lemma follows from Lemma 4 in [4].

Lemma 2.2 *Suppose that aub is a triangle in a 4-connected graph G such that $a, u \in V_{\geq 5}(G)$ and $b \in V_4(G)$. Let $(S, A) \in \mathcal{L}$ and suppose that $u \in A$ and $a, b \in S$, and $|\bar{A}| \geq 2$. Then b is incident with a contractible edge.*

Throughout the rest of this paper, we let G denote a 4-connected graph. The following lemma follows from Lemma 2.6 in [2].

Lemma 2.3 *Let u, a, b, w be four distinct vertices with $ua, ub, ab, aw, bw \in E(G)$ and $\deg(a) = \deg(b) = 4$, and write $N_G(a) = \{u, b, w, x\}$ and $N_G(b) = \{u, a, w, y\}$. Then $x \neq y$, and we have $ax, by \in E_c(G) \cup E_{tn}(G)$.*

The following lemmas appear as Lemmas 2.7 and 2.8 in [2], respectively.

Lemma 2.4 *Under the notation of Lemma 2.3, suppose that* $\deg_G(u), \deg_G(w) \geq 5$. *Then* $ax, by \in E_c(G)$.

Lemma 2.5 *Under the notation of Lemma 2.3, suppose that* $\deg_G(u) \geq 5$ *and* $\deg_G(w) = 4$. *Then one of the following holds:*
(1) $xw \notin E(G)$ *and* $ax \in E_c(G)$; *(2)* $yw \notin E(G)$ *and* $by \in E_c(G)$.

The following lemma appears as Proposition 2 in [1].

Lemma 2.6 *Let* xy *be a* 4*-noncontractible edge such that* $x, y \in V_4(G)$, $|E_c(x)| \leq 1$ *and* $|E_c(y)| \leq 1$. *If* $N_G(x) \cap N_G(y) \cap V_{\geq 5}(G) \neq \emptyset$, *then* $|N_G(x) \cap N_G(y)| \geq 2$.

The following lemma follows from Theorem 2 in [6].

Lemma 2.7 *Suppose that* $|\tilde{E}(G) \cap E_n(G)| = 1$. *Then* $|E_c(G)| \geq 2$.

3 Preliminary results

We continue with the notation of the preceding section. In this section, we state some preliminary results and fix notation for the proof of Theorem 1. Throughout the rest of this paper, we consider a 4-connected graph G with $|\tilde{E}(G)| = 1$, say $\tilde{E}(G) = \{e\}$. If e is 4-noncontractible, then $\hat{G} \simeq K_2$ and Lemma 2.7 assures us that $|E_c(G)| \geq 2$. Hence, to prove Theorem 1, throughout this section we suppose that e is 4-contractible, that is, $|\tilde{E}(G) \cap E_c(G)| = 1$. Let $V^*(G)$ denote the set of end vertices of the edge e. Then we observe that $|V^*(G)| = 2$. Hereafter we write V^* for $V^*(G)$. Moreover, to prove Theorem 1, we assume that $|E_c(x)| \leq 1$ for $x \in V(G) - V^*$. Further, by Lemma 2.1, we may assume $|V_{\geq 5}(G)| \leq 2$. We start with a claim.

Claim 3.1 *Suppose that* $|E_\triangle(G) \cap E_c(G)| \geq 2$. *Then* $|E_c(G)| \geq 3$.

Proof. This claim follows from $|E_c(G)| = |\tilde{E}(G) \cap E_c(G)| + |E_\triangle(G) \cap E_c(G)|$ immediately. □

To state our result, we need some more definitions.

For $u \in V^*$ and $e \in E(G)$, we define $n(u, e)$ and $G(u, e)$, and choose vertices $w_0, w_1, \ldots, w_{n(u,e)}$ of $V(G)$ inductively by the following procedure.

First we let $u = w_0$ and set $W_0 = \{w_0\}$. If there exist $w_1, w_2 \in N_G(w_0) \cap V_4(G)$ such that $w_1 w_2 \in E_n(G)$, then we let $e = w_1 w_2$ and set $W_1 = W_0 \cup \{w_1\}$, where w_1 satisfies one of the following two conditions: if $u \in V_4(G)$, then $N_G(w_1) \cap N_G(w_0) - \{w_2\} \neq \emptyset$ and $N_G(w_2) \cap N_G(w_1) - \{w_0\} \neq \emptyset$; or if $u \in V_{\geq 5}(G)$, then $E_c(w_1) \neq \emptyset$. Otherwise we let $n(u, e) = 0$ and terminate the procedure.

Next we set $W_2 = W_1 \cup \{w_2\}$. Now we let $i \geq 3$, and assume that $w_0, w_1, \ldots, w_{i-1}$ and $W_0, W_1, \ldots, W_{i-1}$ have been defined. If $E_c(w_{i-1}) = \emptyset$ and $(V(G) - W_{i-1} - V^*) \cap (N_G(w_{i-1}) \cap N_G(w_{i-2})) \neq \emptyset$, we let $w_i \in (V(G) - W_{i-1} - V^*) \cap (N_G(w_{i-1}) \cap N_G(w_{i-2}))$ and $W_i = W_{i-1} \cup \{w_i\}$; if $E_c(w_{i-1}) \neq \emptyset$ or $(V(G) - W_{i-1} - V^*) \cap (N_G(w_{i-1}) \cap N_G(w_{i-2})) = \emptyset$, we let $n(u,e) = i - 1$ and terminate the procedure. Note that if $n(u,e) \neq 0$, then $n(u,e) \geq 2$. We set $W_{(u,e)} = W_{n(u,e)} = \{w_0, w_1, \ldots, w_{n(u,e)}\}$, and let $G(u,e) = G[W_{(u,e)}]$. Further set

$$\begin{aligned} \mathcal{J} &= \{(u,e) \mid u \in V^*, e \in E(G), n(u,e) \neq 0\}, \\ \mathcal{J}_0 &= \{(u,e) \in \mathcal{J} \mid u \in V_{\geq 5}(G)\}, \\ \mathcal{J}_1 &= \{(u,e) \in \mathcal{J} \mid u \in V_4(G)\}. \end{aligned}$$

Define an order relation $\preceq$ in $W_{(u,e)}$ for $(u,e) \in \mathcal{J}$ by letting $w_i \preceq w_j$ if and only if $i \leq j$.

Throughout this section, let $W_{(u,e)}, w_0, \ldots, w_{n(u,e)}$ be as above. In the following three claims, we state the properties of $(u,e) \in \mathcal{J}$.

Claim 3.2 *Let $(u,e) \in \mathcal{J}$. Then the following hold.*

(i) For $1 \leq i \leq n(u,e)$, $\deg_G(w_i) = 4$.
(ii) If $n(u,e) = 2$, then $\deg_{G(u,e)}(w_{n(u,e)}) = \deg_{G(u,e)}(w_{n(u,e)-1}) = 2$.
(iii) Suppose that $n(u,e) \geq 3$. Then, for each i with $1 \leq i \leq n(u,e)$,

$$\deg_{G[W_i]}(w_i) = 2,\ \deg_{G[W_i]}(w_{i-1}) = 3 \text{ and } \deg_{G[W_i]}(w_j) = 4 \text{ for } 2 \leq j \leq i-2.$$

(iv) For $1 \leq i \leq n(u,e)$, $N_G(w_i) \cap N_G(w_{i-1}) - W_i \neq \emptyset$.
(v) If $n(u,e) \geq 3$, then $w_{n(u,e)}w_0 \notin E(G)$.

Proof. We have $\deg_G(w_1) = \deg_G(w_2) = 4$ by the definition of w_1 and w_2 immediately. First we suppose that $n(u,e) = 2$. We have $\deg_{G(u,e)}(w_2) = \deg_{G(u,e)}(w_1) = 2$ and $N_G(w_1) \cap N_G(w_0) - W_1 \neq \emptyset$ by the definition of $G(u,e)$. Moreover $N_G(w_2) \cap N_G(w_1) - W_2 \neq \emptyset$ by the definition of $G(u,e)$ or applying Lemma 2.6 to w_1w_2 according as $u \in V_4(G)$ or $u \in V_{\geq 5}(G)$.

Next we suppose that $n(u,e) \geq 3$. We prove the statements (i), (iii), (iv) and (v) by induction on i with $3 \leq i \leq n(u,e)$. We suppose that $i = 3$. We have $w_3w_2, w_3w_1, w_2w_1, w_2w_0, w_1w_0 \in E(G)$, $\deg_{G[W_3]}(w_2) = \deg_{G[W_3]}(w_1) = 3$ by the definition of $G[W_3]$. Since $\deg_G(w_2) = 4$ and $\deg_{G[W_3]}(w_2) = 3$, we may let $N_G(w_2) = \{w_3, w_1, w_0, x\} (x \notin W_3)$. Since $E_c(w_2) = \emptyset$, $xw_2 \in E_{tn}(G)$ and $\deg_G(w_3) = 4$ by Lemmas 2.3 through 2.5 and the definition of w_1. Since $xw_2 \in E_{tn}(G)$, $xw_3 \in E(G)$ by the definition of $G(u,e)$, and hence $N_G(w_3) \cap N_G(w_2) - W_3 \neq \emptyset$. Assume for the moment that $w_3w_0 \in E(G)$. Then $G - \{w_0, w_1, x\}$ is disconnected, which contradicts G being 4-connected. Thus $w_3w_0 \notin E(G)$, and hence $\deg_{G[W_3]}(w_3) = 2$.

Now we suppose that $i \geq 4$, and we assume that the statements (i), (iii), (iv) and (v) hold for $i-1$. We have $w_iw_{i-1}, w_iw_{i-2}, w_{i-1}w_{i-2}, w_{i-1}w_{i-3}, w_{i-2}w_{i-3} \in E(G)$ by the definition of $G[W_i]$. By the induction hypothesis, $\deg_G(w_j) = 4$ for $1 \leq$

$j \leq i-1$ and $\deg_{G[W_{i-1}]}(w_{i-1}) = 2$, $\deg_{G[W_{i-1}]}(w_{i-2}) = 3$, and $\deg_{G[W_{i-1}]}(w_j) = 4$ for $2 \leq j \leq i-3$. Thus we have $\deg_{G[W_i]}(w_{i-1}) = 3$, and $\deg_{G[W_i]}(w_j) = 4$ for $2 \leq j \leq i-2$. Since $\deg_G(w_{i-1}) = 4$ and $\deg_{G[W_i]}(w_{i-1}) = 3$, we may let $N_G(w_{i-1}) = \{w_i, w_{i-2}, w_{i-3}, x\}(x \notin W_i)$. Since $E_c(w_{i-1}) = \emptyset$, $xw_{i-1} \in E_{tn}(G)$ and $\deg_G(w_i) = 4$ by Lemma 2.5. Since $xw_{i-1} \in E_{tn}(G)$, $xw_i \in E(G)$, and hence $N_G(w_i) \cap N_G(w_{i-1}) - W_i \neq \emptyset$. Assume for the moment that $w_iw_0 \in E(G)$ or $w_iw_1 \in E(G)$; then $G - \{w_0, w_1, x\}$ is disconnected, which contradicts G being 4-connected. Thus $w_iw_0 \notin E(G)$, and hence $\deg_{G[W_i]}(w_i) = 2$ and $w_{n(u,e)}w_0 \notin E(G)$. This completes the proof of Claim 3.2. □

Claim 3.3 *Let $(u, e) \in \mathcal{J}$. Then the following hold:*

(i) If $(u, e) \in \mathcal{J}_1$, then $N_G(w_1) \cap N_G(w_0) - W_{(u,e)} \neq \emptyset$.
(ii) If $(u, e) \in \mathcal{J}_0$, then $E_c(w_1) \neq \emptyset$.

Proof. The statement (ii) follows from the definition of $W_{(u,e)}$ immediately. We now prove statement (i). By the definition of $\mathcal{J}_1$, $N_G(w_1) \cap N_G(w_0) - \{w_2\} \neq \emptyset$. Thus we let $x \in N_G(w_1) \cap N_G(w_0) - \{w_2\}$. First we suppose that $n(u, e) = 2$. By Claim 3.2 (ii), we have $x \notin W_{(u,e)}$, and hence $x \in N_G(w_1) \cap N_G(w_0) - W_{(u,e)}$. Next we suppose that $n(u, e) \geq 3$. By Claim 3.2 (i) and (v), we have $x \notin W_{(u,e)}$; hence $x \in N_G(w_1) \cap N_G(w_0) - W_{(u,e)}$. □

Claim 3.4 *Let $(u, e) \in \mathcal{J}$. Then $E_c(w_{n(u,e)}) \neq \emptyset$.*

Proof. Suppose that $E_c(w_{n(u,e)}) = \emptyset$. By Claim 3.2 (iv), $N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{n(u,e)} \neq \emptyset$ and let $x \in N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{n(u,e)}$. Since $E_c(w_{n(u,e)}) = \emptyset$, $x \in V^*$ by the definition of $G(u, e)$. Set $N_G(w_{n(u,e)}) = \{w_{n(u,e)-1}, w_{n(u,e)-2}, x, y\}$ by Claim 3.2 (i). Since $x, w_0 \in V^*$, we have $xw_0 \in \tilde{E}(G) \cap E_c(G)$, and hence $n(u, e) \geq 4$. By Lemma 2.3, $w_{n(u,e)}y \in E_{tn}(G)$. Thus it follows from Claim 3.2 (i) and (iii) that $xy \in E(G)$ since $n(u, e) \geq 4$, and we also have $\deg_G(x) = 4$ by Lemma 2.5. Hence $G - \{w_0, w_1, y\}$ is disconnected, which contradicts G being 4-connected. □

We now define $\psi(u, e)$ and $\varphi(u, e)$. For $(u, e) \in \mathcal{J}$, let $\psi(u, e)$ be a 4-contractible edge incident with $w_{(u,e)}$ with keeping Claim 3.4 in mind. Further, for $(u, e) \in \mathcal{J}_0$, let $\varphi(u, e)$ be a 4-contractible edge incident with w_1 keeping Claim 3.3 (ii) in mind. The following claim plays a crucial role in our proof of Theorem 1.

Claim 3.5 *Let $(u, e) \in \mathcal{J}$. Then the following holds:*

(i) $\psi(u, e) \in E_{\triangle}(G)$.

Further, suppose that $(u, e) \in \mathcal{J}_0$. Then the following hold:

(ii) $\varphi(u, e) \in E_{\triangle}(G)$.
(iii) $\psi(u, e) \neq \varphi(u, e)$.

Proof. It follows from the definition of $\psi(u,e)$ and $\varphi(u,e)$ that the statements (i) and (ii) hold, since $|\tilde{E}(G) \cap E_c(G)| = 1$. We now prove statement (iii). In the case where $2 \leq n(u,e) \leq 3$, we have statement (iii) by the definition of $G(u,e)$, and hence we assume that $n(u,e) \geq 4$ and $\psi(u,e) = \varphi(u,e)$. Since $N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{(u,e)} \neq \emptyset$ by Claim 3.2 (iv), let $x \in N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{(u,e)}$. Then $G - \{w_1, w_0, x\}$ is disconnected, which contradicts G being 4-connected, and hence statement (iii) is proved. $\square$

Throughout the rest of this section, let $V_4^* = V^* \cap V_4(G)$ and $V_{\geq 5}^* = V^* \cap V_{\geq 5}(G)$, and let $W_{(v,f)} = \{w'_0, w'_1, \ldots, w'_{n(v,f)}\}$ for $(v,f) \in \mathcal{J}$, where $w'_i \preceq w'_j$ for $i \leq j$. Recall that $|\tilde{E}(G) \cap E_c(G)| = 1$ and $W_{(u,e)} = \{w_0, w_1, \ldots, w_{n(u,e)}\}$. In the following three claims, we consider the relations between $u \in V^*$ and $v \in V^*$ with $u \neq v$. Note that if $u \in V^*$ and $v \in V^*$ with $u \neq v$, then $uv \in \tilde{E}(G) \cap E_c(G)$ since $V^* = \{u, v\}$.

Claim 3.6 *Let $u \in V^*$ and $v \in V_4^*$ with $u \neq v$. Suppose that $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$. Then there exists $(v,f) \in \mathcal{J}_1$.*

Proof. Set $N_G(v) = \{u, x, y, z\}$ with $x, y, z \in V_4(G)$ since $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$. Then we may assume that $xy, xz \in E(G)$ since $uv \in \tilde{E}(G)$. Thus $N_G(v) \cap N_G(x) - \{y\} \neq \emptyset$ and $N_G(v) \cap N_G(x) - \{z\} \neq \emptyset$. It follows from Lemma 2.3 that either $N_G(x) \cap N_G(y) - \{v\} \neq \emptyset$ or $N_G(x) \cap N_G(z) - \{v\} \neq \emptyset$; hence there exists $(v,f) \in \mathcal{J}_1$ with either $f = xy$ or $f = xz$ by the definition of $\mathcal{J}_1$. $\square$

Claim 3.7 *Let $(u,e), (v,f) \in \mathcal{J}_1$ with $u \neq v$. Then $W_{(u,e)} \cap W_{(v,f)} = \emptyset$.*

Proof. Suppose that $W_{(u,e)} \cap W_{(v,f)} \neq \emptyset$. Then we have $(w_1, w_2, \ldots, w_{n(u,e)}) = (w'_{n(v,f)}, w'_{n(v,f)-1}, \ldots, w'_1)$ since the definition of $W_{(u,e)}$ and $W_{(v,f)}$, and $u \neq v$. Thus $W_{(u,e)} = (W_{(v,f)} - \{v\}) \cup \{u\}$. By Claim 3.3 (i), $N_G(w_1) \cap N_G(w_0) - W_{(u,e)} \neq \emptyset$ and $N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)} \neq \emptyset$. Let $x \in N_G(w_1) \cap N_G(w_0) - W_{(u,e)}$ and $y \in N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)}$. Then $G - \{x, y\}$ is disconnected, which contradicts G being 4-connected. Hence we have $W_{(u,e)} \cap W_{(v,f)} = \emptyset$. $\square$

Claim 3.8 *Suppose that $(u,e), (v,f) \in \mathcal{J}_1$ with $u \neq v$. Then $\psi(u,e) \neq \psi(v,f)$.*

Proof. Suppose that $\psi(u,e) = \psi(v,f)$. By Claim 3.3 (i), $N_G(w_1) \cap N_G(w_0) - W_{(u,e)} \neq \emptyset$ and $N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)} \neq \emptyset$. Suppose that $x \in N_G(w_1) \cap N_G(w_0) - W_{(u,e)}$ and $z \in N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)}$. By Claim 3.4 (iv), $N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{(u,e)} \neq \emptyset$ and $N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)} \neq \emptyset$. Suppose that $y \in N_G(w_{n(u,e)}) \cap N_G(w_{n(u,e)-1}) - W_{(u,e)}$ and $w \in N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)}$. It follows from Claims 3.5 (i) and 3.7 that $w_{n(u,e)} w_{n(v,f)} \in E_{\triangle}(G) \cap E_c(G)$ since $\psi(u,e) = \psi(v,f)$. Thus $y = w$ by Claim 3.2 (i). Note that $u, v \in V_4(G)$ by the definition of $\mathcal{J}_1$ and $uv \in E(G)$. Then $G - \{x, y, z\}$ is disconnected, which contradicts G being 4-connected. $\square$

In the following two claims, we consider a triangle in addition to the relations between u and v as above.

Claim 3.9 *Suppose that aub is a triangle in G with $u \in V^*$ and $(v, f) \in \mathcal{J}_1$ with $u \neq v$. Then $\psi(v, f) \neq ab$ and $\psi(v, f) \neq bu$.*

Proof. By Claim 3.3 (i), $N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)} \neq \emptyset$. Suppose that $x \in N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)}$. By Claim 3.2 (iv), $N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)} \neq \emptyset$. Suppose that $y \in N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)}$. Note that $v \in V_4(G)$ by the definition of $\mathcal{J}_1$. Set $\psi(v, f) = w'_{n(v,f)}z$. By Claim 3.5 (i), we have $\psi(v, f) \in E_\triangle(G)$. Hence $yz \in E(G)$ by Claim 3.2 (i).

First we show that $\psi(v, f) \neq ab$. Suppose that $\psi(v, f) = ab$. Then we have $y = u$. Further we have $z = a$ or $z = b$, hence $G - \{x, y, z\}$ is disconnected, which contradicts G being 4-connected.

Next we show that $\psi(v, f) \neq bu$. Suppose that $\psi(v, f) = bu$. Then we have $y = a$ and $z = u$ since $w_{n(u,e)} \notin V^*$, hence $G - \{x, y, z\}$ is disconnected, which contradicts G being 4-connected. □

Claim 3.10 *Suppose that aub is a triangle in G such that $a \in V_{\geq 5}(G)$ and $u \in V_4^*$ and $b \in V_4(G)$. Suppose that there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$, and there exists $(v, f) \in \mathcal{J}_1$. Then $\psi(v, f) \neq bc$.*

Proof. Suppose that $\psi(v, f) = bc$. By Claim 3.3 (i), $N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)} \neq \emptyset$. Suppose that $x \in N_G(w'_1) \cap N_G(w'_0) - W_{(v,f)}$. By Claim 3.4 (iv), $N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)} \neq \emptyset$. Suppose that $y \in N_G(w'_{n(v,f)}) \cap N_G(w'_{n(v,f)-1}) - W_{(v,f)}$. Set $\psi(v, f) = w'_{n(v,f)}z$. By Claim 3.5 (i), we have $\psi(u, e) \in E_\triangle(G)$, hence $yz \in E(G)$. Then we have $w'_{n(v,f)} = c$, and $y = a$, $z = b$; for otherwise, $w'_{n(v,f)} = b$, and $y = a$, $z = c$, and hence $u = w'_{n(v,f)-1}$, which contradicts the definition of $W_{(v,f)}$. Note that $N_G(a) \cap N_G(b) = \{c, u\}$ and $N_G(b) \cap N_G(u) = \{a, d\}$, and $uv \in E(G)$. Then $G - \{a, d, x\}$ is disconnected, which contradicts G being 4-connected. □

Again, we consider a triangle which contains a vertex in V^* in the following four claims.

Claim 3.11 *Suppose that aub is a triangle in G such that $a \in V_{\geq 5}(G)$ and $u \in V_4^*$ and $b \in V_4(G)$. Suppose that $|E_c(u)| = 1$. Then there exists $(u, e) \in \mathcal{J}_1$, or there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$.*

Proof. Since $b \notin V^*$ and $|E_c(u)| = 1$, we have $bu \in E_n(G)$, and we may assume that $|E_c(b)| \leq 1$. Then there exists $d \in N_G(b) \cap N_G(u) - \{a\}$ by applying Lemma 2.6 to bu. Suppose that $N_G(b) = \{a, c, d, u\}$. Then $bc \in E_{tn}(G) \cup E_c(G)$ by Lemma 2.3.

If $bc \in E_c(G)$, then $ac \in E(G)$ or $cd \in E(G)$ since $bc \in E_\triangle(G)$. Suppose that $ac \in E(G)$. Then this claim holds since $c \neq d, u$. Suppose that $cd \in E(G)$. Then

$d \in V_{\geq 5}(G)$. Thus this claim holds by the assumptions of Claim 3.11 being replaced with the role of a and d each other.

If $bc \in E_{tn}(G)$, then $cd \in E(G)$ since $a \in V_{\geq 5}$ and $N_G(u) = \{a, b, d, v\}$. Thus we have $d \in V_4(G)$. Hence there exists $(u, e) \in \mathcal{J}_1$ with $e = bd$ since $N_G(b) \cap N_G(d) - \{u\} \neq \emptyset$ and $N_G(b) \cap N_G(u) - \{d\} \neq \emptyset$. □

Claim 3.12 *Suppose that aub is a triangle in G such that $a, b \in V_4(G)$ and $u \in V^*_{\geq 5}$. Suppose that $ab \in E_n(G)$. Then there exists $(u, e) \in \mathcal{J}_0$.*

Proof. By applying Lemma 2.6 to ab, there exists $c \in N_G(a) \cap N_G(b) - \{u\}$. Thus we may assume that $E_c(a) \neq \emptyset$ by Lemma 2.5. Hence there exists $(u, e) \in \mathcal{J}_0$ such that $e = ab$ by the definition of $\mathcal{J}_0$. □

Claim 3.13 *Suppose that aub is a triangle in G such that $a \in V_{\geq 5}(G)$ and $u \in V^*_{\geq 5}$ and $b \in V_4(G)$. Suppose that $N_G(a) \cap N_G(b) - \{u\} = \emptyset$. Then either $ab \in E_\triangle(G) \cap E_c(G)$ or $bu \in E_\triangle(G) \cap E_c(G)$.*

Proof. Suppose that $ab \notin E_\triangle(G) \cap E_c(G)$ and $bu \notin E_\triangle(G) \cap E_c(G)$. Let $(S, A) \in \mathcal{L}$ with $\{a, b\} \subset S$.

If $u \in S$, then $|N_G(b) \cap A| = |N_G(b) \cap \bar{A}| = 1$ since $b \in V_4(G)$. Let $x \in N_G(b) \cap A$ and $y \in N_G(b) \cap \bar{A}$. Then $ux, uy \in E(G)$ since $bx, ay \in E_\triangle(G)$ and $N_G(a) \cap N_G(b) - \{u\} = \emptyset$. We may assume that $x, y \in V_4(G)$ by Lemma 2.1. We may also assume that $bx \in E_n(G)$ by Claim 3.1 and symmetry of x, y. Note that $|E_c(b)| \leq 1$ and $|E_c(x)| \leq 1$ since $b, x \notin V^*$. Thus $|N_G(b) \cap N_G(x)| \geq 2$ by applying Lemma 2.6 to bx. Hence we have $ax \in E(G)$, which contradicts the assumption that $N_G(a) \cap N_G(b) - \{u\} = \emptyset$.

Hence we have $u \notin S$. We may assume that $u \in A$. We have $|\bar{A}| \geq 2$ since $N_G(a) \cap N_G(b) - \{u\} = \emptyset$. Thus $|E_c(b)| \geq 1$ by Lemma 2.2. Hence there exists $w \in N_G(b) - \{a, u\}$ such that $bw \in E_\triangle(G) \cap E_c(G)$. Thus it follows from $|V_{\geq 5}(G)| \leq 2$ that $uw \in E(G)$ since $N_G(a) \cap N_G(b) - \{u\} = \emptyset$. Thus $w \in A$ since $bw \in E_c(G)$. Set $N_G(b) = \{a, u, w, z\}$. Then $z \in \bar{A}$ since $N_G(b) \cap \bar{A} \neq \emptyset$; otherwise $G - (S - \{b\})$ is disconnected, which contradicts G being 4-connected. Hence we have $bz \in \tilde{E}(G)$ since $N_G(a) \cap N_G(b) - \{u\} = \emptyset$, which contradicts the assumption that $|\tilde{E}(G)| = 1$. □

Claim 3.14 *Suppose that aub is a triangle in G such that $a \in V_{\geq 5}(G)$ and $u \in V^*_{\geq 5}$ and $b \in V_4(G)$. Suppose that there exists $c \in V_4(G)$ such that $c \in N_G(a) \cap N_G(b) - \{u\}$. Then either $bc \in E_\triangle(G) \cap E_c(G)$ or $bu \in E_\triangle(G) \cap E_c(G)$.*

Proof. We may suppose that $|E_c(b)| = 0$ and $0 \leq |E_c(c)| \leq 1$, or $0 \leq |E_c(b)| \leq 1$ and $|E_c(c)| = 0$ to prove Theorem 1; otherwise we have $|E_c(G)| \geq 3$ since $|E_c(u)| \geq 1$. Suppose that $bc \notin E_\triangle(G) \cap E_c(G)$ and $bu \notin E_\triangle(G) \cap E_c(G)$. Then there exists

$d \in N_G(b) \cap N_G(c) - \{a\}$ by applying Lemma 2.6 to bc. If $d \neq u$, since $ab, bc, bd \notin E_c(G)$, we have $bu \in E_\triangle(G) \cap E_c(G)$ by applying Lemma 2.3, which contradicts the assumption that $bu \notin E_\triangle(G) \cap E_c(G)$. If $d = u$, then we have $|E_c(b)| \geq 1$ and $|E_c(c)| \geq 1$ by Lemma 2.4, which contradicts $|E_c(b)| = 0$ or $|E_c(c)| = 0$. This completes the proof of Claim 3.14. □

Finally we prove the following five claims whose conclusions satisfy $|E_c(G)| \geq 3$.

Claim 3.15 *Suppose that aub is a triangle in G such that $a, b \in V_{\geq 5}(G)$ and $u \in V_4^*$. Then $|E_c(G)| \geq 3$.*

Proof. Suppose that $uv \in \tilde{E}(G) \cap E_c(G)$. We may assume that $v \in V_4^*$ and $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$ by Lemma 2.1. Thus there exists $(v, f) \in \mathcal{J}_1$ by Claim 3.6. Set $N_G(u) = \{a, b, c, v\}$. By symmetry of a, b we may assume that $ac \in E(G)$ since $cu \in E_\triangle(G)$. We may assume that $c \in V_4(G)$ by Lemma 2.1.

If $bu \in E_n(G)$ and $cu \in E_n(G)$, then $bc \in E(G)$ by applying Lemma 2.6 to uc. Thus there exists $d \in N_G(c)$ such that $cd \in E_\triangle(G) \cap E_c(G)$, hence $ad \in E(G)$ or $bd \in E(G)$. Hence we have $|E_c(G)| \geq 3$ since $|\{uv, cd, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.10.

If $bu \in E_c(G)$ or $cu \in E_c(G)$, then we have $|E_c(G)| \geq 3$ since $|\{uv, bu, cu, \psi(v, f)\} \cap E_c(G)| \geq 3$ by Claims 3.5 (i) and 3.9. □

Claim 3.16 *Suppose that aub is a triangle in G such that $a \in V_{\geq 5}(G)$ and $u \in V_{\geq 5}^*$ and $b \in V_4(G)$. Suppose that there exists $c \in V_4(G)$ such that $c \in N_G(a) \cap N_G(b) - \{u\}$ and $bc \in E_\triangle(G) \cap E_c(G)$. Then $|E_c(G)| \geq 3$.*

Proof. Suppose that $N_G(b) = \{a, c, u, x\}$. Then we may assume that $x \in V_4(G)$ by Lemma 2.1. We may also assume that $bx \in E_\triangle(G) \cap E_n(G)$ by Claim 3.1. Note that $cx \notin E(G)$ since $x \in V_4(G)$ and $bc \in E_c(G)$. Thus we have $ax \in E(G)$ and $ux \in E(G)$ by applying Lemma 2.6 to bx. Thus there exists $e \in E_\triangle(G) \cap E_c(x)$ by Lemma 2.4. Note that $e \neq bc$ since $bx \in E_n(G)$. Hence it follows from Claim 3.1 that $|E_c(G)| \geq 3$ since $|\{e, bc\} \cap E_\triangle(G) \cap E_c(G)| = 2$. □

Claim 3.17 *Let $u, v \in V_4^*$ with $u \neq v$. Suppose that $E_\triangle(G) \cap E_c(u) \neq \emptyset$ and there exists $(v, f) \in \mathcal{J}_1$. Then $|E_c(G)| \geq 3$.*

Proof. Suppose that $e \in E_\triangle(G) \cap E_c(u)$. Then we have $|E_c(G)| \geq 3$ since $|\{uv, e, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.9. □

Claim 3.18 *Let $u, v \in V_4^*$ with $u \neq v$. Suppose that $N_G(u) \cap V_{\geq 5}(G) - \{v\} \neq \emptyset$ and $N_G(v) \cap V_{\geq 5}(G) - \{u\} \neq \emptyset$. Then $|E_c(G)| \geq 3$.*

Proof. Suppose that $a \in N_G(u) \cap V_{\geq 5}(G) - \{v\}$. Then there exists $b \in N_G(a) \cap N_G(u) - \{v\}$. We may assume that $b \in V_4(G)$ by Claim 3.15. By the symmetry of u, v there exist $x, y \in N_G(v)$ such that $xy \in E(G)$, $x \in V_{\geq 5}(G)$, and $y \in V_4(G)$. If $|E_c(u)| \geq 2$ and $|E_c(v)| \geq 2$, then we have $|E_c(G)| \geq 3$. Thus we may assume that $|E_c(u)| = 1$. Then $bu \in E_n(G)$. Hence there exists $(u, e) \in \mathcal{J}_1$, or there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$ by Claim 3.11.

Suppose that there exists $(u, e) \in \mathcal{J}_1$. If $E_\triangle(G) \cap E_c(v) \neq \emptyset$, then $|E_c(G)| \geq 3$ by Claim 3.17. Thus we may assume that $|E_c(v)| = 1$. Then $vy \in E_n(G)$. Hence there exists $(v, f) \in \mathcal{J}_1$, or there exist $z \in N_G(x) \cap N_G(y) - \{v\}$ with $yz \in E_\triangle(G) \cap E_c(G)$ and $w \in N_G(y) \cap N_G(v) - \{x, z\}$ by Claim 3.11. If there exists $(v, f) \in \mathcal{J}_1$, then we have $|E_c(G)| \geq 3$ since $|\{uv, \psi(u, e), \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.8. If there exists $z \in N_G(x) \cap N_G(y) - \{v\}$ with $yz \in E_\triangle(G) \cap E_c(G)$ and $w \in N_G(y) \cap N_G(v) - \{x, z\}$, then we have $|E_c(G)| \geq 3$ since $|\{uv, yz, \psi(u, e)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.10.

Suppose that there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$. Then we may assume that $E_\triangle(G) \cap E_c(v) = \emptyset$ by Claim 3.1. Thus we have $|E_c(v)| = 1$, hence there exists $(v, f) \in \mathcal{J}_1$, or there exists $z \in N_G(x) \cap N_G(y) - \{v\}$ with $yz \in E_\triangle(G) \cap E_c(G)$ and there exists $w \in N_G(y) \cap N_G(v) - \{x, z\}$ by Claim 3.11. If there exists $(v, f) \in \mathcal{J}_1$, then we have $|E_c(G)| \geq 3$ since $|\{uv, bc, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.10. If there exists $z \in N_G(x) \cap N_G(y) - \{v\}$ with $yz \in E_\triangle(G) \cap E_c(G)$ and there exists $w \in N_G(y) \cap N_G(v) - \{x, z\}$. Suppose that $bc = yz$. Since $uv \in \tilde{E}(G)$, we have $b = z$ and $c = y$. Note that $a \neq x$ since $uv \in \tilde{E}(G)$. We also have $a = w$ and $d = x$ since $uv \in \tilde{E}(G)$. Then $v \in N_G(a) \cap N_G(c)$, hence $uv \in E_\triangle(G)$, which contradicts the assumption that $uv \in \tilde{E}(G)$. Hence we have $bc \neq yz$. Thus we have $|E_c(G)| \geq 3$ since $|\{uv, bc, yz\} \cap E_c(G)| = 3$. This proves Claim 3.18. □

Claim 3.19 *Suppose that there exists $(v, f) \in \mathcal{J}_0$. Then $|E_c(G)| \geq 3$.*

Proof. Since $v \in V^*$, there exists $u \in V^*$ such that $uv \in \tilde{E}(G) \cap E_c(G)$, and since $(v, f) \in \mathcal{J}_0$, there exist $\varphi(v, f)$ and $\psi(v, f)$. Thus we have $|E_c(G)| \geq 3$ since $|\{uv, \varphi(v, f), \psi(v, f)\} \cap E_c(G)| = 3$ by Claim 3.5 (i) through (iii). □

4 Proof of Theorem 1

We continue with the notation of the preceding section. In this section, we prove Theorem 1. Recall that $|\tilde{E}(G)| = 1$ and $\hat{G} = (\tilde{V}, \tilde{E}(G) \cap E_n(G))$.

In the following three claims, we suppose that $|\tilde{E}(G) \cap E_c(G)| = 1$ and let $uv \in \tilde{E}(G) \cap E_c(G)$, and hence we may clearly assume that $|E_c(x)| \leq 1$ for $x \in V(G) - \{u, v\}$ and $e \in E_\triangle(G)$ for $e \in E(G) - \{uv\}$ to prove Theorem 1 as in Section 3.

Claim 4.1 *Suppose that* $\deg_G(u) \geq 5$ *and* $\deg_G(v) \geq 5$. *Then* $|E_c(G)| \geq 3$.

Proof. Let $a \in N_G(u)-\{v\}$. Since $au \in E_\triangle(G)$, there exists $b \in N_G(u)\cap N_G(a)-\{v\}$. We may assume that $a, b \in V_4(G)$ by Lemma 2.1. By symmetry of u, v and $uv \in \tilde{E}(G)$, there exist $x, y \in N_G(v) \cap V_4(G)$ such that $xy \in E(G) - \{ab\}$. Then we may assume that $ab \in E_n(G)$ or $xy \in E_n(G)$ by Claim 3.1. Thus there exists $(u, e) \in \mathcal{J}_0$ or $(v, f) \in \mathcal{J}_0$ by Claim 3.12, hence we have $|E_c(G)| \geq 3$ by Claim 3.19. □

Claim 4.2 *Suppose that either* $\deg_G(u) \geq 5$ *and* $\deg_G(v) = 4$, *or* $\deg_G(u) = 4$ *and* $\deg_G(v) \geq 5$. *Then* $|E_c(G)| \geq 3$.

Proof. By symmetry of u, v it suffices to analyze the subcase that $\deg_G(u) \geq 5$ and $\deg_G(v) = 4$. Let $a \in N_G(u) - \{v\}$. Since $uv \in \tilde{E}(G)$ and $au \in E_\triangle(G)$, there exists $b \in N_G(u) \cap N_G(a) - \{v\}$. Now we may assume that $b \in V_4(G)$ by symmetry of a, b and Lemma 2.1.

We distinguish two cases, according to the possible degrees of a.

***Case* 1** $a \in V_{\geq 5}(G)$.

We may assume that $N_G(v)\cap V_{\geq 5}(G)-\{u\} = \emptyset$ by Lemma 2.1. Then there exists $(v, f) \in \mathcal{J}_1$ by Claim 3.6.

Suppose that $N_G(a)\cap N_G(b)-\{u\} = \emptyset$. Then $ab \in E_\triangle(G)\cap E_c(G)$ or $bu \in E_\triangle(G)\cap E_c(G)$ by Claim 3.13. Hence we have $|E_c(G)| \geq 3$ since $|\{ab, bu, uv, \psi(v, f)\} \cap E_c(G)| \geq 3$ by Claims 3.5 (i) and 3.9.

Suppose that $N_G(a) \cap N_G(b) - \{u\} \neq \emptyset$. Then there exists $c \in V_4(G)$ such that $c \in N_G(a) \cap N_G(b) - \{u\}$ by Lemma 2.1. Thus we have $bc \in E_\triangle(G) \cap E_c(G)$ or $bu \in E_\triangle(G) \cap E_c(G)$ by Claim 3.14. If $bc \in E_\triangle(G) \cap E_c(G)$, then we have $|E_c(G)| \geq 3$ by Claim 3.16. If $bu \in E_\triangle(G) \cap E_c(G)$, then we have $|E_c(G)| \geq 3$ since $|\{bu, uv, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.9.

***Case* 2** $a \in V_4(G)$.

Suppose that $ab \in E_n(G)$. Then there exists $(u, e) \in \mathcal{J}_0$ by Claim 3.12. Hence we have $|E_c(G)| \geq 3$ by Claim 3.19.

Suppose that $ab \in E_c(G)$. If $N_G(v)\cap V_{\geq 5}(G)-\{u\} = \emptyset$, then there exists $(v, f) \in \mathcal{J}_1$ by Claim 3.6. Hence we have $|E_c(G)| \geq 3$ since $|\{ab, uv, \psi(v, f)\}\cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.9. If $N_G(v)\cap V_{\geq 5}(G)-\{u\} \neq \emptyset$, we let $x \in N_G(v)\cap V_{\geq 5}(G)-\{u\}$. Then it follows from Lemma 2.1 that there exists $y \in V_4(G)$ such that $y \in N_G(x) \cap N_G(v)-\{u\}$ since $uv \in \tilde{E}(G)$ and $vx \in E_\triangle(G)$. If $|E_c(v)| \geq 2$, then $|E_c(G)| \geq 3$ since $ab \in E_\triangle(G) \cap E_c(G)$. Thus we may assume that $|E_c(v)| = 1$. Further we may also assume that $vy \in E_n(G)$ by Claim 3.1. Then there exists $(v, f) \in \mathcal{J}_1$, or there exists $z \in N_G(x)\cap N_G(y)-\{v\}$ with $yz \in E_\triangle(G)\cap E_c(G)$ by Claim 3.11. Suppose that there exists $(v, f) \in \mathcal{J}_1$. Then we have $|E_c(G)| \geq 3$ since $|\{ab, uv, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.9. Suppose that there exists $z \in N_G(x) \cap N_G(y) - \{v\}$ with $yz \in E_\triangle(G) \cap E_c(G)$. Note that $\{y\} \cap \{a, b\} = \emptyset$ since $uv \in \tilde{E}(G)$. Consequently $ab, yz \in E_\triangle(G) \cap E_c(G)$ with $ab \neq yz$, and hence it follows from Claim 3.1 that $|E_c(G)| \geq 3$. □

Claim 4.3 *Suppose that* $\deg_G(u) = \deg_G(v) = 4$. *Then* $|E_c(G)| \geq 3$.

Proof. Suppose that $N_G(u) \cap V_{\geq 5}(G) - \{v\} = \emptyset$ and $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$. Then there exist $(u, e), (v, f) \in \mathcal{J}_1$ with $u \neq v$ by Claim 3.6. Hence we have $|E_c(G)| \geq 3$ since $|\{uv, \psi(u, e), \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.8. Thus we may assume that $N_G(u) \cap V_{\geq 5}(G) - \{v\} \neq \emptyset$ and $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$ by symmetry of u, v and Claim 3.18. We let $a \in N_G(u) \cap V_{\geq 5}(G) - \{v\}$. Then there exists $b \in N_G(u) \cap N_G(a) - \{v\}$ since $uv \in \tilde{E}(G)$ and $au \in E_\triangle(G)$. Further it follows from Claim 3.6 that there exists $(v, f) \in \mathcal{J}_1$ since $N_G(v) \cap V_{\geq 5}(G) - \{u\} = \emptyset$. We may also assume that $b \in V_4(G)$ by Claim 3.15.

Suppose that $|E_c(u)| = 1$. Then we have $bu \in E_n(G)$. Thus there exists $(u, e) \in \mathcal{J}_1$, or there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$ by Claim 3.11. First we suppose that there exists $(u, e) \in \mathcal{J}_1$. Then we have $|E_c(G)| \geq 3$ since $|\{uv, \psi(u, e), \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.8. Next we suppose that there exist $c \in N_G(a) \cap N_G(b) - \{u\}$ with $bc \in E_\triangle(G) \cap E_c(G)$ and $d \in N_G(b) \cap N_G(u) - \{a, c\}$. Then we have $|E_c(G)| \geq 3$ since $|\{uv, bc, \psi(v, f)\} \cap E_c(G)| = 3$ by Claims 3.5 (i) and 3.10.

Suppose that $|E_c(u)| \geq 2$. Then $E_\triangle(G) \cap E_c(u) \neq \emptyset$ since uv is the only 4-contractible edge which is not contained in triangles. Hence we have $|E_c(G)| \geq 3$ by Claim 3.17. $\square$

We are now in a position to complete the proof of Theorem 1. To complete the proof of Theorem 1, it suffices to show the following two cases since $|\tilde{E}(G)| = 1$.

***Case* 1** $\hat{G} \simeq K_2$.

In this case, we have $|\tilde{E}(G) \cap E_n(G)| = 1$. Hence we obtain $|E_c(G)| \geq 2$ by Lemma 2.7.

***Case* 2** $\hat{G} \simeq \emptyset$.

In this case, we have $|\tilde{E}(G) \cap E_n(G)| = 0$. Thus we have $|\tilde{E}(G) \cap E_c(G)| = 1$ since $|\tilde{E}(G)| = 1$. Hence we obtain $|E_c(G)| \geq 3$ by Claims 4.1 through 4.3. This completes the proof of Theorem 1. $\square$

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