

Counting connected sets and connected partitions of a graph

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*Dedicated to the memory of Dan Archdeacon who made graph theory,
and for that matter, everything else, fun.*

Abstract

This paper concerns two related enumeration problems on vertex labeled graphs. Given such a graph G , we investigate the number $C(G)$ of connected subsets of the vertex set and the number $P(G)$ of connected partitions of the vertex set. By *connected* we mean that the induced subgraphs are connected. The numbers $C(G)$ and $P(G)$ can be regarded as the (connected) graph analogs of the number of subsets and the number of set partitions, respectively, of an n -element set.

1 Introduction

Graphical enumeration is a major topic in graph theory; at its origins is the elegant 19th century formula of Cayley for the number of labeled trees on n vertices, and already four decades ago a book [5] appeared on the subject. Not so common is counting under connectivity restrictions. Two such enumeration problems are the topic of this paper. In particular, graph analogs of the Stirling and Bell numbers are introduced and investigated.

Unless otherwise stated, G is a simple graph, no loops or multiple edges, with $V = V(G)$ denoting the vertex set and $E = E(G)$ the edge set. The vertices of G are labeled $V = \{1, 2, \dots, n\}$. The number of subsets of an n -element set is 2^n . A graph analog is the number $C(G)$ of connected subsets of $V(G)$, where a nonempty subset of $V(G)$ is *connected* if its induced subgraph is connected. If G is the complete graph K_n , then $C(G)$ is exactly the number of subsets of an n -element set. Bounds on $C(G)$ in terms of the order and maximum degree of G appear in [2, 6].

The number of set partitions of a labeled n -element set is the Bell number $B(n)$. A graph analog is the number $P(G)$ of partitions of the vertex set $V(G)$ of a graph G of order n , where each part is connected. If G is the complete graph K_n , then $P(G)$ is exactly the Bell number $B(n)$. It should be noted that our notion of partition numbers is different than the “graphical Sterling numbers” defined in [4] and references therein.

The subject of this paper are properties of $C(G)$ and $P(G)$. In particular, if \mathcal{G} is an infinite family of graphs, we examine the growth and growth thresholds of $C(G)$ and $P(G)$ for $G \in \mathcal{G}$. Four basic conjectures remain open.

The paper is organized as follows. Section 2 investigates properties of $C(G)$ while Section 3 investigates properties of $P(G)$. The numbers $C(G)$ and $P(G)$ depend, in large part, on the density of G , in particular on the number of edges of G and on the degrees of the vertices. Theorem 4 provides sharp lower and upper bounds on $C(G)$ in terms of the number of edges of G . The remainder of Section 2 concerns the growth rate (defined at the beginning of Section 2) of $C(G)$ for the graphs G in an infinite family \mathcal{G} of graphs. Exact formulas for $C(G)$ are computed for various families of graphs, providing examples of families with polynomial, subexponential, and exponential growth rates for the number of connected vertex sets. The exponential growth rate of a family cannot exceed 2, and this bound is attained. Theorem 5 provides a threshold between polynomial and exponential growth for any infinite family of graphs, in terms of the degrees of the vertices of the graphs in the family. A more refined condition sufficient for exponential growth is the subject of Conjecture 1. Theorems 6 and 7 confirm the conjecture for special cases.

Results concerning the connected partition number $P(G)$, analogous to those for the connected set number $C(P)$, are the subject of Section 3 and Section 4. According to Propositions 5 and 7 we know that $2^{n-1} \leq P(G) \leq 2^m$, for any graph G with n vertices and m edges, and these bounds are sharp. The sharp upper bound on $P(G)$ in terms of n is clearly the Bell number $B(n)$. A sharp lower bound on $P(G)$ in terms of m is more problematic and is the subject of Conjecture 2. In particular, contrary to what at first may seem reasonable, the complete graph is conjectured to have the minimum number of connected partitions among all graphs with an equal number of edges.

For any infinite family \mathcal{G} of graphs, the growth rate of $P(G)$ of the graphs $G \in \mathcal{G}$ is either exponential or super-exponential. Exact formulas for $P(G)$ are computed in Section 3 for various families of graphs, providing examples of both exponential and super-exponential families. Unlike the situation for $C(G)$, there is no upper bound on the growth rate for $P(G)$ for an exponential family; Example 11 provides families of graphs with arbitrarily high exponential growth rate with respect to the number of connected partitions. Section 4 discusses a threshold, in terms of vertex degrees, between exponential and super-exponential growth for a family of graphs. According to Theorem 13, if the set of average degrees of the graphs in a family \mathcal{G} is bounded, then \mathcal{G} has exponential growth rate. Example 14 shows, however, that the following statement is false: If the set of average degrees of the graphs in a family \mathcal{G} is unbounded, then \mathcal{G} has super-exponential growth rate. We conjecture (Conjecture 3)

that if the set of minimum degrees of the graphs in a family \mathcal{G} is unbounded, then \mathcal{G} has super-exponential growth rate. We provide an approach to proving Conjecture 3 via Conjecture 4.

2 Counting Connected Vertex Sets

An infinite family \mathcal{G} of graphs is said to have *exponential growth rate with respect to the number of connected sets* if

$$\underline{C}(\mathcal{G}) := \liminf_{G \in \mathcal{G}} \sqrt[n]{C(G)} > 1,$$

where n is the number of vertices in G . An infinite family \mathcal{G} of graphs is said to have *subexponential growth rate with respect to number of connected sets* if

$$\overline{C}(\mathcal{G}) := \limsup_{G \in \mathcal{G}} \sqrt[n]{C(G)} = 1.$$

If there is a positive integer a such that

$$\limsup_{G \in \mathcal{G}} C(G) < n^a,$$

then \mathcal{G} has *polynomial growth rate*. If $\overline{C}(\mathcal{G}) = \underline{C}(\mathcal{G})$, then the common value is referred to as the *exponential growth rate* and denoted $C(\mathcal{G})$. The following proposition is clear.

Proposition 1. *For any infinite family \mathcal{G} of graphs, $\overline{C}(\mathcal{G}) \leq 2$.*

Example 1 (Paths, Cycles, Complete Graphs, Complete Bipartite Graphs, Stars, Wheels). For the complete graph K_n , complete bipartite graph $K_{m,n}$, the path P_n , the cycle C_n , the star $K_{1,n}$, and the wheel W_n with n spokes (see Figure 1), it is not hard to determine the number of connected vertex sets:

$$\begin{aligned} C(K_n) &= 2^n - 1 \\ C(K_{m,n}) &= 2^{m+n} - 2^m - 2^n + m + n + 1 \\ C(P_n) &= (n^2 + n)/2 \\ C(C_n) &= n^2 - n + 1 \\ C(K_{n,1}) &= 2^n + n \\ C(W_n) &= 2^n + n^2 - n + 1 \end{aligned}$$

The family of paths and the family of cycles have polynomial growth rate, while the families of complete graphs, of stars, and of wheels have exponential growth rate 2, which is as large as possible. Note that, although all trees on n vertices have the same number of edges and average degree, the family of paths has polynomial growth rate, while the family of star graphs has exponential growth rate.

Example 2 (A family with subexponential, but not polynomial, growth). Let M_n be the tree on n vertices that is the one vertex union of a star graph with $\lfloor \sqrt{n} \rfloor$ spokes and a path with $n - \lfloor \sqrt{n} \rfloor$ vertices. (The single common vertex of the star and the path can be taken to be the hub of the star.) See Figure 1. It is easy to check that the family of such graphs has subexponential, but not polynomial, growth rate.

Example 3 (Ladders). For the ladder L_n with $n = 2r$ vertices and r rungs (see Figure 1), a formula for the number of connected vertex sets can be obtained:

$$C(L_n) = \frac{1}{4} \left(\beta^{\frac{n}{2}+3} + \bar{\beta}^{\frac{n}{2}+3} \right) - n - \frac{7}{2},$$

where $\beta = 1 + \sqrt{2}$ and $\bar{\beta} = 1 - \sqrt{2}$. This was derived as follows. If the rungs are labeled 1 to r , in linear order along the ladder, then the number of connected vertex sets containing a vertex in both rung 1 and rung r is in bijection with the number of sequences of length r using digits 0, 1, 2, for which the digits 1 and 2 cannot appear consecutively. This leads to a recursion to which generating function techniques can be applied. Using the above formula, the exponential growth rate for the family \mathcal{G} of ladders can be obtained:

$$C(\mathcal{G}) = \sqrt{1 + \sqrt{2}} \approx 1.55.$$

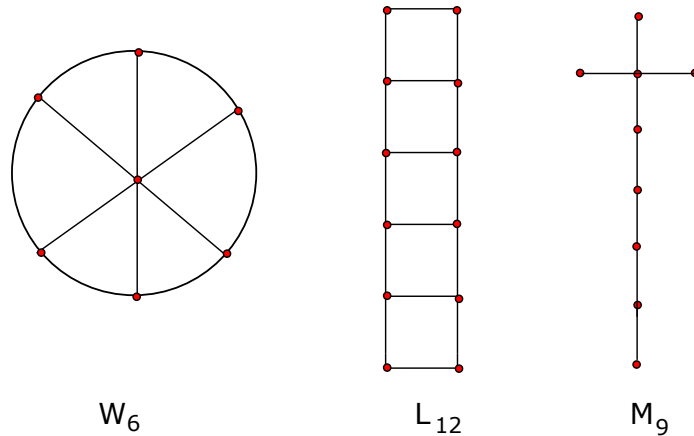


Figure 1: The wheel W_6 , the ladder L_{12} , and the graph M_9 from Example 2.

Theorem 4. For a connected graph G with m edges (independent of the number of vertices),

$$\binom{m+2}{2} \leq C(G) \leq 2^m + m.$$

The upper bound is attained by the star graphs, and the lower bound is attained by paths.

Proof. Concerning the upper bound, $2^m - 1$ is clearly at least as great as the number of non-empty subgraphs with at least one edge, and $m + 1$ is at least as great as the number of one vertex subgraphs.

Concerning the lower bound, assume that G is a connected graph with m edges that minimizes the number of connected subsets of vertices. It is easy to confirm that G is not a complete graph. Assume that G is not a tree. Then there is an edge $e = \{u, v\}$ of G that is contained in a cycle and such that the neighborhood of u is not a complete graph. Consider the graph G' obtained from G by deleting e and adding an edge $e' = \{u, v'\}$, where $\deg(v') = 1$. The number of connected sets in G' not containing v' equals the number of connected sets in G not containing v . The number of connected sets in G' containing v' is less than the number of connected sets in G containing v . This contradicts the minimality of G , thus proving that G must be a tree. The set of connected vertex sets in a path is in bijection with the set of pairs $\{u, v\}$ of (not necessarily distinct) vertices. Hence there are $\binom{m+2}{2}$ connected sets. For a tree, each pair $\{u, v\}$ of vertices determines a distinct connected set, namely the unique path joining u and v . Therefore, of all trees with m vertices, the path minimizes the number of connected sets. \square

Both the star graphs and the paths have average degree essentially 2. Hence, in view of Theorem 4, the average degree of the graphs in a family \mathcal{G} implies little about the growth rate of \mathcal{G} . The remainder of this section concerns a threshold involving vertex degrees that does separate polynomial and exponential growth. The proof of the next lemma is obvious.

Lemma 1. *If H is a subgraph of G , then $C(H) \leq C(G)$.*

For a graph G and vertex v , let $C(G, v)$ denote the number of connected vertex sets of G containing v .

Lemma 2. *If G is a connected graph with minimum degree at least 3, then G has a spanning tree T with an interior vertex v such that $C(T, v) \geq 2^{\frac{n}{4}}$.*

Proof. Let T be a spanning tree with at least $n/4 + 2$ leaves. The existence of such a spanning tree is proved in [7]. Let v be an interior vertex of T . There is an injective map from the set of subsets of the set of leaves into the set of subtrees of T containing v . In this injection a subset of leaves is mapped to the subtree that is the union of all the unique paths joining v to each leaf in the subset. Therefore $C(T, v) \geq 2^{\frac{n}{4}}$. \square

Theorem 5. *Let \mathcal{G} be a family of connected graphs.*

1. *If the minimum degree is at least 3 for all $G \in \mathcal{G}$, then \mathcal{G} has exponential growth rate with respect to the number of connected sets.*
2. *If the maximum degree is 2 for all $G \in \mathcal{G}$, then \mathcal{G} has polynomial growth rate with respect to the number of connected sets.*

Proof. Statement (2) follows from Example 1. Concerning statement (1), for the spanning tree of Lemma 2, and using Lemma 1, we have $C(G) \geq C(T, v) \geq 2^{\frac{n}{4}}$ for any $G \in \mathcal{G}$. \square

In an attempt to refine Theorem 5, we pose the following conjecture.

Conjecture 1. *Let \mathcal{G} be an infinite family of connected graphs. If $N_3(G)$ denotes the number of vertices of degree at least 3 in G , and if*

$$N_3(\mathcal{G}) := \liminf_{G \in \mathcal{G}} \frac{N_3(G)}{n(G)} > 0,$$

then \mathcal{G} has exponential growth.

Remark 1. The converse of Conjecture 1 is not true. The family \mathcal{G} of star graphs has exponential growth, but $N_3(G) = 1$ for all $G \in \mathcal{G}$. The following two results show that the conjecture is true under additional assumptions.

Theorem 6. *If \mathcal{G} is an infinite family of trees for which $N_3(\mathcal{G}) > 0$, then \mathcal{G} has exponential growth.*

Proof. For any tree T we have $n_1(T) \geq N_3(T) + 2$, where $n_1(T)$ is the number of leaves. By assumption, there is a constant $\alpha > 0$ such that for all $\epsilon > 0$ we have $N_3(T) \geq (\alpha - \epsilon)n(T)$ for all but finitely many $T \in \mathcal{G}$. As in the proof of Lemma 2, for all $\epsilon > 0$ we have

$$C(T) \geq 2^{n_1(T)} > 2^{N_3(T)} \geq 2^{(\alpha - \epsilon)n(T)}$$

for all but finitely many $T \in \mathcal{G}$. Therefore $\sqrt[n]{C(T)} \geq 2^{(\alpha - \epsilon)}$ for all but finitely many $T \in \mathcal{G}$, and hence $\underline{C}(\mathcal{G}) \geq 2^\alpha > 1$. \square

Let $\Delta(G)$ denote the maximum degree of the vertices of G . An infinite family of graphs has *bounded degree* if

$$\overline{D}(\mathcal{G}) := \{\Delta(G) : G \in \mathcal{G}\}$$

is a bounded set.

Theorem 7. *If the family \mathcal{G} has bounded degree, and if $N_3(\mathcal{G}) > 0$, then \mathcal{G} has exponential growth.*

Proof. Let B be the maximum of $\overline{D}(\mathcal{G})$. Let $G \in \mathcal{G}$. As in the proof of Lemma 2, let T be a spanning tree of G such that $n_1(T) > \frac{n}{4}$, where n_k , $k \geq 1$, is the number of vertices of degree k of T . Now

$$\sum_{k>1} n_k(T) = 2|E(T)| = 2 \left(\sum_{k>1} n_k(T) - 1 \right),$$

which implies that

$$\frac{n}{4} < n_1(T) = 2 + \sum_{k=3}^B (k - 2) n_k(T) \leq 2 + (B - 2)N_3(T).$$

Therefore $N_3(T) > \frac{n-8}{4(B-2)}$, and hence this theorem follows from Theorem 6. \square

3 Counting Graph Partitions

If G is a graph with labeled vertex set $V = \{1, 2, \dots, n\}$, then a partition of V such that each part is connected will be called a *connected partition*. The number $P(G, k)$ of connected partitions of G into k parts will be called the *connected k -partition number* of G , and the total number of connected partitions

$$P(G) = \sum_{k=1}^n P(G, k)$$

of G will be called the *connected partition number* of G .

For the complete graph $P(K_n, k) = S(n, k)$, the Stirling number of the second kind, and $P(K_n) = B(n)$, the Bell number. For the n -cycle, the connected k -partition number, as will be shown in the next section, equals $\binom{n}{k}$, the binomial coefficient.

The following results are useful.

Proposition 2. *If G is the union of two disjoint graphs G_1 and G_2 , then*

$$P(G_1 \cup G_2, k) = \sum_{i=1}^k P(G_1, i) P(G_2, k - i), \quad P(G_1 \cup G_2) = P(G_1) P(G_2).$$

If G is the union of two graphs G_1 and G_2 that have exactly one vertex in common, then

$$P(G_1 \cup G_2, k) = \sum_{i=1}^k P(G_1, i) P(G_2, k + 1 - i), \quad P(G_1 \cup G_2) = P(G_1) P(G_2).$$

Proof. The proof of the first formula is clear. Concerning the second formula, if v is the common vertex of G_1 and G_2 , then any connected partition of G corresponds to connected partitions P_1 and P_2 of G_1 and G_2 , respectively, where the total number of parts is $k + 1$, and the two parts containing v are combined into one part. \square

Proposition 3. *If H is a subgraph of G , then $P(H, k) \leq P(G, k)$.*

Proof. If U is a set of vertices of H that induces a connected graph, then U obviously induces a connected graph in G . \square

Proposition 4. *The connected partition number $P(G)$ equals the number of subsets S of the edge set of G such that no cycle in G contains exactly one edge in S .*

Proof. The removal of a set S of edges determines a partition, namely the partition obtained by removing those edges (but not the endpoints). This would be a bijection, except in the case that an edge $e \in S$ is the only edge of S on some cycle in G . In this case, removing e is superfluous. \square

An infinite family \mathcal{G} of graphs is said to have *exponential growth rate with respect to the connected partition number* if

$$\overline{P}(\mathcal{G}) := \limsup_{G \in \mathcal{G}} \sqrt[n]{P(G)} < \infty,$$

where n is the number of vertices in G . An infinite family \mathcal{G} of graphs is said to have *super-exponential growth rate with respect to the connected partition number* if

$$\underline{P}(\mathcal{G}) := \liminf_{G \in \mathcal{G}} \sqrt[n]{P(G)} = \infty.$$

If $\overline{P}(\mathcal{G}) = \underline{P}(\mathcal{G}) < \infty$, then the common value is referred to as the *exponential growth rate* and denoted $P(\mathcal{G})$.

Example 8 (Trees, Cycles, Complete Graphs). The connected partition numbers of the complete graph K_n , any tree T_n on n vertices, and the cycle C_n are as follows:

$$P(K_n, k) = S(n, k) \qquad P(T_n, k) = \binom{n-1}{k-1} \qquad P(C_n, k) = \begin{cases} \binom{n}{k} & \text{if } k \geq 2 \\ 1 & \text{if } k = 1. \end{cases}$$

$$P(K_n) = B(n) \qquad P(T_n) = 2^{n-1} \qquad P(C_n) = 2^n - n$$

The first formula is clear because, in the complete graph, any subset of vertices induces a connected subgraph. For a tree, any connected partition into k parts is determined by the removal of $k - 1$ edges. For the cycle, any connected partition into k , $k \geq 2$, parts is determined by the removal of k edges.

For the families of trees and of cycles, the exponential growth rate is 2. For the family of complete graphs, the growth rate is super-exponential. This last fact follows from known asymptotics of the Bell numbers. From a formula of de Bruijn [3], it follows that $B(n) > \left(\frac{n}{e \ln n}\right)^n$, for n sufficiently large. It is shown in [1] that $B(n) < \left(\frac{.792n}{\ln(n+1)}\right)^n$ for all n .

Example 9 (Wheels). Let W_n be the wheel graph with n spokes and $n + 1$ vertices, v denoting the hub of the wheel and c the cycle of n vertices. Then the connected partition number is given by:

$$P(W_n) = f_{2n-1} + f_{2n+1} - n,$$

where f_n is the n^{th} Fibonacci number with $f_0 = 0, f_1 = 1$.

To derive this formula, first note that there are 2 connected partitions for which all n vertices of c are in the same part. To count the other connected partitions first delete $k, 2 \leq k \leq n$, edges on c ; this uniquely determines a connected partition of c . If the k parts are deoted p_1, p_2, \dots, p_k , then a connected partition of W_n is uniquely determined by first choosing any subset S of $P = \{p_1, p_2, \dots, p_k\}$ for which no two cyclicly consecutive parts are in the subset, then taking the union of those parts and v as a single part in the connected partition of W_n . The other parts in the connected partition of W_n are the remaining parts of P . The number of such subsets S is in bijection with the number of circular binary sequences of length n using digits 0, 1 with no two circularly consecutive ones. It is not hard to show that this is equal to $f_{k+2} - f_{k-2} = f_{k+1} + f_{k-1}$. Therefore

$$P(W_n) = 2 + \sum_{k=2}^n \binom{n}{k} (f_{k+1} + f_{k-1}) = 2 + \sum_{k=0}^n \binom{n}{k} (f_{k+1} + f_{k-1}) - n - 2$$

$$= f_{2n-1} + f_{2n+1} - n,$$

the last equality proved by induction. This leads to the exponential growth rate of the family \mathcal{G} of wheels:

$$P(\mathcal{G}) = \tau^2 \approx 2.618,$$

where τ is the golden ratio $(1 + \sqrt{5})/2$.

Example 10 (Ladders). As in Example 3, let L_n be the ladder of order n with $r = n/2$ rungs. Letting a_r and b_r denote the number of connected partitions of L_{2r} for which the two vertices on the last rung are in the same part and in different parts, respectively, we have the recursions: $a_0 = -1, b_0 = 1$, and

$$a_{r+1} = 2a_r + 3b_r, \quad b_{r+1} = 3a_r + 4b_r.$$

Generating function techniques lead to

$$P(L_n) = \frac{1}{\sqrt{10}} (\alpha^{\frac{n}{2}} - \bar{\alpha}^{\frac{n}{2}}),$$

where $\alpha = 3 + \sqrt{10}$ and $\bar{\alpha} = 3 - \sqrt{10}$. The exponential growth rate for the family \mathcal{G} of ladders is

$$P(\mathcal{G}) = \sqrt{3 + \sqrt{10}} \approx 2.48.$$

Example 11 (Exponential families with arbitrarily high growth rate). Let $G_d(k)$ denote the graph that is a chain of k copies of the complete graph K_d , each sharing a single vertex with the next. The graph $G_d(k)$ has $n = k(d - 1) + 1$ vertices. Using Proposition 2 and the bounds on the Bell numbers given at the end of Example 8, there is a constant c such that

$$\frac{d}{\ln d} < \left(\frac{d}{\ln d}\right)^{\frac{kd}{n}} < (P(K_d))^{\frac{k}{n}} = \sqrt[n]{P(G_d(k))} = (P(K_d))^{\frac{k}{n}} < \left(\frac{cd}{\ln d}\right)^{\frac{kd}{n}} < \left(\frac{cd}{\ln d}\right)^2.$$

Therefore, with d fixed but sufficiently large, the family \mathcal{G}_d of graphs $\{G_d(k) : k \geq 1\}$ has exponential growth with rate

$$\underline{P}(\mathcal{G}_d) > \frac{d}{\ln d}.$$

Example 12 (A super-exponential family). On the other hand (referring to the previous example), with k fixed, the family \mathcal{G}_k of graphs $\{G_d(k) : d \geq 3\}$ has super-exponential growth. Moreover, let $d = (d_1, d_2, \dots)$ be a (possibly very slowly) increasing sequence of integers and $k = (k_1, k_2, \dots)$ a (possibly very rapidly) increasing sequence of integers. The family

$$\mathcal{G}_{d,k} = \{G_{d_i}(k_i)\}_{i=1}^\infty$$

has super-exponential growth because, as in Example 11, $\sqrt[n]{P(G_{d_i}(k_i))} > \frac{d_i}{\ln d_i}$.

Proposition 6 below concerns lower and upper bounds on the growth rate for a family of graphs with exponential growth rate. Example 11 verifies the last statement of Proposition 6. The first statement is a consequence of Proposition 5.

Proposition 5. *If G is a connected graph on n vertices, then $P(G) \geq 2^{n-1}$. Equality holds when G is a tree.*

Proof. Let T be any spanning tree of G . By Lemma 3 and Example 11, we have

$$P(G) \geq P(T) = \sum_{k=1}^n P(T, k) = \sum_{k=1}^n \binom{n-1}{k-1} = 2^{n-1}.$$

□

Corollary 1. *There is no infinite family of graphs with subexponential growth rate with respect to the number of connected partitions.*

Proposition 6. *If \mathcal{G} is an infinite family of connected graphs, then $\underline{P}(\mathcal{G}) \geq 2$. There is no finite upper bound on $\underline{P}(\mathcal{G})$ for a family of connected graphs with exponential growth rate.*

Concerning a result similar to Theorem 4 for connected partitions, there is an easy upper bound given in the following proposition. A lower bound is addressed in Conjecture 2.

Proposition 7. *For a graph G with m edges (independent of the number of vertices), we have*

$$P(G) \leq 2^m,$$

with equality for any forest and no other graph.

Proof. The bound and the last statement of the theorem follow from Proposition 4 since, for a forest and only for a forest, any subset of $E(G)$ is a set S of the type in the proposition. □

Concerning a lower bound, let $K(m)$ be the graph with m edges defined as follows. Let p be the largest integer such that $\binom{p}{2} \leq m$, and let $K(m)$ be the graph obtained from the complete graph K_p by adding one additional vertex joined to $m - \binom{p}{2}$ vertices of K_p . Note that $K(m)$ is, up to isomorphism, independent of the particular edges added. The graph $K(12)$ is shown in Figure 2. If $m = \binom{p}{2}$, then $K(m)$ is the complete graph K_p .

Conjecture 2. *For a connected graph G with m edges (independent of the number of vertices), we have*

$$P(G) \geq P(K(m)).$$

In particular, the complete graph K_n has the minimum number of connected partitions among all graphs with an equal number of edges.

Figure 3 shows graphs G_1, G_2 , both with 6 vertices and 11 edges. The graph G_1 is the one that is conjectured extremal (minimum) with respect to the number of connected partitions.

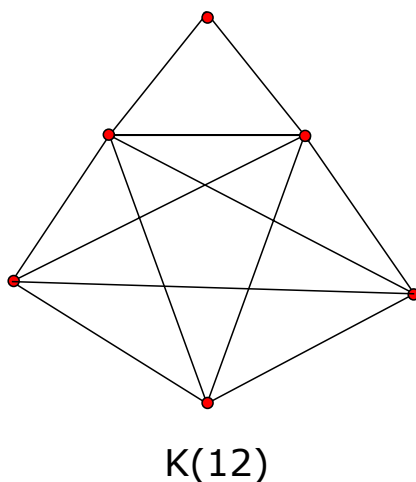


Figure 2: The graph with the minimum number of connected partitions among graphs with 12 edges.

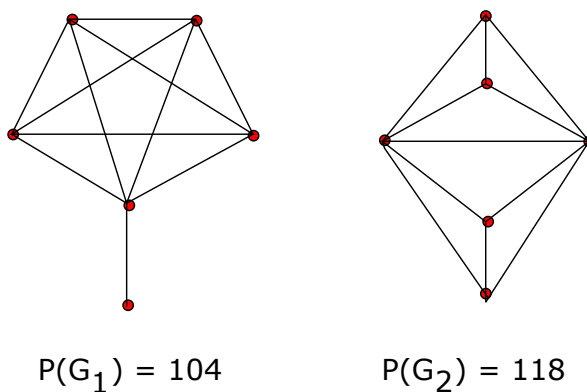


Figure 3: Comparison $P(G_1) < P(G_2)$.

4 A Threshold for Exponential Growth

For a graph G , let $\delta(G)$ and $\bar{d}(G)$ denote the minimum and the average degrees, respectively, of G . Call an infinite family \mathcal{G} of graphs *sparse* if there is a constant c such that $\bar{d}(G) \leq c$ for all $G \in \mathcal{G}$. Call \mathcal{G} *dense* if, for all positive c , we have $\delta(G) > c$ for all but finitely many $G \in \mathcal{G}$. The families of cycles, of wheels, of ladders, and the

family \mathcal{G}_d of Example 11 are sparse. The family of complete graphs and the family $\mathcal{G}_{d,k}$ of Example 12 are dense.

This section concerns a threshold between exponential and super-exponential growth, with respect to connected partitions, for a general family of graphs. The examples in the preceding section show that the families of trees, cycles, wheels, and ladders have exponential growth. The family of complete graphs and the family $\mathcal{G}_{d,k}$ of Example 12 are super-exponential. More generally:

Theorem 13. *If the infinite family \mathcal{G} is sparse, then \mathcal{G} has exponential growth with respect to connected partitions. The converse of this statement is false.*

Proof. Assume that there is a bound c on the average degree $2E(G)/V(G)$ of the graphs in \mathcal{G} . By Theorem 7 the connected partition number is not more than $2^{|E(G)|} \leq 2^{cn/2} = (\sqrt{2}^c)^n$ for every $G \in \mathcal{G}$, where $n = |V(G)|$. Therefore $\overline{P}(\mathcal{G}) < \sqrt{2}^c < \infty$. The second statement in the theorem is shown via the following example. \square

Example 14 (*The converse of Theorem 13 is false*). This is an example of a family of graphs with exponential growth, but whose set of average degrees is unbounded. Let G_n be the graph on n vertices that is the union of a complete graph K_r of order $r = \lceil \sqrt{n} \ln n \rceil$ and the path P of order $n - r + 1$. The graphs K_r and P have exactly one vertex v in common. Let $\mathcal{G} = \{G_n, n \geq 4\}$.

The number of edges in G_n is $\binom{r}{2} + n - r > cn \ln n$ for some constant c independent of n . Therefore the average degree is unbounded. For a connected partition of G_n , there are $(n - r) 2^{r-1}$ possibilities for the part X containing the vertex v . There are at most 2^{n-r} possibilities for the graph partition of P not in X , and, for r sufficiently large, at most $\left(\frac{r}{\ln r}\right)^r$ possibilities for the graph partition of K_r not in X . Therefore,

$$P(G_n) < (n - r) 2^{r-1} 2^{n-r} \left(\frac{r}{\ln r}\right)^r < n 2^n (2\sqrt{n})^{\sqrt{n} \ln n},$$

and

$$\overline{P}(\mathcal{G}) < 2 \sqrt{n} (\sqrt{n})^{\frac{\ln n}{\sqrt{n}}} \rightarrow 2$$

as $n \rightarrow \infty$. Therefore \mathcal{G} has exponential growth.

Conjecture 3. *If \mathcal{G} is a dense family of graphs, then \mathcal{G} has super-exponential growth.*

Remark 2. To conclude that a family \mathcal{G} has super-exponential growth, it is not sufficient to assume that the set of average degrees of the graphs in the family is unbounded. This is shown by Example 14, a family of graphs whose set of average degrees is unbounded but nevertheless has exponential growth.

Consider the super-exponential family $\mathcal{G}_{d,k} = \{G_{d_i}(k_i)\}_{i=1}^\infty$ of Example 12, except now we assume that the copies of K_d are mutually disjoint, not joined at a single vertex. The fact that $G_{d_i}(k_i)$ is now disconnected does not alter the calculation showing that $\mathcal{G}_{d,k}$ is super-exponential. The order of the graph $G_{d_i}(k_i)$ is now $n_i = k_i d_i$, and all vertices have degree $d_i - 1$.

Conjecture 4. *Given $n = k(d + 1)$, the graph $G_{d+1}(k)$ as defined above minimizes $P(G)$ over all graphs G (not necessarily connected) of order n all of whose vertices have degree at least d .*

Proposition 8. *If Conjecture 4 is true, then Conjecture 3 is true.*

Proof. Let \mathcal{G} be a dense family of graphs. For any graph $G \in \mathcal{G}$, of order $n(G)$ and minimum degree $\delta(G)$, let $H(G)$ be the graph in $G_{d_i}(k_i) \in \mathcal{G}_{d+1,k}$ such that $n_i \leq n(G) < n_{i+1}$ and $d_i \leq \delta(G) < d_{i+1}$. Assuming Conjecture 4, we have $P(G) \geq P(H(G))$ for all $G \in \mathcal{G}$. Since the family $\{H(G) : G \in \mathcal{G}\}$ is super-exponential, so is \mathcal{G} . \square

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References

- [1] D. Berend and T. Tassa, Improved bounds on Bell numbers and on moments of sums of random variables, *Probab. Math. Statist.* **30** (2010), 185–205.
- [2] A. Björklund, T. Husfeldt, P. Kaski and M. Koivisto, The traveling salesman problem in bounded degree graphs, *ACM Trans. Algorithms* **8** (2012), Art. 18.
- [3] N. G. de Bruijn, *Asymptotic methods in analysis* (3rd ed.) (1981), Dover, p. 108.
- [4] D. Galvin and D. T. Tranh, Sterling numbers of forests and cycles, *Electron. J. Combin.* **20** (2013), # 73.
- [5] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.
- [6] K. Kangas, P. Kaski, M. Koivisto and J. H. Korhonen, On the number of connected sets in bounded degree graphs, *Graph-theoretic concepts in computer science, Lec. Notes Comp. Sci.* **8747** (2014), 336–347.
- [7] D. Kleitman and D. West, Spanning trees with many leaves, *SIAM J. Disc. Math.* **4** (1991), 99–106.

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