AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **67(2)** (2017), Pages 243–258

Representing graphs in Steiner triple systems - II

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Dedicated to the memory of Dan Archdeacon who introduced the idea for this paper during a visit to the Open University.

Abstract

Let $G = (V, E)$ be a simple graph and let $T = (P, B)$ be a Steiner triple system. Let φ be a one-to-one function from V to P. Any edge $e = \{u, v\}$ has its image $\{\varphi(u), \varphi(v)\}$ in a unique block in B. We also denote this induced function from edges to blocks by φ. We say that T *represents* G if there exists a one-to-one function $\varphi : V \to P$ such that the induced function $\varphi : E \to B$ is also one-to-one; that is, if we can represent vertices of the graph by points of the triple system such that no two edges are represented by the same block.

The concept was introduced in a previous paper [*Graphs Combin.* **30** (2014), 255–266], where various results were proved. When the graph to be represented is a complete graph the concept is equivalent to that of an independent set. In this paper we discuss representing complete bipartite graphs in Steiner triple systems of small order.

By relating the work to configurations in Steiner triple systems we prove that the number of representations of a graph having six or fewer edges in a Steiner triple system of order m is only dependent on the value of m and so is independent of the structure of the system.

1 Introduction

In a previous paper [1], we began the study of graph representations in Steiner triple systems. Precisely what is meant by this is as follows. A *graph* $G = (V, E)$ is a finite set $V(G)$ of *vertices* and a finite set $E(G)$ of *edges*, each edge being an unordered pair of different vertices. Loops and multiple edges are not allowed but G may be disconnected. We will use $n = |V(G)|$ for the *order* of G. A *Steiner triple system* $T = (P, B)$ is a finite set $P(T)$ of *points* and a finite set $B(T)$ of *blocks* each with three elements such that each unordered pair of points occur together in exactly one block. We will use $m = |P(T)|$ for the *order* of T. A Steiner triple system of order m, usually denoted by STS(m), exists if and only if $m \equiv 1, 3 \pmod 6$, [5]. Such values of m are called *admissible*. Throughout this paper we will require that $m \geq n$.

Let $\varphi : V(G) \to P(T)$ be an injection. An edge is a pair of vertices $\{u, v\}$, so the pair $\{\varphi(u), \varphi(v)\}$ determines a unique block in T. We will also call this induced map φ, hopefully without confusion. If the induced $\varphi : E(G) \to B(T)$ is also an injection, then we say that T *represents* G. Representing a complete graph in a Steiner triple system is equivalent to finding an *independent set* or *arc* in the system. This is defined as a subset of points which contain no block. There is a large literature on independent sets in Steiner triple systems, see for example Chapter 17 of [2]. Thus our work is a generalization of this concept. One of the aims in this paper is to present results on representing complete bipartite graphs in Steiner triple systems of small order.

One of the main themes in [1] was to determine in which Steiner triple systems, regular graphs of degree 2 (i.e. unions of cycles), and degree 3 may be represented. A complete answer was obtained for the former class.

Theorem 1.1 *Every disjoint union of cycles G where the total number of vertices is n can be represented in every STS(m) for $m \geq n$ and admissible except for $(G, m) = (C_3, 3)$ and $(C_3 \cup C_4, 7)$.*

For cubic graphs we were able to prove

Theorem 1.2 *If G is a cubic graph of order n, then G can be represented in every STS(m) for $m \geq n + 9$ and admissible.*

In this paper we focus our attention on "small" graphs, i.e. graphs having a small number of edges, and to the number of representations which such a graph has in a Steiner triple system. We prove the following theorem, announced in [1].

Theorem 1.3 *Let G be any graph with $|E(G)| \leq 6$ and let T be any Steiner triple system of order m. Then the number of representations of G in T is independent of the choice of T.*

The proof in Section 3 uses the theory of configurations in Steiner triple systems, see Chapter 13 of [2]. Because all one, two and three-block configurations in Steiner triple systems are *constant* (for the precise definition, see later), the theorem is no surprise for $|E(G)| \leq 3$ but it is certainly unexpected for $4 \leq |E(G)| \leq 6$.

2 Complete bipartite graphs

As we observed in the Introduction, representing a complete graph in a Steiner triple system is equivalent to finding an independent set in the system. Since any subset of an independent set is also independent, we can say that a complete graph K_s represented in an STS(m) is *maximum* if the complete graph K_{s+1} cannot be represented in the same STS(m). In the same vein, we can define the maximum complete bipartite graphs that can be represented in a Steiner triple system of some order. A complete bipartite graph $K_{i,j}$ which can be represented in an STS(m) is said to be *maximum* if the complete bipartite graphs $K_{i+1,j}$ and $K_{i,j+1}$ cannot be represented in the same STS(m). We begin by proving three easy lemmas.

Lemma 2.1 *The complete bipartite graphs $K_{1,(m+1)/2}$ and $K_{2,(m-1)/2}$ cannot be represented in any Steiner triple system of order m.*

Proof: The complete bipartite graph $K_{1,(m+1)/2}$ cannot be represented in an STS(m) because the valency of the vertex in the left partition is $(m+1)/2$ which is greater than $(m-1)/2$, the replication number of the STS(m).
Assume that $K_{2,(m-1)/2}$ is represented in an STS(m). Let x and y be the representation of the two vertices in the left partition of $K_{2,(m-1)/2}$. The total number of blocks through x and through y is $m-2$ since they have one block in common. But the number of edges in $K_{2,(m-1)/2}$ is $m-1$. Contradiction. ■

Lemma 2.2 *The complete bipartite graph $K_{1,(m-1)/2}$ can be represented in every STS(m) and is maximum.*

Proof: Represent the vertex of maximum valency by any point of the system and the edges by the blocks through that point. The graph is maximum by the preceding lemma. ■

Lemma 2.3 *The complete graph $K_{2,(m-3)/2}$ can be represented in every STS(m).*

Proof: Represent the two vertices in the left partition of the graph by x and y. Now consider the graph $G_{x,y}$ whose vertex set is $P \setminus \{x, y, z\}$ where $\{x, y, z\} \in B$ and $\{u, v\}$ is an edge if either $\{x, u, v\}$ or $\{y, u, v\} \in B$. This is the *cycle graph* of the pair x, y and is a union of cycles of even length. Represent the vertices in the right partition by alternate points in each of the cycles. ■

Next, we give the maximum representable bipartite graphs for the Steiner triple systems of order 7, 9, 13, and 15.

Proposition 2.4 *The maximum complete bipartite graphs that can be represented in the STS(7) are $K_{1,3}$ and $K_{2,2}$.*

Proof: This follows immediately from the preceding lemmas by putting $m = 7$. The graphs are illustrated below, using the Steiner triple system on elements of $\mathbb{Z}_7$ obtained by cyclic shifts of $\{0, 1, 3\}$. ■

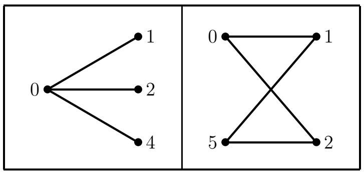

Figure 1: The two maximum complete bipartite graphs in the STS(7).

Proposition 2.5 *The maximum complete bipartite graphs that can be represented in the STS(9) are $K_{1,4}$ and $K_{3,3}$.*

Proof: The graph $K_{1,4}$ can be represented and is maximum by Lemma 2.2. The graph $K_{3,3}$ can be represented as shown below. In order to be maximum, we must show that $K_{3,4}$ cannot be represented. Assume the converse. Since $K_{3,4}$ has 12 edges, each block represents an edge. Consider any two vertices x, y from the same partition. The block $\{x, y, z\}$ containing the two vertices either represents no edge if z is in the same partition or both edges $\{x, z\}$ and $\{y, z\}$ if z is in the opposite partition. Contradiction.

The graphs are illustrated below, using the Steiner triple system with block set $\{0, 1, 2\}$, $\{3, 4, 5\}$, $\{6, 7, 8\}$, $\{0, 3, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{0, 4, 8\}$, $\{1, 5, 6\}$, $\{2, 3, 7\}$, $\{0, 5, 7\}$, $\{1, 3, 8\}$, $\{2, 4, 6\}$. ■

Proposition 2.6 *The maximum complete bipartite graphs that can be represented in an STS(13) are $K_{1,6}$, $K_{2,5}$ and $K_{3,4}$.*

Proof: The graph $K_{1,6}$ can be represented and is maximum by Lemma 2.2. The graph $K_{2,5}$ can be represented by Lemma 2.3. To prove that it is maximum we must show that $K_{2,6}$ and $K_{3,5}$ cannot be represented in an STS(13). The former follows from the second part of Lemma 2.1. Let $X = \{0, 1, 2\}, Y = \{3, 4, 5, 6, 7\}$ and $Z = \{8, 9, 10, 11, 12\}$. Assume that $K_{3,5}$ is represented in an STS(13) as follows:

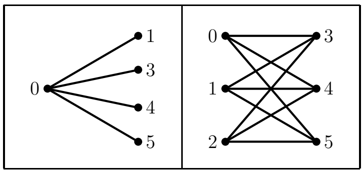

Figure 2: The two maximum complete bipartite graphs in the STS(9).

the three vertices in the small partition by the elements of X, and the five vertices in the large partition by the elements of Y. Therefore, there are 15 blocks of the form $\{x, y, z\}$, $x \in X, y \in Y, z \in Z$, representing the edges of the graph. Since five blocks through each point $0, 1$ and 2 have been used, the block $\{0, 1, 2\}$ is forced as the sixth. The remaining blocks of the system contain elements from the sets Y and Z only. The minimum number of blocks formed by pairs of elements of Y is six; for example $\{3, 4, 5\}$, $\{3, 6, 7\}$, $\{4, 6, z_1\}$, $\{4, 7, z_2\}$, $\{5, 6, z_3\}$, $\{5, 7, z_4\}$, $z_i \in Z$. The minimum number of blocks formed by pairs of elements of Z is also six. This brings the total number of blocks to 28. But an STS(13) has only 26 blocks which leads to a contradiction.

Next, we prove that $K_{3,4}$ is maximum. $K_{3,5}$ has already been proven impossible to represent as above. We must show that $K_{4,4}$ cannot be represented. Let $X = \{0, 1, 2, 3\}$ and $Y = \{4, 5, 6, 7\}$. Represent one partition of $K_{4,4}$ by elements of X and the other by elements of Y. Then there are 16 blocks containing the pairs $\{x, y\}$, $x \in X$, $y \in Y$. Now consider the blocks containing any two elements from the same partition. The blocks containing the elements of X must be one of the following,
(1) $\{0, 1, 2\}$, $\{0, 3, z_1\}$, $\{1, 3, z_2\}$, $\{2, 3, z_3\}$, $z_i \notin X, Y$, without loss of generality, or
(2) $\{0, 1, z_1\}$, $\{0, 2, z_2\}$, $\{0, 3, z_3\}$, $\{1, 2, z_4\}$, $\{1, 3, z_5\}$, $\{2, 3, z_6\}$, $z_i \notin X, Y$.
The replication number of an STS(13) is 6. In both of the above possibilities, at least one element appears more than 6 times in the system. Contradiction.

There are two non-isomorphic STS(13)s and the graphs are illustrated below, using the 22 blocks which they have in common. One of the STS(13)s is cyclic and can be obtained by cyclic shifts of $\{0, 1, 4\}$ and $\{0, 2, 8\}$ on elements of $\mathbb{Z}_{13}$. The other non-cyclic STS(13) can be obtained by choosing any Pasch configuration in the cyclic STS(13) and replacing it with the opposite Pasch configuration. We will choose the blocks $\{2, 3, 6\}$, $\{2, 4, 10\}$, $\{3, 4, 7\}$, $\{6, 7, 10\}$; replacing them with the blocks $\{2, 3, 4\}$, $\{2, 6, 10\}$, $\{3, 6, 7\}$, $\{4, 7, 10\}$. ■

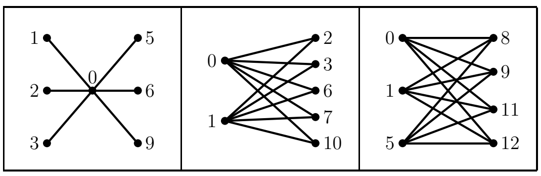

Figure 3: The three maximum complete bipartite graphs in the STS(13)s.

The two Steiner triple systems of order 13 can represent the same maximum complete bipartite graphs. However, this is not true for the Steiner triple systems of order 15. There are 80 nonisomorphic Steiner triple systems of order 15, [6]. The graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, and $K_{4,4}$ can be represented in all 80 systems. By Lemma 2.1 the graphs $K_{1,8}$ and $K_{2,7}$ cannot be represented. However, more than half, 54 to be precise, can also represent $K_{3,6}$. These graphs, i.e. $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{4,4}$ or $K_{1,7}$, $K_{3,6}$, $K_{4,4}$ are the maximum bipartite graphs. To prove maximality we need to show that $K_{4,5}$ cannot be represented.

Proposition 2.7 *The complete bipartite graph $K_{4,5}$ cannot be represented in any STS(15).*

Proof: Assume that $K_{4,5}$ is represented in an STS(15) and without loss of generality assume that the four vertices in the small partition are represented by 0, 1, 2 and 3. The blocks containing the pairs $\{x_1, x_2\}$, $x_1, x_2 \in \{0, 1, 2, 3\}$, $x_1 \neq x_2$, are not used in the representation. These pairs can occur in four or six blocks. If they occur in four blocks then one of the points appears in three of these blocks. Since the replication number of an STS(15) is 7, then there are four more blocks through that point in the system. But this is a contradiction since the valency of the vertex represented by that point is five. If they occur in six blocks then every point appears in three of these blocks. The same argument applies as above. ■

Using the standard listing of the STS(15)s given in [6], Appendix 1 gives the number of representations of the complete bipartite graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{3,6}$ and $K_{4,4}$.

3 Graphs with few edges

This section is devoted to the proof of Theorem 1.3. But first it will be appropriate to recall some basic definitions. An *ℓ-line configuration* in an STS(m) is any collection of ℓ blocks of the Steiner triple system. For some configurations, the number of occurrences in an STS(m) can be expressed as a rational polynomial in m. Thus, for any admissible m this number is the same regardless of the structure of the STS(m). Such configurations are called *constant* whereas other configurations are called *variable*. Information about one, two, three and four line configurations quoted below can be found in [4].

There is only one one-edge graph, i.e. a single edge. In any STS(m) there are $m(m-1)/6$ blocks which cover all $\binom{m}{2} = m(m-1)/2$ single edges which is the number of representations.

There are two two-edge graphs; a pair of disjoint edges denoted by B_1 and a path of length 2 denoted by B_2. The graph B_1 can be obtained by adding an extra edge to the one-edge graph. The number of choices for the extra edge is $\binom{m-2}{2} = (m-2)(m-3)/2$. But the graph B_1 can arise in two ways, hence the number of representations

$$b_1 = [m(m-1)(m-2)(m-3)/4]/2 = m(m-1)(m-2)(m-3)/8.$$

The graph B_2 can be obtained by adjoining an extra edge through one of the two vertices of the one-edge graph. The number of choices for the extra edge is $4[(m-1)/2-1]$ and again the graph B_2 can arise in two ways, hence the number of representations

$$b_2 = 4[(m-1)/2-1]m(m-1)/4 = m(m-1)(m-3)/2.$$

We now give an alternative way of counting the number of representations which will be much easier for graphs with more edges. There are two two-line configurations; a pair of disjoint blocks denoted by B'_1 and a pair of intersecting blocks denoted by B'_2. These configurations are constant and the number of occurrences in any STS(m) is given by

$$b'_1 = m(m-1)(m-3)(m-7)/72, \;\; b'_2 = m(m-1)(m-3)/8.$$

The graph B_1 occurs in B'_1 in nine ways and in B'_2 in five ways. Therefore,

$$b_1 = 9b'_1 + 5b'_2 = m(m-1)(m-2)(m-3)/8.$$

Similarly, the graph B_2 cannot occur in B'_1 but occurs in B'_2 in four ways. Therefore,

$$b_2 = 4b'_2 = m(m-1)(m-3)/2.$$

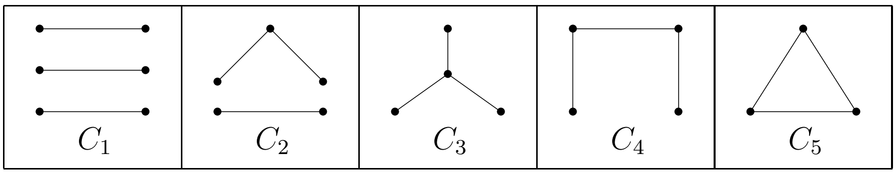

Figure 4: The three-edge graphs.

There are five three-edge graphs; these are shown in Figure 4 and are denoted by $C_1, C_2, \ldots, C_5$. There are also five three-line configurations; these are shown in Figure 5 and are denoted by $C'_1, C'_2, \ldots, C'_5$.

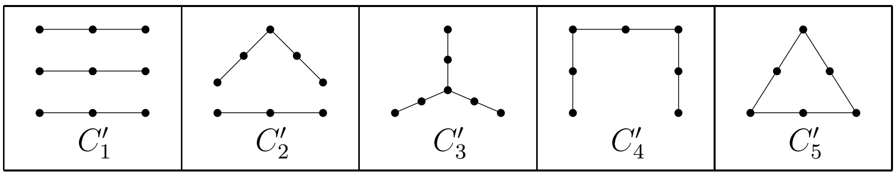

Figure 5: The three-line configurations.

The number of occurrences of each three-line configuration in an STS(m) is

$$c'_1 = m(m-1)(m-3)(m-7)(m^2-19m+96)/1296$$
$$c'_2 = m(m-1)(m-3)(m-7)(m-9)/48$$
$$c'_3 = m(m-1)(m-3)(m-5)/48$$

$$c'_4 = m(m-1)(m-3)(m-7)/8$$
$$c'_5 = m(m-1)(m-3)/6$$

In the table below we list the number of occurrences of every graph in each of the five configurations.

	C_1	C_2	C_3	C_4	C_5
C'_1	27	·	·	·	·
C'_2	15	12	·	·	·
C'_3	7	12	8	·	·
C'_4	7	16	·	4	·
C'_5	2	15	·	9	1

Hence, the number of representations of each three-edge configuration in an STS(m) is

$$c_1 = 27c'_1 + 15c'_2 + 7c'_3 + 7c'_4 + 2c'_5 = m(m-1)(m-2)(m-3)(m-4)(m-5)/48$$
$$c_2 = 12c'_2 + 12c'_3 + 16c'_4 + 15c'_5 = m(m-1)(m-3)^2(m-4)/4$$
$$c_3 = 8c'_3 = m(m-1)(m-3)(m-5)/6$$
$$c_4 = 4c'_4 + 9c'_5 = m(m-1)(m-3)(m-4)/2$$
$$c_5 = c'_5 = m(m-1)(m-3)/6$$

We have now shown that the number of representations of every e-edge graph, when $e \leq 3$, in a Steiner triple system of any order m is constant. This is not a surprise since the number of occurrences of every ℓ-line configuration, when $\ell \leq 3$, in a Steiner triple system of any order m is also constant. However, not all four-line configurations are constant. We next consider graphs with four edges.

There are 16 four-line configurations. These are shown in Figure 6 and are denoted by $D'_1, D'_2, \ldots D'_{16}$. We know from [4] that five of them are constant and all the others are variable. The constant four-line configurations are D'_4, D'_7, D'_8, D'_{11}, and D'_{15}.

Note that D'_{16} is the Pasch configuration, the number of which is denoted by p. The formulae for the numbers of four-line configurations in an STS(m) are given below.

$$d'_1 = m(m-1)(m-3)(m-9)(m-10)(m-13)(m^2-22m+141)/31104 + p$$
$$d'_2 = m(m-1)(m-3)(m-9)(m-10)(m^2-22m+129)/576 - 6p$$
$$d'_3 = m(m-1)(m-3)(m-9)^2(m-11)/128 + 3p$$
$$d'_4 = m(m-1)(m-3)(m-7)(m-9)(m-11)/288$$
$$d'_5 = m(m-1)(m-3)(m-9)(m^2-20m+103)/48 + 12p$$
$$d'_6 = m(m-1)(m-3)(m-9)(m-10)/36 - 4p$$
$$d'_7 = m(m-1)(m-3)(m-5)(m-7)/384$$
$$d'_8 = m(m-1)(m-3)(m-7)(m-9)/16$$
$$d'_9 = m(m-1)(m-3)(m-9)^2/8 - 12p$$

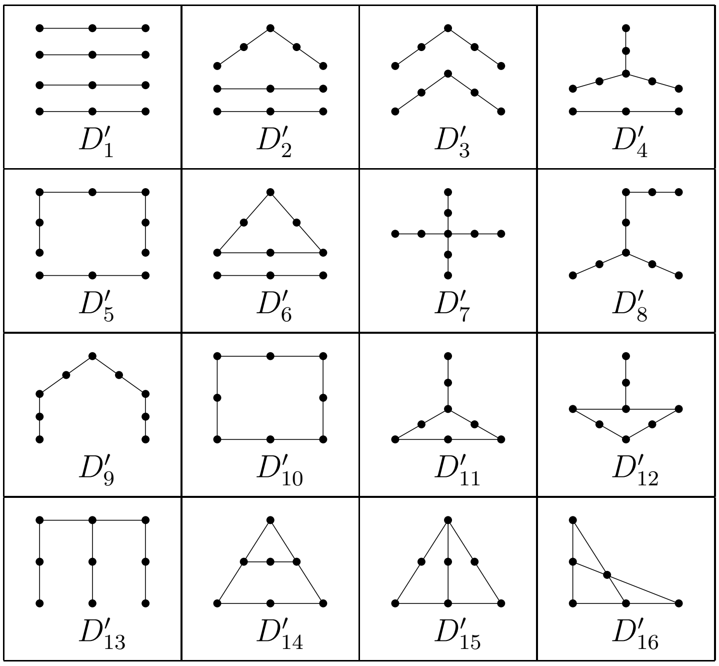

Figure 6: The four-line configurations.

$d_{10}' = m(m-1)(m-3)(m-8)/8 + 3p$
$d_{11}' = m(m-1)(m-3)(m-7)/4$
$d_{12}' = m(m-1)(m-3)(m-9)/4 + 12p$
$d_{13}' = m(m-1)(m-3)(m^2-18m+85)/48 - 4p$
$d_{14}' = m(m-1)(m-3)/4 - 6p$
$d_{15}' = m(m-1)(m-3)/6$
$d_{16}' = p$

The number of occurrences of the Pasch configuration in an STS(m), together with the order m, determines the number of occurrences of all the other variable configurations. There are 11 four-edge graphs and these are shown in Figure 7 and are denoted by D_1, D_2, $\ldots$, D_{11}. But as we will show, the number of representations of each of the four-edge graphs in an STS(m) depends only on m and does not involve p at all.

The table below gives the number of occurrences of every four-edge graph in each of the four-line configurations. These results were obtained by hand and were later checked computationally.

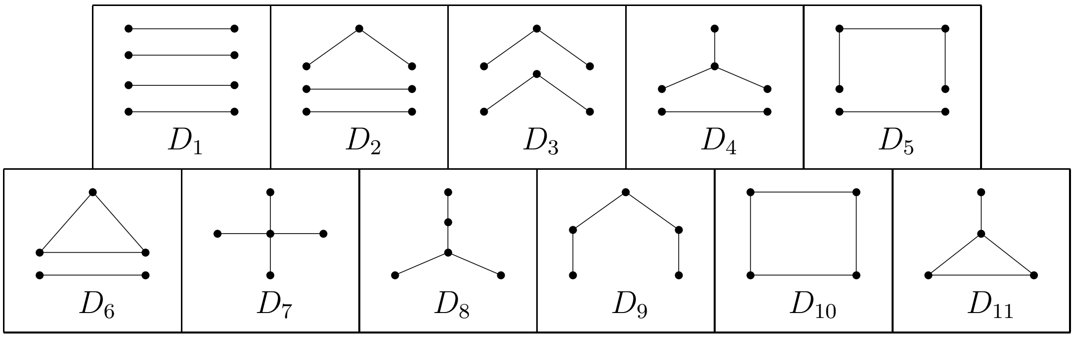

Figure 7: The four-edge graphs.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}
D'_1	81	·	·	·	·	·	·	·	·	·	·
D'_2	45	36	·	·	·	·	·	·	·	·	·
D'_3	25	40	16	·	·	·	·	·	·	·	·
D'_4	21	36	·	24	·	·	·	·	·	·	·
D'_5	21	48	·	·	12	·	·	·	·	·	·
D'_6	6	45	·	·	27	3	·	·	·	·	·
D'_7	9	24	·	32	·	·	16	·	·	·	·
D'_8	9	32	8	16	8	·	·	8	·	·	·
D'_9	9	40	12	·	16	·	·	·	4	·	·
D'_{10}	2	28	18	·	20	·	·	·	12	1	·
D'_{11}	2	23	12	10	15	1	·	12	4	·	2
D'_{12}	2	25	8	·	35	3	·	·	8	·	·
D'_{13}	9	36	·	·	36	·	·	·	·	·	·
D'_{14}	·	10	10	·	34	6	·	·	20	1	·
D'_{15}	·	9	12	6	21	3	·	12	12	·	6
D'_{16}	·	·	6	·	24	12	·	·	36	3	·

Using the formulae for the four-line configurations we can easily obtain the formulae for the numbers of four-edge graphs in an STS(m).

$$d_1 = m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)(m-7)/384$$
$$d_2 = m(m-1)(m-3)^2(m-4)(m-5)(m-6)/16$$
$$d_3 = m(m-1)(m-3)(m^3-13m^2+57m-87)/8$$
$$d_4 = m(m-1)(m-3)(m-4)(m-5)^2/12$$
$$d_5 = m(m-1)(m-3)(m-4)^2(m-5)/4$$
$$d_6 = m(m-1)(m-3)^2(m-4)/12$$
$$d_7 = m(m-1)(m-3)(m-5)(m-7)/24$$
$$d_8 = m(m-1)(m-3)(m-5)^2/2$$
$$d_9 = m(m-1)(m-3)(m^2-9m+21)/2$$
$$d_{10} = m(m-1)(m-3)(m-6)/8$$

$$d_{11} = m(m-1)(m-3)(m-5)/2$$

The results show that the number of any four-edge graph in an STS(m) is constant and thus independent of the number of occurrences of the Pasch configuration in the Steiner triple system.

We now consider five-edge graphs. There are 26 five-edge graphs denoted by E_1, E_2, ... E_{26}. For each of these five-edge graphs the edges are listed in Appendix 2 in ascending order of the number of vertices in each graph. Similarly, also in Appendix 2, we list the blocks of each of the 56 five-line configurations denoted by E'_1, E'_2, ... E'_{56}. These are ordered, as in [3], by ascending order of the number of points in each.

Of the 56 configurations only five are constant; the formulae are given in [3]. Note that E'_1 is the mitre configuration and E'_2 is the mia configuration. The number of mitre and Pasch configurations, together with the order m, determine the number of a variable five-line configuration in an STS(m). The Tables 1 and 2 below give the number of representations of every five-edge graph in each of the five-line configurations together with the coefficients of the Pasch configuration (p), mp, m^2p and the mitre (μ) taken from the formulae.

Using the formulae for the five-line configurations, computational results for the number of representations of a five-edge graph show that the coefficients of the mitre and Pasch configuration in the graph formulae sum to zero. For example for the graph E_2, $((4\times 3)+(1\times -12)+(2\times -12)+(1\times -12)+(2\times -6)+(1\times 36)+(1\times 48)+(1\times -36))p+((6\times 1)+(2\times -6)+(2\times -3)+(1\times 6)+(1\times 12)+(1\times -6))\mu = 0$. For the graph E_3, $((4\times 3)+(2\times -12)+(4\times -21)+(6\times -12)+(2\times 6)+(4\times 24)+(2\times 108)+(2\times -78))p+((4\times 3)+(2\times -12)+(2\times 6))mp = 0$. Hence, the number of representations of any five-edge graph in an STS(m) is constant.

Finally, we must consider graphs on six edges. There are 68 six-edge graphs and 282 six-line configurations. Computational results show that the number of representations of any six-edge graph is also independent of the STS(m). For reasons of space, we do not give details of our calculations here but refer the reader to [7]. The methods used to obtain the results are as above. Thus, based on the above results, Theorem 1.3 is proved.

To summarize, the number of times a graph with six or less edges can be represented in a Steiner triple system is constant even though variable configurations with four, five and six lines exist; indeed most four, five and six-line configurations are variable. Naturally, this gives rise to the following question: What is the smallest number of edges of a variable graph? Clearly it is greater than six but less than or equal to twelve since $K_{2,6}$ is variable (see Appendix 1). However, an investigation into the number of occurrences of $K_{2,4}$ in the two STS(13)s and in the 80 STS(15)s shows that in fact it is less than or equal to eight. The graph $K_{2,4}$ occurs 1989 times in the cyclic STS(13) and 1974 times in the non-cyclic STS(13). For the STS(15)s, we find that the number of representations of $K_{2,4}$ is $11025 + 3p$ where p is the number of Pasch configurations in the STS(15). Hence, the smallest number of edges of a variable graph is seven or eight. Determination of which value is correct and a theory to explain the above observation concerning the representation of $K_{2,4}$ would appear to be the priority for future research in this area.

Appendix 1 Number of representations of the complete bipartite graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{3,6}$ and $K_{4,4}$ in the STS(15)s.

#	$K_{1,7}$	$K_{2,6}$	$K_{3,5}$	$K_{3,6}$	$K_{4,4}$
1	1920	840	840	0	1050
2	1920	648	528	0	570
3	1920	552	372	0	330
4	1920	504	356	0	306
5	1920	504	368	0	370
6	1920	432	324	0	306
7	1920	408	360	0	450
8	1920	432	276	0	206
9	1920	396	268	0	170
10	1920	396	274	0	202
11	1920	356	264	4	178
12	1920	410	272	2	166
13	1920	408	268	0	194
14	1920	432	270	0	174
15	1920	360	258	0	202
16	1920	504	294	0	210
17	1920	360	264	0	234
18	1920	360	252	0	170
19	1920	320	272	8	238
20	1920	338	248	2	130
21	1920	344	254	2	130
22	1920	326	260	8	142
23	1920	324	231	0	104
24	1920	334	232	2	100
25	1920	338	230	2	112
26	1920	356	231	2	114
27	1920	304	229	2	108
28	1920	314	230	4	104
29	1920	332	230	2	100
30	1920	306	231	4	108
31	1920	320	234	0	136
32	1920	306	236	4	96
33	1920	298	228	4	86
34	1920	302	231	4	86
35	1920	308	227	2	82
36	1920	278	218	0	80
37	1920	270	246	0	96
38	1920	290	237	4	100
39	1920	302	232	2	86
40	1920	302	224	2	82
41	1920	298	228	2	86
42	1920	298	255	8	104
43	1920	282	225	0	96
44	1920	274	227	0	72
45	1920	288	234	2	84
46	1920	280	240	4	76
47	1920	288	233	2	78
48	1920	278	230	4	72
49	1920	276	237	4	76
50	1920	270	245	4	112
51	1920	294	239	6	84
52	1920	288	230	4	68
53	1920	288	231	4	78
54	1920	302	239	8	90
55	1920	294	239	6	84
56	1920	282	233	4	72
57	1920	262	241	4	84
58	1920	272	234	4	86
59	1920	320	239	8	82
60	1920	288	253	10	108
61	1920	308	266	14	154
62	1920	266	230	2	58
63	1920	266	239	2	106
64	1920	290	236	8	94
65	1920	280	240	4	76
66	1920	274	242	4	80
67	1920	272	249	4	84
68	1920	268	234	2	64
69	1920	268	244	6	84
70	1920	288	234	4	84
71	1920	262	237	2	68
72	1920	272	249	6	84
73	1920	288	266	8	128
74	1920	272	230	0	88
75	1920	282	249	8	108
76	1920	280	220	0	80
77	1920	252	246	2	48
78	1920	272	250	8	128
79	1920	264	258	0	192
80	1920	240	270	0	120

Appendix 2

E_1 : 01 02 03 13 23	E_2 : 01 04 12 23 34	E_3 : 01 02 03 24 34
E_4 : 01 02 03 14 23	E_5 : 01 02 03 04 34	E_6 : 01 02 03 23 34
E_7 : 01 02 12 34 35	E_8 : 01 02 13 23 45	E_9 : 01 02 03 23 45
E_{10} : 01 02 03 04 05	E_{11} : 01 02 03 04 15	E_{12} : 01 02 03 14 15
E_{13} : 01 02 13 14 45	E_{14} : 01 02 03 14 45	E_{15} : 01 12 23 34 45
E_{16} : 01 02 03 45 46	E_{17} : 01 02 13 45 46	E_{18} : 01 02 12 34 56
E_{19} : 01 02 03 04 56	E_{20} : 01 02 13 24 56	E_{21} : 01 02 03 14 56
E_{22} : 01 02 34 35 67	E_{23} : 01 02 03 45 67	E_{24} : 01 02 13 45 67
E_{25} : 01 02 34 56 78	E_{26} : 01 23 45 67 89	

The five-edge graphs.

E'_1 : 012 034 135 236 456	E'_2 : 012 034 135 236 146	E'_3 : 012 034 135 236 147
E'_4 : 012 034 135 236 457	E'_5 : 012 034 135 245 067	E'_6 : 012 034 135 246 257
E'_7 : 012 034 135 246 567	E'_8 : 012 034 135 067 168	E'_9 : 012 034 135 067 268
E'_{10} : 012 034 135 067 568	E'_{11} : 012 034 135 236 078	E'_{12} : 012 034 135 236 378
E'_{13} : 012 034 135 236 478	E'_{14} : 012 034 135 245 678	E'_{15} : 012 034 135 246 078
E'_{16} : 012 034 135 246 178	E'_{17} : 012 034 135 246 578	E'_{18} : 012 034 135 267 468
E'_{19} : 012 034 156 357 468	E'_{20} : 012 034 056 178 379	E'_{21} : 012 034 135 067 089
E'_{22} : 012 034 135 067 189	E'_{23} : 012 034 135 067 289	E'_{24} : 012 034 135 067 589
E'_{25} : 012 034 135 067 689	E'_{26} : 012 034 135 236 789	E'_{27} : 012 034 135 246 789
E'_{28} : 012 034 135 267 289	E'_{29} : 012 034 135 267 489	E'_{30} : 012 034 135 267 689
E'_{31} : 012 034 156 357 289	E'_{32} : 012 034 156 378 579	E'_{33} : 012 034 056 078 09a
E'_{34} : 012 034 056 078 19a	E'_{35} : 012 034 056 178 19a	E'_{36} : 012 034 056 178 29a
E'_{37} : 012 034 056 178 39a	E'_{38} : 012 034 056 178 79a	E'_{39} : 012 034 135 067 89a
E'_{40} : 012 034 135 267 89a	E'_{41} : 012 034 135 678 69a	E'_{42} : 012 034 156 278 39a
E'_{43} : 012 034 156 357 89a	E'_{44} : 012 034 156 378 59a	E'_{45} : 012 034 056 078 9ab
E'_{46} : 012 034 056 178 9ab	E'_{47} : 012 034 056 789 7ab	E'_{48} : 012 034 135 678 9ab
E'_{49} : 012 034 156 278 9ab	E'_{50} : 012 034 156 378 9ab	E'_{51} : 012 034 156 789 7ab
E'_{52} : 012 034 056 789 abc	E'_{53} : 012 034 156 789 abc	E'_{54} : 012 034 567 589 abc
E'_{55} : 012 034 567 89a bcd	E'_{56} : 012 345 678 9ab cde	

The five-line configurations.

Table 1: Number of occurrences of graphs E_1 to E_{13} in the five-line configurations.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	m^2p	mp	p	μ
E'_1	·	6	·	12	·	·	12	9	24	·	·	·	24	·	·	·	1
E'_2	2	4	4	16	·	12	12	3	24	·	·	2	24	·	·	3	·
E'_3	1	1	2	8	·	10	6	·	16	·	·	5	20	·	·	−12	·
E'_4	·	2	·	6	·	·	6	3	18	·	·	·	20	·	·	−12	−6
E'_5	·	·	4	·	·	·	8	5	12	·	·	·	12	·	3	−21	·
E'_6	·	1	6	4	·	·	8	3	8	·	·	·	10	·	·	−12	·
E'_7	·	2	·	·	·	·	8	9	·	·	·	·	·	·	·	−6	−3
E'_8	1	·	2	4	·	6	2	·	4	·	·	9	20	·	·	6	·
E'_9	·	1	·	4	·	·	4	·	8	·	·	·	16	·	·	36	6
E'_{10}	·	·	4	2	·	·	2	2	4	·	·	·	8	·	·	24	·
E'_{11}	·	·	·	2	·	8	2	·	12	·	·	8	12	·	·	12	·
E'_{12}	·	·	·	·	12	·	·	·	12	·	24	·	·	·	·	·	·
E'_{13}	·	·	·	2	·	·	2	·	16	·	·	·	12	·	·	24	6
E'_{14}	·	·	·	·	·	·	·	9	·	·	·	·	·	1/6	−19/6	14	·
E'_{15}	·	·	·	·	·	·	·	3	12	·	·	·	16	·	−3	27	3
E'_{16}	·	·	2	·	·	·	4	1	6	·	·	·	6	·	−12	108	·
E'_{17}	·	·	·	·	·	·	4	3	·	·	·	·	·	·	−6	66	6
E'_{18}	·	1	·	·	·	·	4	3	·	·	·	·	·	·	·	48	12
E'_{19}	·	·	·	·	·	·	·	9	·	·	·	·	·	·	·	6	2
E'_{20}	·	·	·	·	4	·	·	·	4	·	24	·	·	·	·	·	·
E'_{21}	·	·	·	·	·	4	·	·	4	·	·	12	8	·	·	−12	·
E'_{22}	·	·	·	·	·	·	·	·	6	·	·	·	8	·	12	−156	−12
E'_{23}	·	·	·	·	·	·	·	·	6	·	·	·	·	·	6	−66	·
E'_{24}	·	·	·	2	·	·	2	·	4	·	·	·	12	·	·	−60	−6
E'_{25}	·	·	·	·	·	·	·	·	18	·	·	·	·	·	·	−12	−2
E'_{26}	·	·	·	·	·	·	·	3	·	·	·	·	·	−1	22	−123	−3
E'_{27}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	6	−54	·
E'_{28}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	12	−144	−12
E'_{29}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	12	−192	−18
E'_{30}	·	·	2	·	·	·	·	1	·	·	·	·	6	·	6	−78	·
E'_{31}	·	·	·	·	·	·	·	3	·	·	·	·	·	·	6	−138	−18
E'_{32}	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	−36	−6
E'_{33}	·	·	·	·	·	·	·	·	·	32	·	·	·	·	·	·	·
E'_{34}	·	·	·	·	·	·	·	·	·	·	16	·	·	·	·	·	·
E'_{35}	·	·	·	·	·	·	·	·	·	·	·	16	·	·	·	3	·
E'_{36}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−6	66	3
E'_{37}	·	·	·	·	·	·	·	·	·	·	·	·	8	·	−6	114	6
E'_{38}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−12	132	·
E'_{39}	·	·	·	·	·	·	·	·	6	·	·	·	·	·	−6	102	6
E'_{40}	·	·	·	·	·	·	·	·	·	·	·	·	·	2	−56	414	18
E'_{41}	·	·	·	·	·	·	4	·	·	·	·	·	·	·	−6	90	6
E'_{42}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−24	324	24
E'_{43}	·	·	·	·	·	·	·	3	·	·	·	·	·	1/2	−25/2	108	6
E'_{44}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−24	432	36
E'_{45}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
E'_{46}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	12	−168	−6
E'_{47}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	−33	·
E'_{48}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2/3	56/3	−146	−6
E'_{49}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2/3	74/3	−212	−10
E'_{50}	·	·	·	·	·	·	·	·	·	·	·	·	·	−2	80	−810	−42
E'_{51}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	30	−444	−27
E'_{52}	·	·	·	·	·	·	·	·	·	·	·	·	·	·	−3	39	1
E'_{53}	·	·	·	·	·	·	·	·	·	·	·	·	·	2	−74	690	30
E'_{54}	·	·	·	·	·	·	·	·	·	·	·	·	·	1/2	−67/2	384	18
E'_{55}	·	·	·	·	·	·	·	·	·	·	·	·	·	−1	40	−381	−15
E'_{56}	·	·	·	·	·	·	·	·	·	·	·	·	·	1/6	−37/6	56	2

Table 2: Number of occurrences of graphs E_{14} to E_{26} in the five-line configurations.

	E_{14}	E_{15}	E_{16}	E_{17}	E_{18}	E_{19}	E_{20}	E_{21}	E_{22}	E_{23}	E_{24}	E_{25}	E_{26}	m^2p	mp	p	μ
E'_1	·	66	·	30	6	·	42	12	·	·	·	·	·	·	·	·	1
E'_2	24	40	4	24	4	·	28	16	·	·	·	·	·	·	·	3	·
E'_3	20	17	10	35	3	·	27	32	14	4	12	·	·	·	·	−12	·
E'_4	·	34	·	42	6	·	46	26	14	2	18	·	·	·	·	−12	−6
E'_5	24	24	4	24	16	·	64	16	6	·	24	·	·	·	3	−21	·
E'_6	20	27	6	41	7	·	51	16	15	2	18	·	·	·	·	−12	·
E'_7	·	46	·	50	10	·	78	·	16	·	24	·	·	·	·	−6	−3
E'_8	12	6	18	34	·	·	10	36	43	16	10	10	·	·	·	6	·
E'_9	·	15	·	45	2	·	25	36	39	8	29	11	·	·	·	36	6
E'_{10}	16	12	12	40	1	·	32	18	42	8	29	11	·	·	·	24	·
E'_{11}	16	4	16	24	3	·	20	42	26	12	27	9	·	·	·	12	·
E'_{12}	24	·	24	12	3	12	24	24	24	12	27	9	·	·	·	·	·
E'_{13}	·	12	·	32	7	·	44	34	26	8	39	9	·	·	·	24	6
E'_{14}	·	·	·	·	36	·	108	·	18	·	72	·	·	1/6	−19/6	14	·
E'_{15}	·	16	·	32	6	·	36	40	22	4	46	10	·	·	−3	27	3
E'_{16}	20	8	10	28	8	·	42	24	24	4	46	10	·	·	−12	108	·
E'_{17}	·	16	·	36	14	·	76	·	26	·	58	10	·	·	−6	66	6
E'_{18}	·	23	·	49	5	·	57	·	41	·	48	12	·	·	·	48	12
E'_{19}	·	24	·	48	·	·	60	·	42	·	48	12	·	·	·	6	2
E'_{20}	8	·	24	4	1	20	8	32	32	28	21	31	2	·	·	·	·
E'_{21}	8	·	24	12	1	·	8	40	40	28	21	31	2	·	·	−12	·
E'_{22}	·	4	·	24	3	·	20	44	32	14	53	33	2	·	12	−156	−12
E'_{23}	16	·	16	16	3	·	28	20	36	14	53	33	2	·	6	−66	·
E'_{24}	·	4	·	32	1	·	12	34	58	20	25	35	2	·	·	−60	−6
E'_{25}	·	·	·	·	9	·	36	36	36	18	63	27	·	·	·	−12	−2
E'_{26}	·	·	·	·	18	·	60	·	30	·	102	30	·	−1	22	−123	−3
E'_{27}	16	·	16	20	5	·	16	32	28	8	61	35	2	·	6	−54	·
E'_{28}	·	4	·	24	9	·	52	·	32	·	85	35	2	·	12	−144	−12
E'_{29}	·	8	·	36	5	·	32	·	52	·	65	39	2	·	12	−192	−18
E'_{30}	12	4	18	24	·	·	18	20	56	14	28	38	2	·	6	−78	·
E'_{31}	·	8	·	32	·	·	44	·	54	·	60	40	2	·	6	−138	−18
E'_{32}	·	15	·	45	·	·	25	·	75	·	35	45	2	·	·	−36	−6
E'_{33}	·	·	·	·	·	80	·	·	·	80	·	40	11	·	·	·	·
E'_{34}	·	·	16	·	·	32	·	24	24	56	12	52	11	·	·	·	·
E'_{35}	·	·	32	·	·	·	·	32	32	48	16	56	11	·	·	3	·
E'_{36}	·	·	·	16	·	·	·	48	16	24	68	60	11	·	−6	66	3
E'_{37}	·	·	·	16	·	·	4	32	52	32	24	64	11	·	−6	114	6
E'_{38}	8	·	24	8	·	·	8	16	48	24	28	68	11	·	−12	132	·
E'_{39}	·	·	·	·	3	·	12	36	36	30	45	69	6	·	−6	102	6
E'_{40}	·	·	·	·	9	·	24	·	24	·	105	75	6	2	−56	414	18
E'_{41}	·	·	·	36	5	·	·	·	60	·	45	83	10	·	−6	90	6
E'_{42}	·	·	·	16	·	·	24	·	40	·	80	72	11	·	−24	324	24
E'_{43}	·	·	·	·	·	·	36	·	54	·	60	84	6	1/2	−25/2	108	6
E'_{44}	·	4	·	24	·	·	16	·	68	·	40	80	11	·	−24	432	36
E'_{45}	·	·	·	·	·	48	·	·	·	96	·	72	27	·	·	·	·
E'_{46}	·	·	·	·	·	·	·	24	24	48	24	96	27	·	12	−168	−6
E'_{47}	·	·	32	·	·	·	·	·	48	40	·	88	35	·	3	−33	·
E'_{48}	·	·	·	·	9	·	·	·	·	·	81	135	18	−2/3	56/3	−146	−6
E'_{49}	·	·	·	·	·	·	·	·	·	·	108	108	27	−2/3	74/3	−212	−10
E'_{50}	·	·	·	·	·	·	12	·	36	·	48	120	27	−2	80	−810	−42
E'_{51}	·	·	·	16	·	·	·	·	64	·	20	108	35	·	30	−444	−27
E'_{52}	·	·	·	·	·	·	·	·	·	72	·	108	63	·	−3	39	1
E'_{53}	·	·	·	·	·	·	·	·	·	·	36	144	63	2	−74	690	30
E'_{54}	·	·	·	·	·	·	·	·	48	·	·	120	75	1/2	−67/2	384	18
E'_{55}	·	·	·	·	·	·	·	·	·	·	·	108	135	−1	40	−381	−15
E'_{56}	·	·	·	·	·	·	·	·	·	·	·	·	243	1/6	−37/6	56	2

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(Received 15 Sep 2015)