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A note on inverses of labeled graphs

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Dedicated to the memory of Dan Archdeacon

Abstract

We consider simple labeled graphs, with non-zero labels in a ring. If the adjacency matrix of a labeled graph is invertible, the inverse matrix is a (labeled) adjacency matrix of another graph, called the inverse of the original graph. If the labeling takes place in an ordered ring, then balanced inverses—those with positive products of labels along every cycle—are of interest. We introduce the concept of a derived labeled graph and show how it can be embedded into an inverse. We also prove a new result on balanced inverses of labeled trees and present a construction of new labeled graphs with balanced inverses from old ones.

1 Introduction

Various concepts of inverses of (undirected) graphs have been proposed and studied. A straightforward way of thinking of an inverse of a graph would be to invert its adjacency matrix, provided, of course, that all its eigenvalues are non-zero. It turns out, however [6], that such an inverse has non-negative integral entries if and only if the graph is a union of isolated edges. Another way of thinking has been motivated by lack of suitable bounds for the smallest non-negative eigenvalue of a graph, in contrast with a relative abundance of bounds for the largest eigenvalue. Namely, one can declare an inverse of a graph to be any graph the spectrum of which is obtained by inverting every eigenvalue (including multiplicities) of the original graph. Of course, this again assumes that the graph has no zero eigenvalue. Since every symmetric matrix is diagonalizable (over the field of real numbers, say), the above is equivalent to declaring a graph H to be an inverse of a graph G if the adjacency matrix A_H of H is similar to the inverse of the adjacency matrix A_G of G. Note that since entries of G are non-negative integers, this condition implies that $\det(A_G) = \pm 1$ and hence if H is in this sense an inverse of G, entries of A_H are integral. Such a way of proceeding proved fruitful in a number of ways [5] but suffers from the aesthetical drawback that inverses, if they exist, may not be unique. This can be overcome by restricting similarity to signability, and in this sense a graph H can be declared to be an inverse of G if $A_H = DA_G^{-1}D$ for some diagonal ± 1 matrix D. Such an approach appeared first in [5] and was developed in detail in [8]; in this setting one also has the appealing relation $(G^{-1})^{-1} = G$ if the inverse G^{-1} to G exists. We also refer the reader to [8] regarding more information about the history of investigation of graph inverses.

The main problem with existence of inverse in any of the above sense is the fact that for 'most' graphs with no zero eigenvalue (even if multiple edges are allowed) the inverse of their adjacency matrix is not signable (or similar, in the more general version) to a matrix with *non-negative* entries. Most of the research in [5, 8] therefore focused on sufficient conditions for a graph G to have A_G^{-1} similar or signable to a *non-negative* matrix. To accommodate more objects under the umbrella of 'invertible graphs' a natural step is to consider labeled graphs, which was the case in [8, 10, 1] for labelings by positive integers, real numbers with signs, and elements of general rings, respectively.

Following the third approach, let G be a *simple* graph (that is, with no loops and multiple edges) with edge set E_G and let $\mathcal{K}$ be a (not necessarily commutative) ring. A *labeling* $\alpha: E_G \to \mathcal{K}$ is an arbitrary function that assigns to every edge $e \in E_G$ a non-zero label $\alpha(e) \in \mathcal{K}$. The pair (G, α) is then called a *labeled graph*; the ring $\mathcal{K}$ does not enter the notation as it will always be understood from the context. An adjacency matrix $A_{(G,\alpha)}$ is, as usual, a square matrix with rows and columns indexed by the vertex set of G, whereby the uv-th element a_{uv} of $A_{(G,\alpha)}$ is equal to zero if u and v are not adjacent in G, and $a_{uv} = \alpha(e) \neq 0$ if u and v are joined by the edge e.

An *inverse* of a labeled graph (G, α) is a labeled graph (H, β) with labels in the same ring $\mathcal{K}$ and with adjacency matrix $A_{(H,\beta)}$ equal to $A_{(G,\alpha)}^{-1}$.

It turns out that if G is a bipartite (simple) graph with a unique perfect matching, then the labeled graph (G, α) has an inverse for an arbitrary labeling $\alpha: E_G \to \mathcal{K}$ as introduced above and having the property that all the labels on edges of the perfect matching have multiplicative inverses in $\mathcal{K}$. This has been known in the literature and is equivalent to a matrix multiplication formula given in Theorem 1 of [1] which generalizes a formula of [3], the unweighted Lemma 2.1 of [2] and Theorem 5 of [7]. We nevertheless include a proof of this statement in Section 2 since its restricted version appeared first in the PhD dissertation of the author in 1994 and was never published. In the same place we also show how a 'large part' of a labeled graph can be embedded in its inverse.

In Section 3 we consider labelings in ordered rings and study balanced inverses, in which the product of labels on every closed walk is positive. We prove that balance is equivalent to signability, a fact that does not appear to have been explicitly stated but known to specialists. We also show that trees arbitrarily labeled in an ordered ring always have a balanced inverse. A construction of new labeled graphs with balanced inverses from old such graph is presented in Section 4, and in the final Section 5 we conclude with notes and remarks.

2 Inverses of labeled graphs and derived graphs

Let (G, α) be a labeled graph, with edge labels in a (not necessarily commutative) ring $\mathcal{K}$, such that G is bipartite and has a unique perfect matching. Since G is bipartite, the adjacency matrix $A_{(G,\alpha)}$ may be assumed to have the block form

$$A_{(G,\alpha)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}; \tag{1}$$

here A is usually called a *bipartition matrix* of (G, α). By [5, Lemma 2.1] we know that a simple bipartite graph G as above has a unique perfect matching if and only if its vertex set V_G admits a bipartition such that vertices in both parts can be linearly ordered in such a way that the above bipartition matrix A is triangular; we will assume this from now on. Then the matrices A and $A_{(G,\alpha)}$ are invertible if and only if all diagonal entries of A have multiplicative inverses in $\mathcal{K}$.

Proceeding further, if our bipartite graph G with a unique perfect matching M has $2n$ vertices and its bipartition matrix A is upper triangular, then we may (and we will) without loss of generality assume that the bipartition of V_G has the form $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots, n'\}$, with ii' being the edges of M and with no edge in G of the form ij' for $i > j$, where $i, j \in \{1, 2, \ldots, n\}$. We will briefly refer to this situation by saying that both G and A are in an *upper canonical form.* A *lower canonical form* of G and A is defined analogously. The matched and unmatched edges of G will occasionally be called *horizontal* and *descending*, respectively, as they can be drawn as horizontal and descending segments (from left to right) when the non-dashed and dashed vertices are drawn in two columns next to each other in an obvious way. We will use this notation and terminology throughout from this point on. Note that if (H, β) is the inverse of our labeled graph (G, α) with G in an upper canonical form, then H is automatically represented in a lower canonical form.

We need to introduce one more concept related to our labeled graph (G, α) with a perfect matching M. As usual, by an $u \to v$ path P in G we understand a sequence $u_0 u_1 \ldots u_\ell$ of mutually distinct vertices of G with $u_0 = u$, $u_\ell = v$, and $u_{k-1}u_k \in E_G$ for every $k \in \{1, \ldots, \ell\}$. Such a path P will be called *M-alternating*, or simply *alternating*, if ℓ is odd and $u_{k-1}u_k \in M$ if and only if k is odd, $1 \le k \le \ell$. We will say that an alternating path P is *even* (*odd*) if it contains and even (odd) number of edges not in M. Thus, if P consists of a single edge $e \in M$, then P is even. Letting $\alpha_k = \alpha(u_{k-1}u_k)$ and recalling that labels of edges of M are assumed to have multiplicative inverses in $\mathcal{K}$, for an alternating $u_0 \to u_\ell$ path P in (G, α) as above we define the *value* $\omega_\alpha(P)$ of P to be

$$\omega_\alpha(P) = \alpha_1^{-1} \alpha_2 \alpha_3^{-1} \alpha_4 \ldots \alpha_{\ell-2}^{-1} \alpha_{\ell-1} \alpha_\ell^{-1} \ . \tag{2}$$

That is, to obtain the value $\omega_\alpha(P)$ we multiply through the inverses of labels of matched edges and the original labels of unmatched edges in the order the path is traversed. Finally, for a pair of distinct vertices u, v of G we let $p_M^+(u,v)$ and $p_M^-(u,v)$ be the sum of the values $\omega_\alpha(P)$ of all even and odd alternating $u \to v$ paths P, respectively. Note that the values of $p_M^+(u,v)$ and $p_M^-(u,v)$ are automatically zero if both $u, v \in \{1,2,\ldots,n\}$ or both $u,v \in \{1',2',\ldots,n'\}$. Observe also that, for $i \le j$, alternating $i' \to j$ paths in our graph G will always have the form $i'ir'rs's\ldots t'tj'j$ for some (possibly empty, if $i = j$) set of vertices $r, s, \ldots, t$ such that $i < r < s < \ldots < t < j$.

Generalizing the ideas contained in the (unpublished) PhD dissertation of the author [9] we prove that the labeled graphs considered above automatically have inverses, following the original outline given in [9].

Theorem 1 [9] *Let G be a simple bipartite graph of order $2n$ in am upper canonical form with a unique perfect matching M and let $\alpha : E_G \to \mathcal{K}$ be a labeling in a (not necessarily commutative) ring $\mathcal{K}$ such that the label of every edge in M has a multiplicative inverse in $\mathcal{K}$. Then the labeled graph (G,α) has an inverse (H,β) whose lower canonical form on the vertex set $V_H = V_G$ is given by letting two distinct vertices $i' \in \{1',2',\ldots,n'\}$ and $j \in \{1,2,\ldots,n\}$, $i \le j$, be adjacent in H if and only if $p_M^+(i',j) \ne p_M^-(i',j)$, and by defining $\beta(i'j) = p_M^+(i',j) - p_M^-(i',j)$ for $i'j \in E_H$.*

Proof. Let A and B be bipartition matrices of G and H, respectively; both G and H, in an upper and a lower canonical form, respectively, are assumed to share the bipartition $\{1,2,\ldots,n\}$ and $\{1',2',\ldots,n'\}$ of their common vertex set. To prove the result it is sufficient to show that AB^T is equal to the identity matrix. In view of the above facts this will follow if we prove that the entries $a_{ir'}$ of A and $b_{jr'}$ of B satisfy $\sum_r a_{ir'}b_{jr'} = \delta_{ij}$ for every $i,j \in \{1,2,\ldots,n\}$ such that $i \le j$.

Observe that this is obviously true if $i = j$ since the diagonal entry $b_{ii'}$ is simply equal to $\beta(i'i) = p_M^+(i',i) = \omega_\alpha(P) = \alpha(ii')^{-1} = a_{ii'}^{-1}$ for the unique $i \to i'$ alternating path P formed by the single edge $ii' \in M$. For $1 \le i < j \le n$ we only need to consider the values of r for which $a_{ir'} \ne 0$ and so we let $G(i,j) = \{r;\ i \le r \le j \text{ and } ir' \in E(G)\}$. Using $a_{ir'} = \alpha(ir')$ and $b'_{jr'} = \beta(r'j) = p_M^+(r',j) - p_M^-(r',j)$ by the definition of H, the rest of the equality to be proved can be written in the form $\sum_{r\in G(i,j)} \alpha(ir')(p_M^+(r',j) - p_M^-(r',j)) = 0$, or, equivalently,

$$\sum_{r\in G(i,j)} \alpha(ir')p_M^+(r',j) = \sum_{r\in G(i,j)} \alpha(ir')p_M^-(r',j) \quad \text{for } i < j\ . \tag{3}$$

Take a vertex $r \in G(i,j)$ for a fixed pair i,j such that $i < j$ and let P be an even alternating $r' \to j$ path in G. If $i < r \le j$, then the even path P must have the form $r'r$ if $r = j$, or $r'rs'st't\ldots j'j$ for some $s,t,\ldots \in \{1,2,\ldots,n\}$ such that $r < s < t < \ldots < j$. Since $r \in G(i,j)$, we have $ir' \in E_G$ and so we may extend P by adding the pair of edges $i'i$ and ir' to an odd $i' \to j$ alternating path $P' = i'iP$ with value $\omega_\alpha(P') = \alpha(i'i)^{-1}\alpha(ir')\omega_\alpha(P)$. In the case when $r = i$ the even path P has the form $i'is's\ldots j'j$ as above for some $s,\ldots$ such that $i < s < \ldots < j$, and this time we form an odd alternating $s' \to j$ path P' by removing the edges $i'i$ and is' from P, with $\omega_\alpha(P) = \alpha(i'i)^{-1}\alpha(is')\omega_\alpha(P')$.

It follows that for every r with $i < r \leq j$ and every even alternating $r' \to j$ path P there exists an odd alternating $i' \to j$ path P' that extends P by two edges and such that $\alpha(ir')\omega_\alpha(P) = \alpha(i'i)\omega_\alpha(P')$, and for every even alternating $i' \to j$ path P there exists an odd alternating $s' \to j$ path P' for some s, $i < s < j$, shorter by two edges, such that $\alpha(i'i)\omega_\alpha(P) = \alpha(is')\omega_\alpha(P')$. It is easy to check that this correspondence $P \to P'$ between even and odd alternating $r' \to j$ paths in G for $r \in G(i,j)$ is one-to-one and onto. Summing up the equations $\alpha(ir')\omega_\alpha(P) = \alpha(i'i)\omega_\alpha(P')$, $j \leq r < i$, together with the equations $\alpha(i'i)\omega_\alpha(P) = \alpha(is')\omega_\alpha(P')$, in both cases with P ranging over all even alternating $r' \to j$ paths for $r \in G(i,j)$, we obtain (3). This completes the proof. □

As stated in the introduction, various versions of Theorem 1 have appeared after the 1994 submission of the author's PhD dissertation [9]; the result is equivalent to the formula given in Theorem 1 of [1] and generalizes earlier unweighted versions of [3, 2, 7].

Let (H, β) be the inverse of (G, α) as in Theorem 1. The graphs G and H have the same vertex set but their edge sets have just the edges of the unique perfect matching M in common because of an upper and a lower canonical form of G and H, respectively. We now show that (at least part of) G can be embedded in H. Let G' be the subgraph of G on the same vertex set $V_{G'} = V_G$ with the edge set $E_{G'} = \{i'j;\ ij' \in E_G;\ \beta(i'j) \neq 0\}$. We make G' into a labeled graph (G', α') by letting $\alpha'(i'j) = \beta(i'j)$ and we will call (G', α') the *derived graph* of (G, α). Note that G' is isomorphic to a subgraph of G via the bijection interchanging i with i', $1 \leq i \leq n$. The derived graph (G', α') is a *labeled subgraph* of the inverse (H, β) of (G, α) in the sense that G' is a subgraph of H and the labelings α' and β coincide on edges of G'. Observe also that all edges $e \in M$ appear in both G' and H, with labels $\alpha'(e) = \beta(e) = \alpha(e)^{-1}$. We sum up these facts as follows.

Lemma 1 *Let G be a simple bipartite graph with a unique perfect matching and with a labeling in a ring $\mathcal{K}$ assigning to every matched edge an invertible label, and let (G, α) be a labeled graph in an upper canonical form. Then the derived graph (G', α') is a labeled subgraph of the inverse of (G, α).* □

Suppose, for example, that the underlying graph G of our labeled graph (G, α) considered above is a tree T. For every unmatched edge $ij' \in E_T$ $(i < j)$ we then have a unique (and, as it happens, odd) alternating $i' \to j$ path $P = i'ij'j$ and so $p^+_M(i', j) - p^-_M(i', j) = -\omega_\alpha(P) = -\alpha(ii')^{-1}\alpha(ij')\alpha(jj')^{-1} \neq 0$. It follows that $i'j$ is an edge of the derived graph (T', α'), so that T' can be identified with T, and $\alpha'(i'j) = -\alpha(ii')^{-1}\alpha(ij')\alpha(jj')^{-1}$. Of course, for every matched edge ii' of $T = T'$ we have $\alpha'(ii') = \alpha(ii')^{-1} = a^{-1}_{ii'}$. Lemma 1 then shows that the inverse of a labeled tree can be considered to be a super-graph of the tree. We state these observations for a later use.

Lemma 2 *Let T be a tree of order $2n$ with a unique perfect matching and let (T, α) be a labeled graph, given in an upper canonical form. The derived graph (T', α') has*

T' isomorphic to T, with $\alpha'(ii') = \alpha(ii')^{-1}$ for every matched edge and $\alpha'(i'j) = -\alpha(ii')^{-1}\alpha(ij')\alpha(jj')^{-1}$ for every unmatched edge ($i < j$) of T. Moreover, (T', α') is a labeled subgraph of the inverse of (T, α). □

We conclude with an observation about connectivity of the derived graph.

Lemma 3 *Let (G, α) be a bipartite labeled graph with a perfect matching M and with invertible labels on edges of M, and let (G', α') be its derived graph. Then G is connected if and only if G' is.*

Proof. Obviously, if G' is connected, then so is G, and we therefore focus on the reverse direction. Let G be connected and let i', j be vertices of G with $i < j$; it is sufficient to show that i', j are contained in a path of G'. To do so, consider an alternating $i' \to j$ path in G of largest length among all such $i' \to j$ paths (note that there may be several of these). For every unmatched edge rs' of P with $r < s$ we have $\alpha'(r's) = -\alpha(rs') \neq 0$ since $r'rss'$ is the *unique* (and odd) alternating $r' \to s$ path in G because of the largest length of P. Thus, for every unmatched edge rs' in an alternating $i' \to j$ path in G of largest length the edge $r's$ appears in G'. Since edges of all alternating $i' \to j$ paths in G, taken over all $i < j$, clearly span a connected subgraph of G, it follows that the derived graph G' is connected. □

3 Balanced inverses of labeled graphs

A ring $\mathcal{K}$ is *ordered* if it contains a subset $\mathcal{P}$ of *positive elements* which has the following three properties: $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$, $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$, and for every $x \in \mathcal{K}$ we have the trichotomy that either $x = 0$, or $x \in \mathcal{P}$, or else $-x \in \mathcal{P}$. Non-zero elements not in $\mathcal{P}$ are *negative*. The three properties imply that for every k elements $x_1 x_2, \ldots, x_k \in \mathcal{K}$ the product $x_1 x_2 \cdots x_k$ is positive if and only if the k elements contain an even number of negative entries and the remaining entries are positive. In particular, if π is an arbitrary permutation of the subscript set $\{1, 2, \ldots, k\}$, then $x_1 x_2 \cdots x_k \in \mathcal{P}$ if and only if $x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)} \in \mathcal{P}$. Observe also that if an element $x \in \mathcal{K}$ has a multiplicative inverse, then either both x, x^{-1} are positive or they both are negative. For a theory of ordered rings we refer to the monograph [4].

In Theorem 1 we saw how to calculate the inverse of a labeled graph (G, α). If, however, the ring $\mathcal{K}$ in which the labeling α takes place is ordered, with a set $\mathcal{P}$ of positive elements, then the inverse may have extra appealing properties. A property of this kind appears to be balance, which we explain next. Let $W = u_0 u_1 \ldots u_{\ell-1} u_\ell$ be a walk in G, i.e., $u_{k-1} u_k \in E_G$, $1 \le k \le \ell$ and we do not require vertices or edges of W to be mutually distinct; recall that W is closed if $u_\ell = u_0$. We will say that the *α-sign* or simply *sign* $\sigma_\alpha(W)$ of W is $+1$ (-1), or positive (negative), if the product $\alpha(u_0 u_1)\alpha(u_1 u_2) \cdots \alpha(u_{\ell-1} u_\ell)$ is positive (negative), respectively; we bear in mind that the sign depends neither on the choice of the initial vertex nor on the sense the cycle is traced (even if $\mathcal{K}$ is not commutative). A positive walk (or a closed walk, a path, or a cycle) will also be called *balanced*. The labeled graph (G, α) itself is called *balanced* if every cycle in the graph is balanced (or, equivalently, if every closed walk in the graph is balanced).

It turns out that the concept of a balance is closely related to signability mentioned in the introduction. We say that a labeled graph (G, α) with labels in an *ordered* ring $\mathcal{K}$ is *signable* if there is a diagonal ± 1 matrix D such that $DA_{(G,\alpha)}D$ has no negative entry.

Proposition 1 *A labeled graph (G, α) is balanced if and only if its adjacency matrix is signable.*

Proof. We first need to introduce a local operation on labeled graphs. Let v be a vertex of a labeled graph (G, α). We may create a new labeling α_v from α by *switching at* v, that is, by changing the sign on the label of every edge incident with the vertex v. Switching can be applied step-by-step to an arbitrary non-empty subset $U \subset V$ and the resulting labeling α_U does not depend on the order of switchings applied to individual vertices of U.

Now let (G, α) be a labeled graph from the statement of our result. By Lemma 3.1 of [11] the graph (G, α) is balanced if and only if there is a subset U of its vertex set such that the adjacency matrix of (G, α_U) has no negative entry. But, for any $v \in U$, producing (G, α_v) from (G, α) by switching at a vertex v is equivalent to forming $A_{(G,\alpha_v)}$ from $A_{(G,\alpha)}$ by conjugation by the diagonal matrix D_v with v-th diagonal entry equal to -1 and all other entries equal to $+1$; that is, $A_{(G,\alpha_v)} = D_v A_{(G,\alpha)} D_v$. Letting $D_U = \prod_{v \in U} D_v$ and observing that the product does not depend on the order of multiplication we conclude that the matrix $A_{(G,\alpha_U)} = D_U A_{(G,\alpha)} D_U$ has no negative entries. The argument is clearly reversible. □

With the help of Proposition 1 we now generalize Theorem 2.2 of [5]. If M is a perfect matching in a simple graph G, by G/M we denote the graph obtained from G by contracting every edge of M and replacing all the resulting multiple edges by simple edges.

Theorem 2 *Let (G, α) be a labeled graph with labels in an ordered ring. Assume that G is simple, bipartite, and containing a unique perfect matching M such that the label of every edge of M has a multiplicative inverse. If G/M is bipartite and α does not have negative values, then (G, α) has a balanced inverse.*

Proof. Let (H, β) be the inverse of (G, α) from Theorem 1; we will refer to the notation introduced earlier. By Proposition 1 it is sufficient to show that (H, β) is balanced. The key observation is that the assumption of G/M being bipartite implies that for any two vertices i', j of G with $i \leq j$, either all alternating $i' \to j$ paths are even, or all these paths are odd. Accordingly, for the labeling β we have $\beta(i'j) = p_M^+(i', j)$ in the 'even' case and $\beta(i'j) = -p_M^-(i', j)$ in the 'odd' case.

Let C be a cycle in the graph H. By our assumption that all values of α are positive and by the facts in the above paragraph, for every edge $i'j$ of C, $i \leq j$, the value $\beta(i'j)$ is positive (negative) if and only if there is an even (odd) alternating

$i'\!\to\! j$ path in G. We select one such path $P_{i',j}$ for every edge $i'j$ of C and form a closed walk W in G by concatenating all the paths $P_{i',j}$. Further, let C^*/M be the closed walk in G/M obtained by contracting every edge of C contained in M. Obviously, the same closed walk C^*/M is also obtained by concatenating the paths $P_{i',j}/M$ obtained form $P_{i',j}$ by contracting edges in M.

Note that the length of $P_{i',j}/M$ is even (odd) if and only if $P_{i',j}$ is even (odd). Now, C^*/M has even length, since G/M is assumed to be bipartite. But this implies that C^*/M must contain an even number of paths $P_{i',j}/M$ of odd length. Equivalently, the closed walk W must contain an even number of odd alternating paths $P_{i',j}$. By the above correspondence this means that the cycle C must contain an even number of edges $i'j$ such that $\beta(i'j') < 0$, that is, C is balanced. Since these considerations are valid for every cycle C of H, we conclude that the labeled graph (H, β) is balanced. □

Note that Theorem 2.2 of [5] follows from the above by letting $\alpha(uv) = 1$ for every edge uv of G. For trees we offer an even stronger result that does not require restriction on labels of unmatched edges.

Theorem 3 *Let (T, α) be a labeled graph with labels in some ordered ring, where T is a tree containing a unique perfect matching M such that the label of every edge of M has a multiplicative inverse. Then (T, α) has a balanced inverse.*

Proof. As usual we assume that (T, α) is in an upper canonical form. By Proposition 1 and it is sufficient to show that the inverse (H, β) of (T, α) is balanced. Let (T', α') be the derived graph of (T, α). Lemma 2 tells us that (T', α') is a labeled subgraph of the inverse (H, β) of (T, α). We may identify T' with its isomorphic copy T through the bijection φ interchanging i and i', $1 \le i \le n$, that is, we let $T' = \varphi(T)$. By standard cycle space arguments, to prove balance of (H, β) it suffices to show that every cycle of H containing exactly one edge not in T' is balanced.

Let C be a cycle of H containing exactly one edge $i'j$ not in T' for some $i < j$. Such an edge can only be in H if there is an alternating $i'\!\to\! j$ path P in the original tree T. But if P exists, it is unique (since T is a tree) and has the form $P = i'Qj$ for a unique $i\!\to\! j'$ sub-path $Q = e_1 f_1 e_2 f_2 \ldots e_{\ell-1} f_{\ell-1} e_\ell$ of P in which e_k and f_k are the unmatched and matched edges, respectively. The cycle C may now be assumed to consist of the edge $i'j$ traversed from j to i' and followed by edges of $\varphi(Q)$. In the above notation the number of unmatched edges in P and also in Q and $\varphi(Q)$ is equal to ℓ. Let m be the number of negative labels on edges of P.

By Theorem 1 we have $\beta(i'j) = p^+_M(i', j) - p^-_M(i', j) = \omega_\alpha(P)$ and so for the β-sign of the path consisting solely of the edge $i'j$ we have $\sigma_\beta(i'j) = (-1)^{\ell+m}$. The product of β-labels on edges of $\varphi(Q)$ when traversed from i' to j is $\beta(\varphi(e_1))\beta(\varphi(f_1)) \cdots \beta(\varphi(e_{\ell-1}))\beta(\varphi(f_{\ell-1}))\beta(\varphi(e_\ell))$. We saw in Lemma 2 that for the matched edges $f_k = \varphi(f_k)$ of the form tt' we have $\beta(tt') = \alpha(tt')^{-1}$ and for the unmatched edges e_k of the form $e_k = rs'$ for $r < s$ we have $\beta(\varphi(rs') = \beta(r's) = -\alpha(r'r)^{-1}\alpha(rs')\alpha(s's)^{-1}$; in particular, note that for the first edge e_1 of the form ir' and for the last edge e_ℓ of the form sj' of Q we have $\beta(i'r) = -\alpha(i'i)^{-1}\alpha(ir')\alpha(r'r)^{-1}$ and $\beta(s'j) =$

$-\alpha(s's)^{-1}\alpha(sj')\alpha(j'j)^{-1}$. It follows that the β-sign of $\varphi(Q)$ is $\sigma_\beta(Q) = (-1)^{\ell+m}$, which is the same as the β-sign of the edge $i'j$. Thus, the β-sign of the cycle C consisting of the edge $i'j$ followed by edges of $\varphi(Q)$ is positive, completing the proof. □

4 New bipartite graphs with balanced inverses from old by means of edge overlapping

In our next auxiliary result we again refer to the notation of Theorem 1 and to the labeling β in particular.

Lemma 4 *Let (G, α) be a labeled bipartite graph of order $2n$ with a unique perfect matching M labeled by invertible elements, and in an upper canonical form. Let $e = rr'$ be a matched edge of G and let i, j' be arbitrary vertices of G such that $1 \le i \le r \le j \le n$. Further, let $p^+_M(i', j; e)$ and $p^-_M(i', j; e)$ denote the sum of the α-values of all even and odd, respectively, alternating $i' \to j$ paths in G containing the edge e. Then*

$$p^+_M(i', j; e) - p^-_M(i', j; e) = \beta(i', r)\alpha(e)\beta(r', j) .$$

Proof. Every alternating $i' \to j$ path in G through the edge $e = rr'$ has the form $P = i'i \ldots r'r \ldots j'j$. If $P_1 = i'i \ldots r'r$ and $P_2 = r'r \ldots j'j$ are the $i' \to r$ and $r' \to j$ subpaths of P, then we have $\alpha(P) = \alpha(P_1)\alpha(e)\alpha(P_2)$, since the rightmost and leftmost factors in the product of α-labels determining $\alpha(P_1)$ and $\alpha(P_2)$ are both equal to $\alpha(r'r)^{-1} = \alpha(e)^{-1}$. An obvious parity consideration leads to the following pair of equations:

$$p^+_M(i', j; e) = p^+_M(i', r)\alpha(e)p^+_M(r', j) + p^-_M(i', r)\alpha(e)p^-_M(r', j) ,$$
$$p^-_M(i', j; e) = p^+_M(i', r)\alpha(e)p^-_M(r', j) + p^-_M(i', r)\alpha(e)p^+_M(r', j) .$$

Subtracting the second equation from the first, rearranging terms, and using the definition of the labeling β yields the result. □

Let G_1 and G_2 be simple graphs with pendant vertices u_1 and u_2 and with pendant edges u_1v_1 and u_2v_2, respectively. Let $G_1 * G_2$ be the simple graph obtained from G_1 and G_2 by identifying the edges u_1v_1 and u_2v_2 into a single edge e in such a way that u_1 is identified with v_2 and u_2 with v_1. We will loosely say that $G_1 *_e G_2$ is obtained from G_1 and G_2 by *pendant overlapping* (at e). The situation is depicted in Figs. 1 and 2.

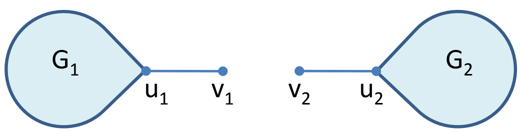

Figure 1: The graphs G_1 and G_2 with pendant edges u_1v_1 and u_2v_2.

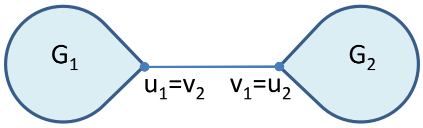

Figure 2: The graph that arises by pendant overlapping from G_1 and G_2 in Fig. 1.

If G_1 and G_2 have unique perfect matchings M_1 and M_2, then $u_1v_1 \in M_1$, $u_2v_2 \in M_2$ and $G_1 * G_2$ also has a unique perfect matching containing the edge e obtained by identifying u_1v_1 with u_2v_2. If, in addition, (G_1, α_1) and (G_2, α_2) are labeled graphs with labels in a ring such that $\alpha(u_1v_1) = \beta(u_2v_2)$, pendant overlapping naturally leads to a labeled graph $(G_1 *_e G_2, \alpha_1 *_e \alpha_2)$ in which the new labeling is defined as the one that coincides with α and β on edges of G_1 and G_2, respectively.

The operation of pendant overlapping is useful in constructions of new labeled graphs with balanced inverses, as we show in our next result. Before its statement let us recall the well known fact that if a bipartite graph has a unique perfect matching, then the matching contains at least two pendant edges of the graph.

Theorem 4 *Let (G_1, α_1) and (G_2, α_2) be labeled bipartite graphs, both with a unique perfect matching, with labels in an ordered ring $\mathcal{K}$ such that matched edges receive invertible labels. Assume that the graphs G_1 and G_2 contain pendant vertices u_1 and u_2 together with matched pendant edges u_1v_1 and u_2v_2, respectively, such that $\alpha_1(u_1v_1) = \alpha_2(u_2v_2)$. If both (G_1, α_1) and (G_2, α_2) have balanced inverses and α_1 and α_2 coincide on the edge e obtained by identifying u_1 with v_2 and u_2 with v_1, then the labeled graph $(G_1 *_e G_2, \alpha_1 *_e \alpha_2)$ also has a balanced inverse.*

Proof. Let (H_1, β_1) and (H_2, β_2) be balanced inverses of (G_1, α_1) and (G_2, α_2) obtained as in Theorem 1; we also let $G = G_1 *_e G_2$ and $\alpha = \alpha_1 *_e \alpha_2$. Let (H, β) be an inverse of (G, α); our first goal is to explicitly calculate the labeling β with the help of Theorem 1 and Lemma 4. Letting M_1 and M_2 denote the unique perfect matchings in G_1 and G_2, we let $M = (M_1 \backslash \{u_1v_1\}) \cup (M_2 \backslash \{v_1v_2\}) \cup \{e\}$ be the unique perfect matching of G_3. To conform with the earlier notation, let G_1 and G_2 have orders $2m$ and $2n$ and let them be in an upper canonical form with vertex sets $\{1, 2, \ldots, m\} \cup \{1', 2', \ldots, m'\}$ and $\{m, \ldots, m+n-1\} \cup \{m', \ldots, (m+n-1)'\}$, presuming without loss of generality that m and m' are results of identification of u_1 with v_2 and u_2 with v_1, so that the overlap edge e is identified with the edge mm'.

Clearly, if i', j are vertices of G_1 such that $i \leq j \leq m$, then $\beta(i', j) = \beta_1(i', j)$. Similarly, we have $\beta(i', j) = \beta_2(i', j)$ if i', j are vertices of G_1 with $m \leq i \leq j \leq m+n-1$. It can be seen that the remaining situation to be considered is when i, j are such that $1 \leq i < m < j \leq m+n-1$. Now, an alternating $i' \to j$ path P in G exists if and only if there is an $i' \to m$ alternating path in G_1 and an $m' \to j$ alternating path in G_2; of course, every such path must contain the edge e. Applying Lemma 4 we obtain

$$\beta(i', j) = \beta_1(i', m)\alpha(e)\beta_2(m', j) \ . \tag{4}$$

By our assumptions the inverses (H_1, β_1) and (H_2, β_2) are balanced. Let φ be the bijection interchanging i with i' for every $i \in \{1, 2, \ldots, m+n-1\}$. To show that (H, β) is balanced as well we only need to show that addition of edges $i'j$ such that $1 \leq i < m < j \leq m+n-1$ never creates a cycle with a negative product of β-values on its edges. Thus, let $i'j$ be such an edge of H, which implies that there exists an alternating $i' \to j$ path P in G. Let Q be the $i \to j'$ sub-path of P obtained by removing is end-vertices i' and j. By standard arguments concerning cycle spaces, balance of (H, β) will follow by showing that for every such edge $i'j$ and a corresponding $i \to j'$ path Q the cycle $C = \varphi(Q) \cup i'j$ obtained by adding the edge $i'j$ to $\varphi(Q)$ is balanced, which we do next.

For the edge $i'j$ and the $i' \to j$ path $\varphi(Q)$ we also have the existence of edges $i'm$ in H_1 and $m'j$ in H_2. The path Q must have the form $Q = Q_1 e Q_2$ where Q_1 and Q_2 are an $i \to m'$ and an $m \to j'$ sub-path of Q, respectively. Our assumptions imply that the cycles $C_1 = \varphi(Q_1) \cup i'm$ and $C_2 = \varphi(Q_2) \cup m'j$ are both balanced, as they are contained in H_1 and H_2. To finish the proof it is now sufficient to show that the 4-cycle formed by the edges $i'j$, $i'm$, $m'j$ and $mm' = e$ is balanced under the labeling β. By (4), however, we have $\beta(i'j) = \beta_1(i'm)\alpha(e)\beta_2(m'j)$ and so the product of the four labels $\beta(i'j)$, $\beta(i'm) = \beta_1(i'm)$, $\beta(m'j) = \beta_2(m'j)$ and $\beta(mm') = \alpha(e)^{-1}$ on the edges of our 4-cycle is clearly positive. This completes the proof. □

5 Remarks

Literature on inverses of graphs is, for a major part, dealing with calculating (and determining properties of) inverses of bipartite graphs with a unique perfect matching, since these can be assumed to be in a canonical form and thus admit well developed tools for their study. The concept of an inverse defined here, however, applies to any labeled graph with a non-singular adjacency matrix and has not been investigated in full generality. Restricting ourselves to labels in the ring of integers, examples of *integrally invertible* graphs that are non-bipartite and with a single perfect matching, or bipartite with multiple perfect matchings, or even non-bipartite with multiple perfect matchings exist; see Figs. 3, 4, 5 in which all edges carry the label 1. Note that the non-bipartite unique-matching case was studied in [10] for the so-called stellated graphs and in [8] for the so-called coronas of graphs.

For an invertible graph (G, α) with labeling in an ordered ring it is also tempting to think of a possible connection between balance of the derived graph $(G'\alpha')$ and balance of the inverse (H, α) of (G, α). While balance of (H, α) implies balance of (G', α') by Lemma 1, the converse need not be true, as demonstrated by the examples in Figs. 6, 7, 8.

Investigation of inverses of labeled graphs in the non-bipartite or non-unique-matching setting and the study of their balance may generate further interesting results and applications.

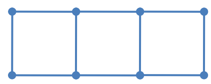

Figure 3: An integrally invertible bipartite graph with more than one perfect matching.

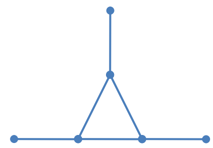

Figure 4: An integrally invertible non-bipartite graph with a unique perfect matching.

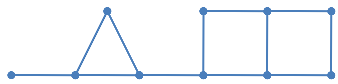

Figure 5: An integrally invertible non-bipartite graph with more than one perfect matching.

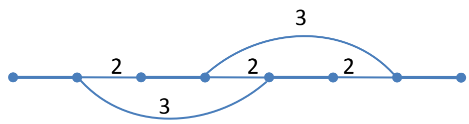

Figure 6: An integrally labeled graph (G, α).

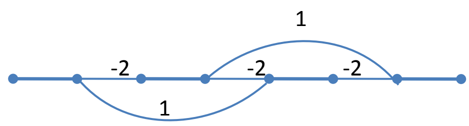

Figure 7: The balanced derived graph (G', α') of the graph in Fig. 6.

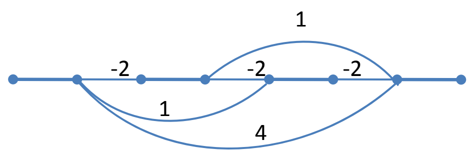

Figure 8: The unbalanced inverse of the graph (G, α) from Fig. 6.

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