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Self-embeddings of doubled affine Steiner triple systems

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Abstract

Given a properly face two-coloured triangulation of the graph K_n in a surface, a Steiner triple system can be constructed from each of the colour classes. The two Steiner triple systems obtained in this manner are said to form a biembedding. If the systems are isomorphic to each other it is a self-embedding.

In the following, for each $k \geq 2$, we construct a self-embedding of the doubled affine Steiner triple system $\mathrm{AG}(k, 3)$ in a nonorientable surface. We also make use of a construction due to Grannell, Griggs and Širáň to obtain a biembedding of $\mathrm{AG}(k, 3)$ in a nonorientable surface that is not a self-embedding for $k > 2$.

1 Introduction

A *Steiner triple system of order* v, an $\mathrm{STS}(v)$, is an ordered pair $(V, \mathcal{B})$ where V is a set of cardinality v and $\mathcal{B}$ is a collection of *triples* which has the property that each pair of distinct elements of V occurs in precisely one triple. The necessary and sufficient condition for the existence of an $\mathrm{STS}(v)$ is that $v \equiv 1$ or $3 \pmod 6$ [14].

Now consider a triangulation of the complete graph K_n in a surface. Such a triangulation exists in an orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and a nonorientable surface if and only if $n \equiv 0, 1, 3$ or $4 \pmod 6$ and $n \geq 9$ [15]. Suppose that the triangulation we are considering satisfies the additional property that its faces can be properly 2-coloured. Then the set of faces of each colour class forms a Steiner triple system of order n, and we say that the two Steiner triple systems obtained in this manner are *biembedded* in the surface.

For such a biembedding to exist the number of faces around each vertex $(n-1)$ must be even. So if the surface is orientable, a necessary condition is that $n \equiv 3$ or $7 \pmod{12}$, whereas if the surface is nonorientable, a necessary condition is that $n \equiv 1$ or $3 \pmod 6$. These necessary conditions are also sufficient, the orientable case $n \equiv 3 \pmod{12}$ was established in [15]; the orientable case $n \equiv 7 \pmod{12}$ in [18]; the nonorientable case $n \equiv 9 \pmod{12}$ in [15] and a remark is made, although no details are given, that the method described also works for the case $n \equiv 3 \pmod{12}$, with the details made explicit in [1]; finally the nonorientable case $n \equiv 1 \pmod 6$, where $n \geq 13$ was established in [9]. A comprehensive survey of results on biemebeddings of Steiner triple systems can be found in [6]. For background on topological graph embeddings we refer the reader to [13].

In this paper we will investigate biembeddings of Steiner triple systems that satisfy the additional condition that the two systems are isomorphic to each other; such a biembedding is called a *self-embedding* of a Steiner triple system. For example consider the face 2-coloured triangulation of K_7 in the torus depicted in Figure 1; the corresponding Steiner triple systems are $S_1 = (V, \mathcal{B}_1)$ and $S_2 = (V, \mathcal{B}_2)$ where

$$\mathcal{B}_1 = \big\{\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}\big\}$$

and

$$\mathcal{B}_2 = \big\{\{0,1,5\},\{1,2,6\},\{2,3,0\},\{3,4,1\},\{4,5,2\},\{5,6,3\},\{6,0,4\}\big\}.$$

The map $\phi : V \to V$ given by $v \mapsto -v \pmod 7$ is an isomorphism from S_1 to S_2. Thus the biembedding is a self-embedding.

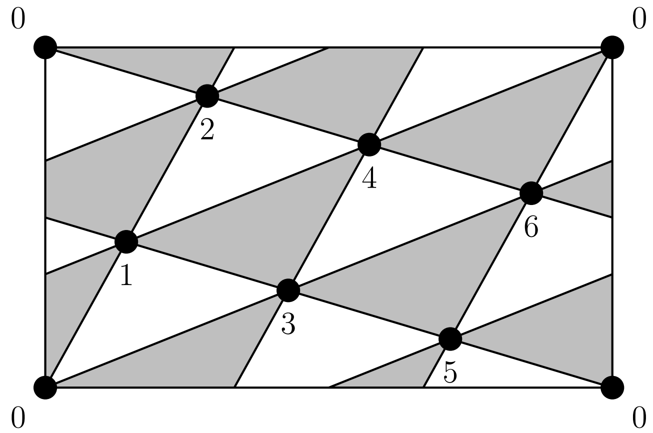

Figure 1: A biembedding of a pair of STS(7) in the torus.

Steiner triple systems obtained from the Bose construction have self-embeddings in both orientable and nonorientable surfaces [4, 8]. Further investigation of self-embeddings of Steiner triple systems was carried out in [10], in which the existence of self-embeddings in nonorientable surfaces of Steiner triple systems from a a large and well-structured class of systems (2-rotational Steiner triple systems) was established.

Self-embeddings of other combinatorial designs have also been studied and infinite families established [3, 12, 17].

Consider a triangular embedding of the complete tripartite graph $K_{n,n,n}$. Such an embedding is face 2-colourable if and only if the surface is orientable [7]. In this case the faces in each colour class can be regarded as the triples of a *transveral design with block size three* and we say the face 2-coloured embedding is a *biembedding* of the two transversal designs; such biembeddings exist for all n [16]. If the two transversal designs forming a biembedding are isomorphic the biembedding is called a *self-embedding*. The existence of self-embeddings of transversal designs has been studied in [3] and [17].

Next consider a properly face two-coloured embedding of the complete graph K_n in a surface in which all the faces are cycles of length m. Then the faces of each colour class form an *m-cycle system of order n* and we say that the embedding is a biembedding of the two cycle systems. Necessary and sufficient conditions for the existence of biembeddings of m-cycle system of order $2m+1$ for both orientable and nonorientable surfaces were established in [12]. In the orientable case the biembeddings provided in [12] are self-embeddings (the cycle systems obtained from the face colour classes are isomorphic).

Recall that in the case of transversal designs self-embeddings are necessarily in orientable surfaces. Although biembeddings of symmetric cycle systems are not necessarily in orientable surfaces the only known results for self-embeddings of symmetric systems are for orientable surfaces.

The above results on self-embeddings of transversal designs and symmetric cycle systems along with many biembedding (and self-embedding) results for Steiner triple systems rely on the systems (or fixed subsystems) containing a cyclic automorphism, e.g. the results in [9, 10, 15, 18]. In this paper, rather than using the structure provided by a cyclic automorphism of a (sub)system, we use the structure of the subsystems isomorphic to affine Steiner triple systems of order $v = 3^n$ (which have the automorphism group $A\Sigma L(n, 3)$, the affine semilinear group [2]).

2 Steiner triple system constructions

Before establishing the self-embeddings we are concerned with we will discuss some well known constructions of Steiner triple systems. Details on these and many other constructions of Steiner triple systems can be found in [2].

Let $(V, \mathcal{B})$ be a Steiner triple system and suppose that $\{x, y, z\} \in \mathcal{B}$; then, for ease of notation, we will often denote this triple by $xyz \in \mathcal{B}$.

Let $V = \mathbb{Z}_3^k$ and let $\mathcal{B}$ be the set of triples $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\mathbf{x}+\mathbf{y}+\mathbf{z} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \neq \mathbf{x}$. Then $(V, \mathcal{B})$ forms an STS(3^k) called an *affine Steiner triple system* and is denoted by AG$(k, 3)$.

We now consider two recursive constructions of Steiner triple systems.

Let $S = (V, \mathcal{B})$ be a Steiner triple system of order v. Let $W = (V \times \mathbb{Z}_2) \cup \{\infty\}$. For ease of notation, for $x \in V$ we will represent the element $(x,0)$ as x and the element $(x,1)$ by $\overline{x}$. Let

$$\mathcal{C} = \{xyz, x\overline{y}\overline{z}, \overline{x}y\overline{z}, \overline{x}\overline{y}z \mid xyz \in \mathcal{B}\} \cup \{\infty x\overline{x} \mid x \in V\}.$$

Then $(W,\mathcal{C})$ is an STS$(2v+1)$. We say that $(W,\mathcal{C})$ is a *doubled Steiner triple system*, more specifically it is a *doubling* of the system $S=(V,\mathcal{B})$ and we will denote it by $\delta(S)$.

Let $S=(V,\mathcal{B})$ be a Steiner triple system of order v. Let $U = V \times \mathbb{Z}_3$ and

$$\begin{aligned}\mathcal{D} \;=\; \{&(x,0)(x,1)(x,2),(y,0)(y,1)(y,2),(z,0)(z,1)(z,2),\\ &(x,0)(y,0)(z,0),(x,0)(y,1)(z,2),(x,0)(y,2)(z,1),\\ &(x,1)(y,0)(z,2),(x,1)(y,1)(z,1),(x,1)(y,2)(z,0),\\ &(x,2)(y,0)(z,1),(x,2)(y,1)(z,0),(x,2)(y,2)(z,2) \mid xyz \in \mathcal{B}\}.\end{aligned}$$

Then $(U,\mathcal{D})$ is an STS$(3v)$; we say that $(W,\mathcal{C})$ is a *tripling* of the Steiner triple system $S=(V,\mathcal{B})$. (The version of tripling described here is a particular instance of the more general *singular direct product* of triple systems, see [2, pg 39].)

Observe that the tripling of AG$(k,3)$ yields a Steiner triple system isomorphic to AG$(k+1,3)$. Before proving the main result of this paper we use this observation to prove that for every affine Steiner triple system AG$(k,3)$ there exists a biembedding of a pair of Steiner triple systems in which one of the systems is isomorphic to AG$(k,3)$. When $k \geq 2$ these biembeddings are in nonorientable surfaces and when $k \geq 3$ they are not self-embeddings.

Given a pair of Steiner triple systems of order v, say $S=(V,\mathcal{B})$ and $T=(V,\mathcal{C})$, construct a pseudo-surface P in the following manner. Each triple $xyz \in \mathcal{B}\cup\mathcal{C}$ corresponds to a face with vertices x, y and z and edges between each pair of vertices. The pseudo-surface is obtained by identifying edges in these triples if they have the same end-vertices. If the rotation at each vertex, $v \in V$ say, is a *full rotation*, i.e. the rotation at v consists of a single permutation that contains each vertex in $V \setminus \{v\}$ exactly once, then the pseudo-surface is a surface. See [13] for details.

The following theorem is a special case of the nonorientable analogue of [8, Construction 6]. We provide a simplified proof for this special case.

Theorem 2.1 (Grannell, Griggs & Širáň, [8])**.** *Suppose that S and T are a pair of STS(n) on the same vertex set V such that S and T form a biembedding in a nonorientable surface and where T contains a parallel class. Then there exists a biembedding of a pair of STS($3n$) such that one of the STS($3n$), S' say, can be obtained by tripling S and the second STS($3n$), T' say, contains a parallel class.*

Proof. First note that for Steiner triple system to contain a parallel class its order must be 3 modulo 6. Let $S=(V,\mathcal{B})$ and $T=(V,\mathcal{C})$ be Steiner triple systems of order $n=6m+3$. Suppose that $\mathcal{P}\subset\mathcal{C}$ is parallel class of triples in T. Define S' to be the tripling of S and define $T'=(V\times\mathbb{Z}_3,\mathcal{C}')$ where, denoting $(v,i)\in V\times\mathbb{Z}_3$ as

v^i,

$$\begin{aligned}\mathcal{C}' &= \{a^ib^jc^k \mid abc \in \mathcal{C} \setminus \mathcal{P} \text{ and } i+j+k \equiv 0 \pmod 3\} \cup \\ &\quad \{a^0a^1c^2,\ a^0a^2c^1,\ a^0b^0b^2,\ a^0b^1c^0,\ a^1a^2c^0,\ a^1b^0b^1, \\ &\quad a^1b^2c^1,\ a^2b^1b^2,\ a^2b^0c^2,\ b^0c^0c^1,\ b^1c^1c^2,\ b^2c^0c^2 \mid abc \in \mathcal{P}\}.\end{aligned}$$

Note that the choice of a, b and c for each triple in the parallel class is independent of the other triples.

Let $abc \in \mathcal{P}$; we show that there is a full rotation around each of the vertices in $\{a,b,c\} \times \mathbb{Z}_3$. Note that for any triple $def \in (\mathcal{B} \cup \mathcal{C}) \setminus \mathcal{P}$, all of the triples of the form $d^ie^jf^k$ for $i+j+k \equiv 0 \pmod 3$ are in either S' or in T'. Denote the rotations around a, b and c in the biembedding of S and T as

$$\begin{aligned} a: &\quad (b\, x_1\, x_2\, x_3\, \ldots\, x_{6m}\, c) \\ b: &\quad (c\, y_1\, y_2\, y_3\, \ldots\, y_{6m}\, a) \\ c: &\quad (a\, z_1\, z_2\, z_3\, \ldots\, z_{6m}\, b) \end{aligned}$$

Then the rotations around each vertex in $\{a,b,c\} \times \mathbb{Z}_3$ in the pseudo-surface obtained from S' and T' are given below.

$$\begin{aligned} a^0: &\quad (b^0\, x_1^0\, x_2^0\, x_3^0\, \ldots\, x_{6m}^0\, c^0\, b^1\, x_1^2\, x_2^1\, x_3^2\, \ldots\, x_{6m}^1\, c^2\, a^1\, a^2\, c^1\, x_{6m}^2\, \ldots\, x_3^1\, x_2^2\, x_1^1\, b^2) \\ a^1: &\quad (b^1\, x_1^1\, x_2^1\, x_3^1\, \ldots\, x_{6m}^1\, c^1\, b^2\, x_1^0\, x_2^2\, x_3^0\, \ldots\, x_{6m}^2\, c^0\, a^2\, a^0\, c^2\, x_{6m}^0\, \ldots\, x_3^2\, x_2^0\, x_1^2\, b^0) \\ a^2: &\quad (b^2\, x_1^2\, x_2^2\, x_3^2\, \ldots\, x_{6m}^2\, c^2\, b^0\, x_1^1\, x_2^0\, x_3^1\, \ldots\, x_{6m}^0\, c^1\, a^0\, a^1\, c^0\, x_{6m}^1\, \ldots\, x_3^0\, x_2^1\, x_1^0\, b^1) \end{aligned}$$

$$\begin{aligned} b^0: &\quad (c^1\, y_1^2\, y_2^1\, y_3^2\, \ldots\, y_{6m}^1\, a^2\, c^2\, y_1^1\, y_2^2\, y_3^1\, \ldots\, y_{6m}^2\, a^1\, b^1\, b^2\, a^0\, y_{6m}^0\, \ldots\, y_3^0\, y_2^0\, y_1^0\, c^0) \\ b^1: &\quad (c^2\, y_1^0\, y_2^2\, y_3^0\, \ldots\, y_{6m}^2\, a^0\, c^0\, y_1^2\, y_2^0\, y_3^2\, \ldots\, y_{6m}^0\, a^2\, b^2\, b^0\, a^1\, y_{6m}^1\, \ldots\, y_3^1\, y_2^1\, y_1^1\, c^1) \\ b^2: &\quad (c^0\, y_1^1\, y_2^0\, y_3^1\, \ldots\, y_{6m}^0\, a^1\, c^1\, y_1^0\, y_2^1\, y_3^0\, \ldots\, y_{6m}^1\, a^0\, b^0\, b^1\, a^2\, y_{6m}^2\, \ldots\, y_3^2\, y_2^2\, y_1^2\, c^2) \end{aligned}$$

$$\begin{aligned} c^0: &\quad (a^2\, z_1^1\, z_2^2\, z_3^1\, \ldots\, z_{6m}^2\, b^1\, a^0\, z_1^0\, z_2^0\, z_3^0\, \ldots\, z_{6m}^0\, b^0\, c^1\, c^2\, b^2\, z_{6m}^1\, \ldots\, z_3^2\, z_2^1\, z_1^2\, a^1) \\ c^1: &\quad (a^0\, z_1^2\, z_2^0\, z_3^2\, \ldots\, z_{6m}^0\, b^2\, a^1\, z_1^1\, z_2^1\, z_3^1\, \ldots\, z_{6m}^1\, b^1\, c^2\, c^0\, b^0\, z_{6m}^2\, \ldots\, z_3^0\, z_2^2\, z_1^0\, a^2) \\ c^2: &\quad (a^1\, z_1^0\, z_2^1\, z_3^0\, \ldots\, z_{6m}^1\, b^0\, a^2\, z_1^2\, z_2^2\, z_3^2\, \ldots\, z_{6m}^2\, b^2\, c^0\, c^1\, b^1\, z_{6m}^0\, \ldots\, z_3^1\, z_2^0\, z_1^1\, a^0) \end{aligned}$$

As each of these are single cycles, i.e. full rotations, we have that the pseudo-surface is indeed a surface and hence is a biemebdding of S' and T'.

Note that the collection $\{a^0a^1c^2,\ a^2b^1b^2,\ b^0c^0c^1 \mid abc \in \mathcal{P}\}$ is a parallel class of T'. Hence, all that remains is to show that the surface we have obtained is nonorientable. To do so it is sufficient to show that the vertex rotations can not be consistently directed.

Suppose, for a contradiction, that the surface is orientable, in which case the surface satisfies Ringel's Rule Δ^*, see [13]. Without loss of generality we can choose the orientation of the face with vertices a^0, a^1 and a^2 so that in the facial walk these vertices appear in the cyclic order (a^0, a^1, a^2). This implies that the rotations at a^0,

a^1 and a^2 have the following, consistent, orderings.

$$\begin{array}{ll} a^0: & (\ldots\, b^2\, b^0 \ldots\, c^0\, b^1\, \ldots\, c^2\, a^1\, a^2\, c^1\, \ldots) \\ a^1: & (\ldots\, b^0\, b^1 \ldots\, c^1\, b^2\, \ldots\, c^0\, a^2\, a^0\, c^2\, \ldots) \\ a^2: & (\ldots\, b^1\, b^2 \ldots\, c^2\, b^0\, \ldots\, c^1\, a^0\, a^1\, c^0\, \ldots) \end{array}$$

The rotation at a^2 implies that the facial walk of the face containing vertices a^2, c^2 and b^0 has the cyclic ordering (a^2, c^2, b^0). So in order to be consistent with the rotation at a^2, the rotation at b^0 should be

$$b^0: \quad (\ldots\, c^0\, c^1 \ldots\, a^2\, c^2\, \ldots\, a^1\, b^1\, b^2\, a^0\, \ldots).$$

Now the rotation at a^1 implies that the facial walk of the face containing vertices a^1, b^0 and b^1 has the cyclic ordering (a^1, b^0, b^1), while the rotation at b^0 implies that facial walk should have cyclic ordering (b^0, a^1, b^1). □

Corollary 2.2. *For all $k \geq 2$ there exists a biembedding of a pair of STS(3^k) in a nonorientable surface in which one of the systems, S say, is isomorphic to AG($k,3$) and the other system, T say, contains a parallel class. Moreover, T is isomorphic to AG($k,3$) if and only if $k = 2$.*

Proof. Up to isomorphism there is a unique STS(9), and by [15] there exists a biembedding of a pair of STS(9) in a nonorientable surface (no orientable biembedding can exist as $9 \not\equiv 3$ or $7 \pmod{12}$). Hence, this biembedding is a self-embedding of AG$(2,3)$. Also note that AG$(2,3)$ contains a parallel class.

Repeated applications of Theorem 2.1 establishes the existence of the desired biembeddings. Recall that tripling AG$(k-1,3)$ yields an STS(3^k) isomorphic to AG$(k,3)$, so we have that one of the systems, S', in the constructed biembedding is isomorphic to AG$(k,3)$.

In order to complete the proof we need to show that for $k > 2$ the second system, T', is not isomorphic to AG$(k,3)$. To do so we make use of the observation that AG$(k,3)$ is a Hall triple system; in particular if $U = (V, \mathcal{D})$ is an STS(3^k) isomorphic to AG$(k,3)$, then if $abc, ade, bdf, ceg \in \mathcal{D}$, so is afg.

For $k \geq 2$, let $T = (V, \mathcal{C})$ with parallel class $\mathcal{P}$ be the STS(3^{k-1}) that biembeds with a copy of AG$(k-1,3)$ constructed as above. Consider a triple $abc \in \mathcal{P}$, then there exist triples $ade, bdf, ceg \in \mathcal{C} \setminus \mathcal{P}$. Hence $T' = (V \times \mathbb{Z}_3, \mathcal{C}')$, where $\mathcal{C}'$ is defined as in Theorem 2.1, contains the triples $a^0b^1c^0$, $a^0d^0e^0$, $b^1d^0f^2$ and $c^0e^0g^0$, but not $a^0f^2g^0$ (as the subscripts do not sum to zero modulo three). Thus T' is not isomorphic to AG$(k,3)$. □

It is our understanding that in [8] Grannell, Griggs and Širáň were unaware of the above application of the nonorientable analogue of their Construction 6 [11].

Note that when k is even any biembedding in which one of the systems is isomorphic to AG$(k,3)$ must necessarily be in a nonorientable surface (if k is even, then $3^k \equiv 9 \pmod{12}$). If a biembedding in an orientable surface in which one of the Steiner triple systems is isomorphic to AG$(3,3)$ can be found and the other system contains a parallel class, then repeated applications of Construction 6 from [8] could be used to show that there exist biembeddings in orientable surfaces in which one of the systems is isomorphic to AG$(k,3)$ for all odd k.

3 Self-embeddings

In order to construct a family of self-embeddings of Steiner triple systems we will use the following approach. We start with a well understood system (the affine system AG$(k, 3)$). We then take a second copy of the system and perform a simple derangement (the map ϕ below) on the point/vertex set to yield a second Steiner triple system. At this stage, in the manner described above, a pseudo-surface can be constructed from these two systems.

We then double the first system (the system AG$(k, 3)$) and perform a variant of the doubling construction on the second system (which yields a system isomorphic to the doubled AG$(k, 3)$). We choose this second doubling so that, together with the doubling of AG$(k, 3)$, we 'unpick' the pinch points of the pseudo-surface and yield the desired self-embedding.

In order to simplify the proof we express the second system as a doubling of AG$(k, 3)$ to which a permutation of the point/vertex set has been applied (we do so in such a manner that this corresponds to the variant of the doubling construction being applied to the image of AG$(k, 3)$ under ϕ).

In the following, when no confusion is likely, we will suppress commas when writing elements of $\mathbb{Z}_3^k$; that is if $\mathbf{x} = (x_1, x_2, x_3, \dots, x_k) \in \mathbb{Z}_3^k$, we write $\mathbf{x} = x_1x_2x_3 \dots x_k$. We also maintain the same notation as established when discussing doubling Steiner triple systems in Section 2; that is, we will denote

- the element $(\mathbf{x}, 0) = (x_1, x_2, x_3, \dots, x_k, 0) \in \mathbb{Z}_3^k \times \mathbb{Z}_2$ by $\mathbf{x} = x_1x_2x_3 \dots x_k$; and
- the element $(\mathbf{x}, 1) = (x_1, x_2, x_3, \dots, x_k, 1) \in \mathbb{Z}_3^k \times \mathbb{Z}_2$ by $\overline{\mathbf{x}} = \overline{x_1x_2x_3 \dots x_k}$.

Consider the group $\mathbb{Z}_3^k$. Let $i^*_{\mathbf{0}} = k$, and for $\mathbf{0} \neq \mathbf{x} = x_1x_2x_3 \dots x_k \in \mathbb{Z}_3^k$ let $i^*_{\mathbf{x}} = \min\{i \mid x_i \neq 0\}$ denote the first non-zero coordinate of $\mathbf{x}$. Define the map $\phi : \mathbb{Z}_3^k \to \mathbb{Z}_3^k$ by

$$\phi : \mathbf{x} \mapsto (x_1 + 1)(x_2 + 1) \dots (x_{i^*_{\mathbf{x}}-1} + 1)(x_{i^*_{\mathbf{x}}} + 1)x_{i^*_{\mathbf{x}}+1} \dots x_k,$$

where the computations in the brackets are performed modulo three.

Consider the bijection $\iota : \mathbb{Z}_3^k \to \mathbb{Z}_{3^k}$ where

$$\iota : \mathbf{x} \mapsto (x_1 - 1) + (x_2 - 1)3 + (x_3 - 1)3^2 + \dots + (x_k - 1)3^{k-1},$$

where once again computations within brackets are performed modulo three. Observe that $\iota \circ \phi(\mathbf{x}) = \iota(\mathbf{x}) + 1 \pmod{3^k}$, and therefore ϕ is a permutation of length 3^k of the elements of $\mathbb{Z}_3^k$.

Example 3.1. Consider the case where $k = 2$, then under the map ϕ

$$11 \mapsto 21 \mapsto 01 \mapsto 12 \mapsto 22 \mapsto 02 \mapsto 10 \mapsto 20 \mapsto 00 \mapsto 11,$$

and

$$\iota(11) = 0,\ \iota(21) = 1,\ \iota(01) = 2,\ \iota(12) = 3,\ \iota(22) = 4,$$
$$\iota(02) = 5,\ \iota(10) = 6,\ \iota(20) = 7,\ \iota(00) = 8.$$

Let $W = (\mathbb{Z}_3^k \times \mathbb{Z}_2) \cup \{\infty\}$. We extend the map ϕ to a map $\psi : W \to W$ given by

$$\begin{aligned}\psi \quad : \quad & x_1x_2x_3\dots x_k0 \mapsto \phi(x_1x_2x_3\dots x_k)1\\ & x_1x_2x_3\dots x_k1 \mapsto x_1x_2x_3\dots x_k0\\ & \infty \mapsto \infty.\end{aligned}$$

In a slight abuse of notation we will often write $\psi(\mathbf{x}) = \overline{\phi(\mathbf{x})}$.

We are now ready to describe our construction. Let $(\mathbb{Z}_3^k, \mathcal{B})$ be the affine system AG$(k,3)$ and take $(W,\mathcal{C})$ to be the resulting doubled system $\delta(\text{AG}(k,3))$. We will show that $(W,\mathcal{C})$ has a biembedding with the Steiner triple system $(W,\mathcal{D})$, where $\mathcal{D} = \{\psi(\mathbf{x})\psi(\mathbf{y})\psi(\mathbf{z}) \mid \mathbf{xyz} \in \mathcal{C}\}$. Denote the properly face 2-coloured pseudo-surface generated by $(W,\mathcal{C})$ and $(W,\mathcal{D})$ as $\mathcal{S}$; then, as before, establishing that there is a full rotation at each vertex would imply that $\mathcal{S}$ is a surface. We begin with the rotation about the vertex ∞.

Lemma 3.1. *In the pseudo-surface $\mathcal{S}$ there is full rotation about the vertex ∞.*

Proof. For each $\mathbf{x} \in \mathbb{Z}_3^k$, $\{\infty, \mathbf{x}, \overline{\mathbf{x}}\} \in \mathcal{C}$ and $\{\infty, \mathbf{x}, \overline{\phi(\mathbf{x})}\} \in \mathcal{D}$. As ϕ is a cycle of length 3^k we have that the rotation

$$\infty : \ \overline{\mathbf{0}}\ \ \mathbf{0}\ \ \overline{\phi(\mathbf{0})}\ \ \phi(\mathbf{0})\ \ \overline{\phi^2(\mathbf{0})}\ \ \phi^2(\mathbf{0})\ \ \dots\ \ \overline{\phi^{3^k-1}(\mathbf{0})}\ \ \phi^{3^k-1}(\mathbf{0})$$

is a cycle of length $2(3^k)$. □

For a fixed $\mathbf{x} = x_1x_2x_3\dots x_k \in \mathbb{Z}_3^k$, define the following maps, for $1 \le r \le k$, $\gamma_r : \mathbb{Z}_3^r \to \mathbb{Z}_3^r$, and $\gamma : \mathbb{Z}_3 \to \mathbb{Z}_3$ by

$$\begin{aligned}\gamma_r \quad &: \quad \mathbf{y} \mapsto (-x_1 - y_1)(-x_2 - y_2)(-x_3 - y_3)\dots(-x_r - y_r)\\ \gamma \quad &: \quad z \mapsto -x_k - z,\end{aligned}$$

where the computations are made modulo three. Note that γ_r and γ are involutions.

Using these maps we can re-express the subsets $\mathcal{C}'$ and $\mathcal{D}'$ of triples from $\mathcal{C}$ and $\mathcal{D}$, respectively, that do not contain the vertex ∞:

$$\mathcal{C}' = \Big\{\{\mathbf{x},\mathbf{y},\gamma_k(\mathbf{y})\}, \{\mathbf{x},\overline{\mathbf{y}},\overline{\gamma_k(\mathbf{y})}\}, \{\overline{\mathbf{x}},\mathbf{y},\overline{\gamma_k(\mathbf{y})}\}, \{\overline{\mathbf{x}},\overline{\mathbf{y}},\gamma_k(\mathbf{y})\} \mid \mathbf{x},\mathbf{y} \in \mathbb{Z}_3^k, \mathbf{x} \neq \mathbf{y}\Big\};$$

$$\begin{aligned}\mathcal{D}' = \Big\{&\{\overline{\phi(\mathbf{x})}, \overline{\phi(\mathbf{y})}, \overline{\phi(\gamma_k(\mathbf{y}))}\}, \{\overline{\phi(\mathbf{x})}, \mathbf{y}, \gamma_k(\mathbf{y})\}, \{\mathbf{x}, \overline{\phi(\mathbf{y})}, \gamma_k(\mathbf{y})\},\\ &\{\mathbf{x},\mathbf{y},\overline{\phi(\gamma_k(\mathbf{y}))}\} \mid \mathbf{x},\mathbf{y} \in \mathbb{Z}_3^k, \mathbf{x} \neq \mathbf{y}\Big\}.\end{aligned}$$

For the remainder of the paper we consider $\mathbf{x} = x_1x_2x_3\dots x_k \in \mathbb{Z}_3^k$ to be a fixed vertex in the pseudo-surface $\mathcal{S}$. In Subsection 3.1 we prove that the rotation about vertex $\mathbf{x}$ is a single cycle, and in Subsection 3.2 we prove that the rotation about vertex $\overline{\phi(\mathbf{x})}$ is a single cycle. Together with Lemma 3.1 these results imply that $\mathcal{S}$ is a surface. Finally in Subsection 3.3 we verify that $\mathcal{S}$ is nonorientable.

3.1 The rotation about x

We will write ϕ_{k-1} and ϕ_k to denote the map ϕ when it is applied to a vector in $\mathbb{Z}_3^{k-1}$ and a vector in $\mathbb{Z}_3^k$, respectively. The subset of triples from $\mathcal{C}$ containing $\mathbf{x}$ is

$$\Big\{\{\mathbf{x},\mathbf{y},\gamma_k(\mathbf{y})\},\{\mathbf{x},\overline{\mathbf{y}},\overline{\gamma_k(\mathbf{y})}\}\mid \mathbf{y}\in\mathbb{Z}_3^k,\mathbf{y}\neq\mathbf{x}\Big\}\cup\Big\{\{\mathbf{x},\infty,\overline{\mathbf{x}}\}\Big\},$$

and, as γ_k is an involution, the subset of triples from $\mathcal{D}$ containing $\mathbf{x}$ is

$$\Big\{\{\mathbf{x},\mathbf{y},\overline{\phi_k(\gamma_k(\mathbf{y}))}\}\mid \mathbf{y}\in\mathbb{Z}_3^k,\mathbf{y}\neq\mathbf{x}\Big\}\cup\Big\{\{\mathbf{x},\infty,\overline{\phi_k(\mathbf{x})}\}\Big\}.$$

Thus for some j the rotation about $\mathbf{x}$ contains the following cycle.

$$\begin{aligned}\mathbf{x}:\quad &\infty\xrightarrow{\mathcal{C}}\overline{\mathbf{x}}=\overline{\mathbf{b}_1}\xrightarrow{\mathcal{D}}\gamma_k(\phi_k^{-1}(\mathbf{b}_1))\xrightarrow{\mathcal{C}}\phi_k^{-1}(\mathbf{b}_1)\xrightarrow{\mathcal{D}}\overline{\phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_1)))}=\overline{\mathbf{b}_2}\\
&\xrightarrow{\mathcal{C}}\overline{\gamma_k(\mathbf{b}_2)}=\overline{\mathbf{b}_3}\xrightarrow{\mathcal{D}}\gamma_k(\phi_k^{-1}(\mathbf{b}_3))\xrightarrow{\mathcal{C}}\phi_k^{-1}(\mathbf{b}_3)\xrightarrow{\mathcal{D}}\overline{\phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_3)))}=\overline{\mathbf{b}_4}\\
&\xrightarrow{\mathcal{C}}\overline{\gamma_k(\mathbf{b}_4)}=\overline{\mathbf{b}_5}\xrightarrow{\mathcal{D}}\cdots\xrightarrow{\mathcal{C}}\overline{\gamma_k(\mathbf{b}_{2j-2})}=\overline{\mathbf{b}_{2j-1}}\xrightarrow{\mathcal{D}}\gamma_k(\phi_k^{-1}(\mathbf{b}_{2j-1}))\\
&\xrightarrow{\mathcal{C}}\phi_k^{-1}(\mathbf{b}_{2j-1})\xrightarrow{\mathcal{D}}\overline{\phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_{2j-1})))}=\overline{\mathbf{b}_{2j}}\xrightarrow{\mathcal{C}}\overline{\gamma_k(\mathbf{b}_{2j})}=\overline{\mathbf{b}_{2j+1}}=\overline{\phi_k(\mathbf{x})}\\
&\xrightarrow{\mathcal{D}}\infty\end{aligned}$$

Where '$\xrightarrow{\mathcal{C}}$' and '$\xrightarrow{\mathcal{D}}$' indicate from which set of triples the face yielding this part of the rotation is obtained.

We will prove that the rotation around $\mathbf{x}$ consists precisely of this cycle; i.e. that the cycle has length $2(3^k)$. We will use the following notation, which is introduced above, for all $2\leq 2\ell\leq 2j$,

$$\mathbf{b}_{2\ell}=\phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_{2\ell-1})))\quad\text{ and }\quad\mathbf{b}_{2\ell+1}=\gamma_k(\mathbf{b}_{2\ell}).$$

It suffices to show that $\pi_k^{\mathbf{x}}=(\infty\,\mathbf{b_1}\,\mathbf{b_2}\,\cdots\,\mathbf{b_{2j}}\,\mathbf{b_{2j+1}})$ is a permutation of length 3^k+1; we do so by induction (the base case, for AG(2, 3), is easily established). Let $m=3^{k-1}$ and assume that for $\mathbf{x}'=x_1x_2\cdots x_{k-1}$ the permutation $\pi_{k-1}^{\mathbf{x}'}$ has length $m+1$ and denote it as

$$\pi_{k-1}^{\mathbf{x}'}=(\,\infty\;\;\phi_{k-1}(\mathbf{a}_1)\;\;\phi_{k-1}(\mathbf{a}_2)\;\;\cdots\;\;\phi_{k-1}(\mathbf{a}_m)\,)\,,$$

where $\phi_{k-1}(\mathbf{a}_1)=\mathbf{x}'$ and $\mathbf{a}_m=\mathbf{x}'$. We now partition the set $\{2,3,...,3m\}$ into two disjoint sets I_0 and I_1 as follows. There is a unique $1\leq r\leq m$ such that $\mathbf{a}_r=\mathbf{0}$; write $r=2s$ or $r=2s+1$ depending on the parity of r. If $r=1$, set $I_0=\{2m+1\}$; otherwise, set

$$I_0=\{2s+1,2m+2s+1,2m-2s+1\}.$$

In either case, let $I_1=\{2,3,...,3m\}\setminus I_0$.

Claim 3.2. *The permutation $\pi_k^{\mathbf{x}}$ is given by the cycle*

$$\begin{pmatrix}\infty & \phi_k(\mathbf{a}_1 f_1) & \phi_k(\mathbf{a}_2 f_2) & \dots & \phi_k(\mathbf{a}_{m-1} f_{m-1}) & \phi_k(\mathbf{a}_m f_m) \\ & \phi_k(\mathbf{a}_m f_{m+1}) & \phi_k(\mathbf{a}_{m-1} f_{m+2}) & \dots & \phi_k(\mathbf{a}_2 f_{2m-1}) & \phi_k(\mathbf{a}_1 f_{2m}) \\ & \phi_k(\mathbf{a}_1 f_{2m+1}) & \phi_k(\mathbf{a}_2 f_{2m+2}) & \dots & \phi_k(\mathbf{a}_{m-1} f_{3m-1}) & \phi_k(\mathbf{a}_m f_{3m})\end{pmatrix}$$

where

$$f_1 = \begin{cases} x_k & \text{if } \mathbf{a}_1 \neq \mathbf{0} \\ x_k - 1 & \text{if } \mathbf{a}_1 = \mathbf{0}, \end{cases}$$

and for $i > 1$

$$f_i = \begin{cases} \gamma(f_{i-1}) - 1 & \text{if } i \in I_0 \text{ and } \mathbf{a}_1 \neq \mathbf{0} \\ \gamma(f_{i-1}) - 2 & \text{if } i \in I_0 \text{ and } \mathbf{a}_1 = \mathbf{0} \\ \gamma(f_{i-1}) & \text{if } i \in I_1. \end{cases}$$

The following identities will be useful when establishing the value of each f_i in the permutation $\pi_k^{\mathbf{x}}$.

From $\phi_{k-1}(\mathbf{a}_{2\ell}) = \phi_{k-1}(\gamma_{k-1}(\phi_{k-1}^{-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1})))) = \phi_{k-1}(\gamma_{k-1}(\mathbf{a}_{2\ell-1}))$, we obtain, for $2 \leq 2\ell \leq m-1$,

$$\mathbf{a}_{2\ell} = \gamma_{k-1}(\mathbf{a}_{2\ell-1}) \text{ and } \gamma_{k-1}(\mathbf{a}_{2\ell}) = \mathbf{a}_{2\ell-1}. \tag{1}$$

Similarly, for $2 \leq 2\ell \leq m-1$,

$$\phi_{k-1}(\mathbf{a}_{2\ell+1}) = \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell})) \text{ and } \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell+1})) = \phi_{k-1}(\mathbf{a}_{2\ell}). \tag{2}$$

Finally, we note that $-x_i - x_i = -2x_i = x_i$ for all $x_i \in \mathbb{Z}_3$, so we have

$$\gamma_{k-1}(\mathbf{x}') = \mathbf{x}' \text{ and } \gamma(x_k) = x_k. \tag{3}$$

We proceed by induction on i. First note that if $\mathbf{a}_1 \neq \mathbf{0}$, then $\mathbf{b}_1 = \mathbf{x} = \mathbf{x}'x_k = \phi_{k-1}(\mathbf{a}_1)x_k = \phi_k(\mathbf{a}_1 x_k)$, so $f_1 = x_k$; and if $\mathbf{a}_1 = \mathbf{0}$, then $\mathbf{b}_1 = \mathbf{x} = \mathbf{x}'x_k = \phi_{k-1}(\mathbf{a}_1)x_k = \phi_k(\mathbf{a}_1(x_k - 1))$, so $f_1 = x_k - 1$. Assume now that Claim 3.2 holds for all $1 \leq i \leq n-1$; we will show the claim holds for $i = n$.

Lemma 3.3. *Suppose that $n \in I_0$. Then*

$$f_n = \begin{cases} \gamma(f_{n-1}) - 1; & \text{if } \mathbf{a}_1 \neq \mathbf{0} \\ \gamma(f_{n-1}) - 2; & \text{if } \mathbf{a}_1 = \mathbf{0}. \end{cases}$$

Proof. Recall that r is the unique index such that $\mathbf{a}_r = \mathbf{0}$. We will consider three cases, when $r = 1$, when r is even ($r = 2s$), and when r is odd ($r = 2s + 1$) and not equal to one.

Case 1: $r = 1$. Then $n = 2m + 1$ and

$$\begin{aligned}\mathbf{b}_{2m+1} &= \gamma_k(\mathbf{b}_{2m}) = \gamma_k(\phi_k(\mathbf{a}_1 f_{2m})) = \gamma_k(\phi_{k-1}(\mathbf{a}_1)(f_{2m}+1)) = \gamma_{k-1}(\mathbf{x}')\gamma(f_{2m}+1) \\ &\overset{(3)}{=} \mathbf{x}'(\gamma(f_{2m}) - 1) = \phi_{k-1}(\mathbf{a}_1)(\gamma(f_{2m}) - 1) = \phi_k(\mathbf{a}_1(\gamma(f_{2m}) - 2)).\end{aligned}$$

Which implies that $f_{2m+1} = \gamma(f_{2m}) - 2$.

Case 2: $r = 2s$; that is $\mathbf{a}_{2s} = \mathbf{0}$. So $\mathbf{a}_{2s+1} \neq \mathbf{0}$. First assume that $n \in \{2s+1, 2m+2s+1\}$. Then

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2s} f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2s})(f_{n-1}+1)) \\ &= \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2s}))\gamma(f_{n-1}+1) \overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2s+1})(\gamma(f_{n-1})-1) \\ &= \phi_k(\mathbf{a}_{2s+1}(\gamma(f_{n-1})-1)).\end{aligned}$$

Which implies that $f_n = \gamma(f_{n-1}) - 1$.

So now assume that $n = 2m - 2s + 1$; then

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2s+1} f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2s+1}) f_{n-1}) \\ &= \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2s+1}))\gamma(f_{n-1}) \overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2s})\gamma(f_{n-1}) = \phi_k(\mathbf{a}_{2s}(\gamma(f_{n-1})-1)).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1}) - 1$.

Case 3: $r = 2s+1$ and $r \neq 1$; that is $\mathbf{a}_{2s+1} = \mathbf{0}$. So $\mathbf{a}_{2s} \neq \mathbf{0}$. First assume that $n \in \{2s+1, 2m+2s+1\}$. Then

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2s} f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2s}) f_{n-1}) = \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2s}))\gamma(f_{n-1}) \\ &\overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2s+1})\gamma(f_{n-1}) = \phi_k(\mathbf{a}_{2s+1}(\gamma(f_{n-1})-1)).\end{aligned}$$

Thus, $f_n = \gamma(f_{n-1}) - 1$.

Now assume that $n = 2m - 2s + 1$. Then

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2s+1} f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2s+1})(f_{n-1}+1)) \\ &= \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2s+1}))\gamma(f_{n-1}+1) \overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2s})(\gamma(f_{n-1})-1) \\ &= \phi_k(\mathbf{a}_{2s}(\gamma(f_{n-1})-1)).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1}) - 1$. □

Lemma 3.4. *Suppose that $n \in I_1$. Then $f_n = \gamma(f_{n-1})$.*

Proof. We consider three cases, when n is even and $n \neq m+1$, when n is odd, and when $n = m+1$.

Case 1: n is even and $n \neq m+1$. First assume that $n = 2\ell$ or $n = 2m + 2\ell$ for $2 \leq 2\ell \leq m-1$. Then

$$\begin{aligned}\mathbf{b}_n &= \phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_{n-1}))) = \phi_k(\gamma_k(\phi_k^{-1}(\phi_k(\mathbf{a}_{2\ell-1} f_{n-1})))) = \phi_k(\gamma_k(\mathbf{a}_{2\ell-1} f_{n-1})) \\ &= \phi_k(\gamma_{k-1}(\mathbf{a}_{2\ell-1})\gamma(f_{n-1})) \overset{(1)}{=} \phi_k(\mathbf{a}_{2\ell}\gamma(f_{n-1})).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1})$.

Now assume that $n = 2m - 2\ell + 2$ for $2 \leq 2\ell \leq m-1$. Then

$$\begin{aligned}\mathbf{b}_n &= \phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_{n-1}))) = \phi_k(\gamma_k(\phi_k^{-1}(\phi_k(\mathbf{a}_{2\ell} f_{n-1})))) = \phi_k(\gamma_k(\mathbf{a}_{2\ell} f_{n-1})) \\ &= \phi_k(\gamma_{k-1}(\mathbf{a}_{2\ell})\gamma(f_{n-1})) \overset{(1)}{=} \phi_k(\mathbf{a}_{2\ell-1}\gamma(f_{n-1})).\end{aligned}$$

Thus $f_n = \gamma(f_{n-1})$.

Case 2: n is odd. First assume that $n = 2\ell+1$ or $n = 2m+2\ell+1$ for $2 \le 2\ell \le m-1$. Note that, as $n \in I_1$, $\ell \ne s$, so $\mathbf{a}_{2\ell+1} \ne \mathbf{0}$ and $\mathbf{a}_{2\ell} \ne \mathbf{0}$. So,

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2\ell}f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2\ell})f_{n-1}) = \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell}))\gamma(f_{n-1}) \\ &\overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2\ell+1})\gamma(f_{n-1}) = \phi_k(\mathbf{a}_{2\ell+1}\gamma(f_{n-1})).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1})$.

Next assume that $n = 2m - 2\ell + 1$ for $2 \le 2\ell \le m - 1$. Again, as $n \in I_1$, $\ell \ne s$, so $\mathbf{a}_{2\ell+1} \ne \mathbf{0}$ and $\mathbf{a}_{2\ell} \ne \mathbf{0}$. Hence,

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\phi_k(\mathbf{a}_{2\ell+1}f_{n-1})) = \gamma_k(\phi_{k-1}(\mathbf{a}_{2\ell+1})f_{n-1}) \\ &= \gamma_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell+1}))\gamma(f_{n-1}) \overset{(2)}{=} \phi_{k-1}(\mathbf{a}_{2\ell})\gamma(f_{n-1}) = \phi_k(\mathbf{a}_{2\ell}\gamma(f_{n-1})).\end{aligned}$$

So, $f_n = \gamma(f_{n-1})$.

Finally assume that $n = 2m + 1$. Then, as $\mathbf{a}_1 \ne \mathbf{0}$,

$$\begin{aligned}\mathbf{b}_{2m+1} &= \gamma_k(\mathbf{b}_{2m}) = \gamma_k(\phi_k(\mathbf{a}_1 f_{2m})) = \gamma_k(\phi_{k-1}(\mathbf{a}_1)f_{2m}) = \gamma_{k-1}(\mathbf{x}')\gamma(f_{2m}) \\ &\overset{(3)}{=} \mathbf{x}'\gamma(f_{2m}) = \phi_{k-1}(\mathbf{a}_1)\gamma(f_{2m}) = \phi_k(\mathbf{a}_1\gamma(f_{2m})).\end{aligned}$$

We have that $f_{2m+1} = \gamma(f_{2m})$.

Case 3: $n = m + 1$. If $f_m = x_k$, then $\mathbf{a}_m f_m = \mathbf{x}$ and ∞ follows $\mathbf{b}_m = \phi_k(\mathbf{x})$ in the rotation around $\mathbf{x}$. Thus, we first show that $f_m \ne x_k$. Note that $\gamma(y) = x_k$ if and only if $y = x_k$. Recall that r is the unique index such that $\mathbf{a}_r = \mathbf{0}$, and set $r^* = r$ if r is odd and set $r^* = r + 1$ if r is even (note that, as $m = 3^{k-1}$, $r^* \le m$). From above we have that $f_i = \gamma(f_{i-1})$ for all $2 \le i \le m$, $i \ne r^*$. Thus, $f_i = x_k$ for $1 \le i \le r^* - 1$ and $f_i \ne x_k$ for $r^* \le i \le m$. Recall that $\mathbf{a}_m = \mathbf{x}'$; then

$$\begin{aligned}\mathbf{b}_{m+1} &= \phi_k(\gamma_k(\phi_k^{-1}(\mathbf{b}_m))) = \phi_k(\gamma_k(\phi_k^{-1}(\phi_k(\mathbf{x}'f_m)))) = \phi_k(\gamma_k(\mathbf{x}'f_m)) \\ &= \phi_k(\gamma_{k-1}(\mathbf{x}')\gamma(f_m)) \overset{(3)}{=} \phi_k(\mathbf{x}'\gamma(f_m)).\end{aligned}$$

Hence, $f_{m+1} = \gamma(f_m)$. □

From Lemmas 3.3 and 3.4, we know $\pi_k^{\mathbf{x}}$ is a permutation of length at least $3m + 1 = 3^k + 1$. Since $(W, \mathcal{C})$ and $(W, \mathcal{D})$ are Steiner triple systems, we know $\pi_k^{\mathbf{x}}$ is a permutation of length at most $3^k + 1$. Thus, Claim 3.2 is established, and the next result follows immediately.

Corollary 3.5. *There is a full rotation around the vertex* $\mathbf{x}$.

3.2 The rotation about $\overline{\phi(\mathbf{x})}$

The argument for the rotation about $\overline{\phi(\mathbf{x})}$ is of the same style as that for $\mathbf{x}$. We continue to use γ, γ_k, ϕ_{k-1}, and ϕ_k as in Subsection 3.1. We will also denote $\phi(\mathbf{x}) = \phi_k(\mathbf{x})$ by $\hat{x}_1\hat{x}_2\cdots\hat{x}_k$.

Define the following maps, for $1 \le r \le k$, $\hat{\gamma}_r : \mathbb{Z}_3^r \to \mathbb{Z}_3^r$; and $\hat{\gamma} : \mathbb{Z}_3 \to \mathbb{Z}_3$ by

$$\begin{aligned}\hat{\gamma}_r &: \mathbf{y} \mapsto (-\hat{x}_1 - y_1, -\hat{x}_2 - y_2, ..., -\hat{x}_r - y_r) \\ \hat{\gamma} &: z \mapsto -\hat{x}_k - z,\end{aligned}$$

where the computations are made modulo three. Note that $\hat{\gamma}_r$ and $\hat{\gamma}$ are involutions.

As $\hat{\gamma}_k$ is an involution, the subset of triples from $\mathcal{C}$ containing $\overline{\phi_k(\mathbf{x})}$ is

$$\left\{\{\overline{\phi_k(\mathbf{x})}, \mathbf{y}, \overline{\hat{\gamma}_k(\mathbf{y})}\} \mid \mathbf{y} \in \mathbb{Z}_3^k, \mathbf{y} \neq \phi_k(\mathbf{x})\right\} \cup \left\{\{\overline{\phi_k(\mathbf{x})}, \infty, \phi_k(\mathbf{x})\}\right\},$$

and, as $\mathbf{z} = \phi_k(\mathbf{y})$ implies that $\phi_k(\gamma_k(\mathbf{z})) = \phi_k(\gamma_k(\phi_k^{-1}(\mathbf{y})))$, the subset of triples from $\mathcal{D}$ containing $\overline{\phi_k(\mathbf{x})}$ is

$$\left\{\{\overline{\phi_k(\mathbf{x})}, \mathbf{y}, \gamma_k(\mathbf{y})\}, \{\overline{\phi_k(\mathbf{x})}, \overline{\mathbf{y}}, \overline{\phi_k(\gamma_k(\phi_k^{-1}(\mathbf{y})))}\} \mid \mathbf{y} \in \mathbb{Z}_3^k, \mathbf{y} \neq \phi_k(\mathbf{x})\right\} \cup$$

$$\left\{\{\overline{\phi_k(\mathbf{x})}, \infty, \mathbf{x}\}\right\}.$$

Hence, the rotation around $\overline{\phi_k(\mathbf{x})}$ contains the following cycle.

$$\overline{\phi_k(\mathbf{x})}: \quad \infty \xrightarrow{\mathcal{C}} \phi_k(\mathbf{x}) = \mathbf{b}_1 \xrightarrow{\mathcal{D}} \gamma_k(\mathbf{b}_1) = \mathbf{b}_2 \xrightarrow{\mathcal{C}} \overline{\hat{\gamma}_k(\mathbf{b}_2)} \xrightarrow{\mathcal{D}} \overline{\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_2))))}$$

$$\xrightarrow{\mathcal{C}} \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_2))))) = \mathbf{b}_3 \xrightarrow{\mathcal{D}} \gamma_k(\mathbf{b}_3) = \mathbf{b}_4 \xrightarrow{\mathcal{C}} \overline{\hat{\gamma}_k(\mathbf{b}_4)}$$

$$\xrightarrow{\mathcal{D}} \overline{\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_4))))} \xrightarrow{\mathcal{C}} \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_4))))) = \mathbf{b}_5 \xrightarrow{\mathcal{D}} \cdots$$

$$\xrightarrow{\mathcal{C}} \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{2j-2}))))) = \mathbf{b}_{2j-1} \xrightarrow{\mathcal{D}} \gamma_k(\mathbf{b}_{2j-1}) = \mathbf{b}_{2j} \xrightarrow{\mathcal{C}} \overline{\hat{\gamma}_k(\mathbf{b}_{2j})}$$

$$\xrightarrow{\mathcal{D}} \overline{\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{2j}))))} \xrightarrow{\mathcal{C}} \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{2j}))))) = \mathbf{x} \xrightarrow{\mathcal{D}} \infty$$

We will prove that the rotation around $\overline{\phi_k(\mathbf{x})}$ consists precisely of this cycle. We will use the following notation, which is introduced above, for all $2 \leq 2\ell \leq 2j$,

$$\mathbf{b}_{2\ell} = \gamma_k(\mathbf{b}_{2\ell-1}) \quad \text{and} \quad \mathbf{b}_{2\ell+1} = \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{2\ell}))))).$$

It suffices to show that $\omega_k^{\overline{\phi_k(\mathbf{x})}} = (\infty\, \mathbf{b_1}\, \mathbf{b_2}\, \cdots\, \mathbf{b_{2j}}\, \mathbf{b_{2j+1}})$ is a permutation of length 3^k+1; we do so by induction (once again, the base case, for AG$(2,3)$, is easily established). Let $m = 3^{k-1}$ and assume that for $\phi_{k-1}(\mathbf{x}') = \hat{x}_1\hat{x}_2\cdots\hat{x}_{k-1}$ the permutation $\omega_{k-1}^{\overline{\phi_{k-1}(\mathbf{x}')}}$ has length $m+1$ and denote it as

$$\omega_{k-1}^{\overline{\phi_{k-1}(\mathbf{x}')}} = (\,\infty \;\; \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1)) \;\; \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_2)) \;\; \ldots \;\; \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\,),$$

where $\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1)) = \phi_{k-1}(\mathbf{x}')$ and $\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m)) = \mathbf{x}'$.

We now partition the set $\{2, 3, ..., 3m\}$ into two disjoint sets J_0 and J_1 as follows. There is a unique $1 \leq r \leq m$ such that $\mathbf{a}_r = \mathbf{0}$; write $r = 2s$ or $r = 2s-1$ depending on the parity of r. If $r = m$, set $J_0 = \{m+1\}$; otherwise, set

$$J_0 = \{2s, 2m+2s, 2m-2s+2\}.$$

In either case let $J_1 = \{2, 3, ..., 3m\} \setminus J_0$.

Claim 3.6. *The permutation* $\omega_k^{\overline{\phi_k(\mathbf{x})}}$ *is given by the cycle*

$$\begin{pmatrix}\infty & \hat{\gamma}_k(\phi_k(\mathbf{a}_1 f_1)) & \hat{\gamma}_k(\phi_k(\mathbf{a}_2 f_2)) & \dots & \hat{\gamma}_k(\phi_k(\mathbf{a}_{m-1} f_{m-1})) & \hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m)) \\ & \hat{\gamma}_k(\phi_k(\mathbf{a}_m f_{m+1})) & \hat{\gamma}_k(\phi_k(\mathbf{a}_{m-1} f_{m+2})) & \dots & \hat{\gamma}_k(\phi_k(\mathbf{a}_2 f_{2m-1})) & \hat{\gamma}_k(\phi_k(\mathbf{a}_1 f_{2m})) \\ & \hat{\gamma}_k(\phi_k(\mathbf{a}_1 f_{2m+1})) & \hat{\gamma}_k(\phi_k(\mathbf{a}_2 f_{2m+2})) & \dots & \hat{\gamma}_k(\phi_k(\mathbf{a}_{m-1} f_{3m-1})) & \hat{\gamma}_k(\phi_k(\mathbf{a}_m f_{3m}))\end{pmatrix}$$

where

$$f_1 = \begin{cases} \hat{x}_k & \text{if } \mathbf{a}_1 \neq \mathbf{0} \\ \hat{x}_k - 1 & \text{if } \mathbf{a}_1 = \mathbf{0}, \end{cases}$$

and for $i > 1$

$$f_i = \begin{cases} \gamma(f_{i-1}) - 1 & \text{if } i \in J_0, \mathbf{x}' \neq \mathbf{0}, \text{ and } \mathbf{a}_m \neq \mathbf{0} \\ \gamma(f_{i-1}) - 2 & \text{if } i \in J_0, \mathbf{x}' \neq \mathbf{0}, \text{ and } \mathbf{a}_m = \mathbf{0} \\ \gamma(f_{i-1}) + 1 & \text{if } i \in J_1, i \text{ is even, and } \mathbf{x}' = \mathbf{0} \\ \gamma(f_{i-1}) & \text{otherwise.} \end{cases}$$

We again make use of Equation (3) from Subsection 3.1; additionally, the following new identities will be useful when establishing the value of each f_i in the permutation $\omega_k^{\overline{\phi_k(\mathbf{x})}}$. From

$$\begin{aligned} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell+1}) &= \hat{\gamma}_{k-1}(\phi_{k-1}(\gamma_{k-1}(\phi_{k-1}^{-1}(\hat{\gamma}_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell})))))) \\ &= \hat{\gamma}_{k-1}(\phi_{k-1}(\gamma_{k-1}(\mathbf{a}_{2\ell}))), \end{aligned}$$

we obtain, for $2 \leq 2\ell \leq m-1$,

$$\mathbf{a}_{2\ell+1} = \gamma_{k-1}(\mathbf{a}_{2\ell}) \text{ and } \gamma_{k-1}(\mathbf{a}_{2\ell+1}) = \mathbf{a}_{2\ell}. \tag{4}$$

Similarly, we obtain, for $2 \leq 2\ell \leq m-1$,

$$\begin{aligned} &\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell})) = \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1}))) \text{ and} \\ &\gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell}))) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1})). \end{aligned} \tag{5}$$

Finally, we note that $-\hat{x}_i - \hat{x}_i = -2\hat{x}_i = \hat{x}_i$ for all $\hat{x}_i \in \mathbb{Z}_3$, so

$$\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{x}')) = \phi_{k-1}(\mathbf{x}') \text{ and } \hat{\gamma}(\hat{x}_k) = \hat{x}_k. \tag{6}$$

Recall that $\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1)) = \phi_{k-1}(\mathbf{x}')$, so $\hat{\gamma}_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1))) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{x}'))$. Hence, by Equation (6), $\phi_{k-1}(\mathbf{a}_1) = \phi_{k-1}(\mathbf{x}')$, and therefore $\mathbf{a}_1 = \mathbf{x}'$.

We proceed by induction on i.

If $\mathbf{a}_1 \neq \mathbf{0}$, then

$$\mathbf{b}_1 = \phi_k(\mathbf{x}) = \phi_{k-1}(\mathbf{x}')\hat{x}_k \overset{(6)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1))\hat{\gamma}(\hat{x}_k) = \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_1)\hat{x}_k) = \hat{\gamma}_k(\phi_k(\mathbf{a}_1\hat{x}_k)),$$

so $f_1 = \hat{x}_k$.

If $\mathbf{a}_1 = \mathbf{0}$, then

$$\begin{aligned} b_1 &= \phi_k(\mathbf{x}) = \phi_{k-1}(\mathbf{x}')\hat{x}_k \overset{(6)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1))\hat{\gamma}(\hat{x}_k) \\ &= \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_1)\hat{x}_k) = \hat{\gamma}_k(\phi_k(\mathbf{a}_1(\hat{x}_k - 1))), \end{aligned}$$

so $f_1 = \hat{x}_k - 1$.

Assume now that Claim 3.6 holds for all $1 \leq i \leq n-1$; we will show the claim holds for n.

Lemma 3.7. *Suppose that $n \in J_0$ and $\mathbf{x}' \neq \mathbf{0}$. Then*

$$f_n = \begin{cases} \gamma(f_{n-1}) - 1 & \text{if } \mathbf{a}_m \neq \mathbf{0} \\ \gamma(f_{n-1}) - 2 & \text{if } \mathbf{a}_m = \mathbf{0}. \end{cases}$$

Proof. Recall that r is the unique index such that $\mathbf{a}_r = \mathbf{0}$; since $\mathbf{a}_1 = \mathbf{x}' \neq \mathbf{0}$, we know $r > 1$ and $\phi_k(\mathbf{x}) = \phi_{k-1}(\mathbf{x}')x_k$, so $\hat{x}_k = x_k$. From this we obtain, for all $y \in \mathbb{Z}_3$,

$$\gamma(\hat{\gamma}(y)) = \gamma(-\hat{x}_k - y) = -x_k + \hat{x}_k + y = y \text{ and}$$
$$\hat{\gamma}(\gamma(y)) = \hat{\gamma}(-x_k - y) = -\hat{x}_k + x_k + y = y,$$

so γ and $\hat{\gamma}$ commute.

We consider three cases, when $r = m$, when r is even ($r = 2s$), and when r is odd ($r = 2s - 1$) and not equal to m.

Case 1: $r = m$. Then $n = m + 1$ and

$$\begin{aligned} \mathbf{b}_{m+1} &= \gamma_k(\mathbf{b}_m) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_m)(f_m + 1)))) \\ &= \gamma_k(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(f_m + 1)) = \gamma_{k-1}(\mathbf{x}')\gamma(\hat{\gamma}(f_m + 1)) \overset{(3)}{=} \mathbf{x}'\hat{\gamma}(\gamma(f_m + 1)) \\ &= \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1))\hat{\gamma}(\gamma(f_m) - 1) = \gamma_k(\phi_k(\mathbf{a}_1(\gamma(f_m) - 2))). \end{aligned}$$

Thus, $f_n = \gamma(f_{n-1}) - 2$.

Case 2: $r = 2s$; that is $\mathbf{a}_{2s} = \mathbf{0}$. So, $\mathbf{a}_{2s-1} \neq \mathbf{0}$. First assume that $n \in \{2s, 2m+2s\}$. Then

$$\begin{aligned} \mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2s-1} f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s-1}) f_{n-1})) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s-1})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s}))\hat{\gamma}(\gamma(f_{n-1})) \\ &= \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s})(\gamma(f_{n-1}))) = \hat{\gamma}_k(\phi_k(\mathbf{a}_{2s}(\gamma(f_{n-1}) - 1))). \end{aligned}$$

Hence, $f_n = \gamma(f_{n-1}) - 1$.

Now assume that $n = 2m - 2s + 2$; then

$$\begin{aligned} \mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2s} f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s})(f_{n-1} + 1))) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s})))\gamma(\hat{\gamma}(f_{n-1} + 1)) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s-1}))\hat{\gamma}(\gamma(f_{n-1} + 1)) \\ &= \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s-1})(\gamma(f_{n-1} + 1))) = \hat{\gamma}_k(\phi_k(\mathbf{a}_{2s-1}(\gamma(f_{n-1}) - 1))). \end{aligned}$$

So, $f_n = \gamma(f_{n-1}) - 1$.

Case 3: $r = 2s - 1$ and $r \neq m$; that is $\mathbf{a}_{2s-1} = \mathbf{0}$. So, $\mathbf{a}_{2s} \neq \mathbf{0}$. First assume that $n \in \{2s, 2m + 2s\}$. Then

$$\begin{aligned} \mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2s-1} f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s-1})(f_{n-1} + 1))) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s-1})))\gamma(\hat{\gamma}(f_{n-1} + 1)) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s}))\hat{\gamma}(\gamma(f_{n-1} + 1)) \\ &= \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s})(\gamma(f_{n-1} + 1))) = \hat{\gamma}_k(\phi_k(\mathbf{a}_{2s}(\gamma(f_{n-1}) - 1))). \end{aligned}$$

Hence, $f_n = \gamma(f_{n-1}) - 1$.

Now assume that $n = 2m - 2s + 2$; then

$$\begin{aligned} \mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2s} f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s}) f_{n-1})) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2s-1}))\hat{\gamma}(\gamma(f_{n-1})) \\ &= \hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2s-1}(\gamma(f_{n-1}))) = \hat{\gamma}_k(\phi_k(\mathbf{a}_{2s-1}(\gamma(f_{n-1}) - 1))). \end{aligned}$$

Thus, $f_n = \gamma(f_{n-1}) - 1$. □

Lemma 3.8. *Suppose that $n \in J_1$ and $\mathbf{x}' \neq \mathbf{0}$. Then $f_n = \gamma(f_{n-1})$.*

Proof. Since $\mathbf{x}' \neq \mathbf{0}$, following the same argument as that given at the beginning of the proof of Lemma 3.7, we have that γ and $\hat{\gamma}$ commute. Once again, we will consider three cases, when n is even and $n \neq m+1$, when n is odd, and when $n = m+1$.

Case 1: n is even and $n \neq m+1$. First assume that $n = 2\ell$ or $n = 2m+2\ell$ for $2 \leq 2\ell \leq m-1$. Note that, as $n \in J_1$, $\ell \neq s$, so $\mathbf{a}_{2\ell-1} \neq 0$ and $\mathbf{a}_{2\ell} \neq 0$. So

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell-1}f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2\ell-1})f_{n-1})) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell}))\hat{\gamma}(\gamma(f_{n-1})) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}\gamma(f_{n-1}))).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1})$.

Now assume that $n = 2m-2\ell+2$. Again, as $n \in J_1$, $\ell \neq s$, so $\mathbf{a}_{2\ell-1} \neq 0$ and $\mathbf{a}_{2\ell} \neq 0$. Hence

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2\ell})f_{n-1})) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1}))\hat{\gamma}(\gamma(f_{n-1})) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell-1}\gamma(f_{n-1}))).\end{aligned}$$

Thus, $f_n = \gamma(f_{n-1})$.

Case 2: n is odd. First assume that $n = 2\ell+1$ or $n = 2m+2\ell+1$. Then

$$\begin{aligned}\mathbf{b}_n &= \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{n-1}))))) = \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}f_{n-1}))))))) \\ &= \hat{\gamma}_k(\phi_k(\gamma_k(\mathbf{a}_{2\ell}f_{n-1}))) = \hat{\gamma}_k(\phi_k(\gamma_{k-1}(\mathbf{a}_{2\ell})\gamma(f_{n-1}))) \overset{(4)}{=} \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell+1}\gamma(f_{n-1}))).\end{aligned}$$

So, $f_n = \gamma(f_{n-1})$.

Next, assume that $n = 2m-2\ell+1$. Then

$$\begin{aligned}\mathbf{b}_n &= \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{n-1}))))) = \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell+1}f_{n-1}))))))) \\ &= \hat{\gamma}_k(\phi_k(\gamma_k(\mathbf{a}_{2\ell+1}f_{n-1}))) = \hat{\gamma}_k(\phi_k(\gamma_{k-1}(\mathbf{a}_{2\ell+1})\gamma(f_{n-1}))) \overset{(4)}{=} \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}\gamma(f_{n-1}))).\end{aligned}$$

Thus, $f_n = \gamma(f_{n-1})$.

Finally, assume that $n = 2m+1$. Then, recalling that $\mathbf{a}_1 = \mathbf{x}'$,

$$\begin{aligned}\mathbf{b}_{2m+1} &= \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\mathbf{b}_{2m}))))) = \hat{\gamma}_k(\phi_k(\gamma_k(\phi_k^{-1}(\hat{\gamma}_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_1 f_{2m}))))))) \\ &= \hat{\gamma}_k(\phi_k(\gamma_k(\mathbf{a}_1 f_{2m}))) = \hat{\gamma}_k(\phi_k(\gamma_{k-1}(\mathbf{x}')\gamma(f_{2m}))) \overset{(3)}{=} \hat{\gamma}_k(\phi_k(\mathbf{x}'\gamma(f_{2m}))) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_1\gamma(f_{2m}))).\end{aligned}$$

Hence, $f_{2m+1} = \gamma(f_{2m})$.

Case 3: $n = m+1$. If $\hat{\gamma}(f_m) = x_k$, then

$$\hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m)) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(f_{2m}) = \mathbf{x}' x_k = \mathbf{x}.$$

However, this implies ∞ follows $\mathbf{b}_m = \mathbf{x}$ in the rotation around $\overline{\phi_k(\mathbf{x})}$. Thus, we first show that $\hat{\gamma}(f_m) \neq x_k$. Since $\mathbf{x}' \neq 0$, $x_k = \hat{x}_k$ and it suffices to show $f_m \neq \hat{x}_k$.

Note that $\gamma(y) = \hat{\gamma}(y) = \hat{x}_k$ if and only if $y = \hat{x}_k$. Recall that r is the unique index such that $\mathbf{a}_r = \mathbf{0}$, and set $r^* = r-1$ if r is odd and set $r^* = r$ if r is even. From above we have that $f_i = \gamma(f_{i-1})$ for all $2 \le i \le m$, $i \ne r^*$. Thus, $f_i = \hat{x}_k$ for $1 \le i \le r^*-1$ and $f_i \ne \hat{x}_k$ for $r^* \le i \le m$. As $\mathbf{a}_m \ne \mathbf{x}'$ and recalling that $\mathbf{x}' = \gamma_{k-1}(\mathbf{x}') = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))$, we have that

$$\begin{aligned}\mathbf{b}_{m+1} &= \gamma_k(\mathbf{b}_m) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m))) = \gamma_k(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(f_m)) = \gamma_k(\mathbf{x}'\hat{\gamma}(f_m)) \\ &= \gamma_{k-1}(\mathbf{x}')\gamma(\hat{\gamma}(f_m)) \overset{(3)}{=} \mathbf{x}'\hat{\gamma}(\gamma(f_m)) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(\gamma(f_m)) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_m\gamma(f_m))).\end{aligned}$$

Hence, $f_{m+1} = \gamma(f_m)$. □

Lemma 3.9. *Suppose that $n \in J_0$ and $\mathbf{x}' = \mathbf{0}$. Then $f_n = \gamma(f_{n-1})$.*

Proof. We have that $\mathbf{a}_1 = \mathbf{x}'$, so $r = 1$ and $J_0 = \{2, 2m+2, 2m\}$. Since $\mathbf{x}' = \mathbf{0}$, we have $\phi_k(\mathbf{x}) = \phi_{k-1}(\mathbf{x}')(x_k+1)$, so $\hat{x}_k = x_k + 1$. From this we obtain, for all $y \in \mathbb{Z}_3$,

$$\begin{aligned}&\gamma(\hat{\gamma}(y)) = \gamma(-\hat{x}_k - y) = -x_k + \hat{x}_k + y = y+1 \text{ and} \\ &\hat{\gamma}(\gamma(y)+1) = \hat{\gamma}(-x_k - y + 1) = -\hat{x}_k + x_k + y - 1 = y - 2 = y + 1,\end{aligned}$$

so $\gamma(\hat{\gamma}(y)) = \hat{\gamma}(\gamma(y)+1)$.

Note that $\mathbf{a}_2 \ne \mathbf{0}$. First assume that $n \in \{2, 2m+2\}$. Then, as $\mathbf{0} = \mathbf{x}' = \mathbf{a}_1$,

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_1 f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_1)(f_{n-1}+1))) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1)))\gamma(\hat{\gamma}(f_{n-1}+1)) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_2))\hat{\gamma}(\gamma(f_{n-1}+1)+1) \\ &= \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_2))\hat{\gamma}(-x_k - f_{n-1} - 1 + 1) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_2))\hat{\gamma}(\gamma(f_{n-1}) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_2(\gamma(f_{n-1})))).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1})$.

So assume that $n = 2m$; then

$$\begin{aligned}\mathbf{b}_{2m} &= \gamma_k(\mathbf{b}_{2m-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_2 f_{2m-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_2)f_{2m-1})) \\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_2)))\gamma(\hat{\gamma}(f_{2m-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_1))\hat{\gamma}(\gamma(f_{2m-1})+1) \\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_1(\gamma(f_{2m-1})))).\end{aligned}$$

Thus $f_{2m} = \gamma(f_{2m-1})$. □

Lemma 3.10. *Suppose that $n \in J_1$ and $\mathbf{x}' = \mathbf{0}$. Then*

$$f_n = \begin{cases} \gamma(f_{n-1}) + 1 & \text{if } n \text{ is even} \\ \gamma(f_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $\mathbf{x}' = \mathbf{0}$, as in the proof Lemma 3.9, we have that $\hat{x}_k = x_k + 1$ and $\gamma(\hat{\gamma}(y)) = \hat{\gamma}(\gamma(y)+1)$. We consider three cases, when n is even and $n \ne m+1$, when n is odd and when $n = m+1$.

Case 1: n is even and $n \neq m+1$. First assume that $n = 2\ell$ or $n = 2m + 2\ell$. Note that, as $\mathbf{0} = \mathbf{x}' = \mathbf{a}_1$ and $n \in J_1$, $\ell \neq s = 1$, so $\mathbf{a}_{2\ell-1} \neq \mathbf{0}$ and $\mathbf{a}_{2\ell} \neq \mathbf{0}$. Hence

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell-1}f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2\ell-1})f_{n-1}))\\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell}))(\hat{\gamma}(\gamma(f_{n-1})+1))\\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}(\gamma(f_{n-1})+1))).\end{aligned}$$

So, $f_n = \gamma(f_{n-1}) + 1$.

Now assume that $n = 2m - 2\ell + 2$. Again, as $n \in J_1$, $\ell \neq s = 1$, so $\mathbf{a}_{2\ell-1} \neq \mathbf{0}$ and $\mathbf{a}_{2\ell} \neq \mathbf{0}$. Hence

$$\begin{aligned}\mathbf{b}_n &= \gamma_k(\mathbf{b}_{n-1}) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell}f_{n-1}))) = \gamma_k(\hat{\gamma}_k(\phi_{k-1}(\mathbf{a}_{2\ell})f_{n-1}))\\ &= \gamma_{k-1}(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell})))\gamma(\hat{\gamma}(f_{n-1})) \overset{(5)}{=} \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_{2\ell-1}))\hat{\gamma}(\gamma(f_{n-1})+1)\\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_{2\ell-1}(\gamma(f_{n-1})+1))).\end{aligned}$$

Hence, $f_n = \gamma(f_{n-1}) + 1$.

Case 2: n is odd. The proof is identical to Case 2 in the proof of Lemma 3.8.

Case 3: $n = m + 1$. If $\hat{\gamma}(f_m) = x_k$, then

$$\hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m)) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(f_m) = \mathbf{x}'x_k = \mathbf{x}.$$

However this implies that ∞ follows $\mathbf{b}_m = \mathbf{x}$ in the rotation around $\overline{\phi_k(\mathbf{x})}$. Thus, we first show that $\hat{\gamma}(f_m) \neq x_k$. Since $m \equiv 3 \pmod 6$, it suffices to show that $\hat{\gamma}(f_i) = x_k + 1$ for all $3 \leq i \leq m$ satisfying $i \equiv 3 \pmod 6$. We note that $\mathbf{a}_1 = \mathbf{0}$, so $f_1 = \hat{x}_k - 1 = x_k$ and $J_0 = \{2, 2m+2, 2m\}$. Thus, by Lemma 3.9 $f_2 = \gamma(f_1) = x_k$, and, from the above, for all $3 \leq i \leq m$, $f_i = \gamma(f_{i-1}) + 1$ if i is even and $f_i = \gamma(f_{i-1})$ if i is odd. We compute as a base case that $f_3 = \gamma(f_2) = x_k$. If i is odd, then

$$f_i = \gamma(f_{i-1}) = \gamma(\gamma(f_{i-2})+1) = \gamma(-x_k - f_{i-2} + 1) = -x_k + x_k + f_{i-2} - 1 = f_{i-2} - 1.$$

Now assume that $f_i = x_k$ for all $3 \leq i \leq m'$ (where $m' < m$) satisfying $i \equiv 3 \pmod 6$. We obtain the sequence

$$\begin{aligned}f_{i+2} &= f_i - 1 = x_k - 1\\ f_{i+4} &= f_{i+2} - 1 = x_k - 2\\ f_{i+6} &= f_{i+4} - 1 = x_k.\end{aligned}$$

Thus, for all $3 \leq i \leq m$ satisfying $i \equiv 3 \pmod 6$, we have that $f_i = x_k$ and $\hat{\gamma}(f_i) = -\hat{x}_k - x_k = -x_k + 1 - x_k = -2x_k + 1 = x_k + 1$. Therefore $\hat{\gamma}(f_m) \neq x_k$.

Hence, as $\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m)) = \mathbf{x}'$ and $\mathbf{a}_m \neq \mathbf{0}$,

$$\begin{aligned}\mathbf{b}_{m+1} &= \gamma_k(\mathbf{b}_m) = \gamma_k(\hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m))) = \gamma_k(\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(f_m)) = \gamma_k(\mathbf{x}'\hat{\gamma}(f_m))\\ &= \gamma_{k-1}(\mathbf{x}')\gamma(\hat{\gamma}(f_m)) \overset{(3)}{=} \mathbf{x}'\hat{\gamma}(\gamma(f_m)+1) = \hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m))\hat{\gamma}(\gamma(f_m)+1)\\ &= \hat{\gamma}_k(\phi_k(\mathbf{a}_m(\gamma(f_m)+1))).\end{aligned}$$

Thus $f_{m+1} = \gamma(f_m) + 1$. □

From Lemmas 3.7, 3.8, 3.9 and 3.10, we know $\omega_k^{\overline{\phi_k(\mathbf{x})}}$ is a permutation of length at least $3m+1 = 3^k+1$. Since $(W, \mathcal{C})$ and $(W, \mathcal{D})$ are Steiner triple systems, we know $\omega_k^{\overline{\phi_k(\mathbf{x})}}$ is a permutation of length at most 3^k+1. Thus, Claim 3.6 is established, and the next result follows immediately.

Corollary 3.11. *There is a full rotation around the vertex* $\overline{\phi(\mathbf{x})}$.

3.3 Nonorientability of $\mathcal{S}$

Theorem 3.12. *For all* $k \geq 2$, $\delta(AG(k,3))$ *has a self-embedding in a nonorientable surface.*

Proof. The construction of the pseudo-surface $\mathcal{S}$ is from two Steiner triple systems isomorphic to $\delta(\mathrm{AG}(k,3))$. By Lemma 3.1 and Corollaries 3.5 and 3.11, $\mathcal{S}$ is a surface and hence the accompanying face 2-coloured triangulation is a self-embedding of $\delta(\mathrm{AG}(k,3))$.

It remains to show that this embedding is in a nonorientable surface. Assume to the contrary, so that Ringel's Rule Δ^* holds. Let $\mathbf{0}_{k-1} = 00\ldots0, \mathbf{1}_{k-1} = 11\ldots1, \mathbf{2}_{k-1} = 22\ldots2 \in \mathbb{Z}_3^{k-1}$, and take the rotation around ∞ to be orientated as written in Lemma 3.1, that is

$$\infty : \overline{\mathbf{0}_{k-1}0}\ \ \mathbf{0}_{k-1}0\ \ \ldots\ \ \overline{\mathbf{1}_{k-1}2}\ \ \mathbf{1}_{k-1}2\ \ \ldots\ .$$

For $\mathbf{x} = \mathbf{0}_{k-1}0$, let $\mathbf{b}_i$, $\mathbf{a}_i$, and f_i be defined as in Section 3.1. The triples

$$\{\mathbf{0}_{k-1}0, \mathbf{0}_{k-1}1, \mathbf{0}_{k-1}2\} \in \mathcal{C} \text{ and } \{\mathbf{0}_{k-1}0, \mathbf{0}_{k-1}1, \overline{\mathbf{1}_{k-1}0}\}, \{\mathbf{0}_{k-1}0, \mathbf{0}_{k-1}2, \overline{\mathbf{1}_{k-1}2}\} \in \mathcal{D}$$

imply that the rotation around $\mathbf{0}_{k-1}0$ is either

$$\mathbf{0}_{k-1}0 : \ldots\ \overline{\mathbf{1}_{k-1}0}\ \ \mathbf{0}_{k-1}1\ \ \mathbf{0}_{k-1}2\ \ \overline{\mathbf{1}_{k-1}2}\ \ \ldots \text{ or}$$
$$\mathbf{0}_{k-1}0 : \ldots\ \overline{\mathbf{1}_{k-1}2}\ \ \mathbf{0}_{k-1}2\ \ \mathbf{0}_{k-1}1\ \ \overline{\mathbf{1}_{k-1}0}\ \ \ldots\ .$$

Recall that $\mathbf{a}_m = \mathbf{x}' = \mathbf{0}_{k-1}$. So, by Claim 3.2, we have that $f_1 = 0$, $f_i = \gamma(f_{i-1}) = 0$ for all $2 \leq i \leq m-1$ and $f_m = \gamma(f_{m-1}) - 1 = 2$. Hence, again by Claim 3.2, $\mathbf{b}_m = \phi(\mathbf{a}_m 2) = \mathbf{1}_{k-1}0$ and $\mathbf{b}_{m+1} = \phi(\mathbf{a}_m \gamma(2)) = \mathbf{1}_{k-1}2$.

With the rotation around ∞ orientated as above, in order to satisfy Δ^* the rotation from Claim 3.2 must be oriented so that $\mathbf{0}_{k-1}0 : \infty\ \overline{\mathbf{0}_{k-1}0}\ \ \ldots$. As $\mathbf{b}_m = \mathbf{1}_{k-1}0$ and $\mathbf{b}_{m+1} = \mathbf{1}_{k-1}2$, it follows that

$$\mathbf{0}_{k-1}0 : \infty\ \ \overline{\mathbf{0}_{k-1}0}\ \ \ldots\ \ \overline{\mathbf{1}_{k-1}0}\ \ \mathbf{0}_{k-1}1\ \ \mathbf{0}_{k-1}2\ \ \overline{\mathbf{1}_{k-1}2}\ \ \ldots\ .$$

Now let $\mathbf{b}_i$, $\mathbf{a}_i$, and f_i be defined as in Subsection 3.2 for $\overline{\phi_k(\mathbf{x})}$, where $\mathbf{x} = \mathbf{0}_{k-1}1$ and $\phi_k(\mathbf{0}_{k-1}1) = \mathbf{1}_{k-1}2$. Observe that $\{\mathbf{0}_{k-1}0, \mathbf{0}_{k-1}2, \overline{\mathbf{1}_{k-1}2}\} \in \mathcal{D}$. Hence, either

$$\overline{\mathbf{1}_{k-1}2} : \ldots\ \mathbf{0}_{k-1}0\ \ \mathbf{0}_{k-1}2\ \ \ldots \text{ or}$$
$$\overline{\mathbf{1}_{k-1}2} : \ldots\ \mathbf{0}_{k-1}2\ \ \mathbf{0}_{k-1}0\ \ \ldots\ .$$

From Case 3 in the proof of Lemma 3.10 we have that $f_m = x_k = 1$. By Claim 3.6, $b_m = \hat{\gamma}_k(\phi_k(\mathbf{a}_m f_m))$. Also recall that $\hat{\gamma}_{k-1}(\phi_{k-1}(\mathbf{a}_m)) = \mathbf{x} = \mathbf{0}_{k-1}$, hence $\phi_{k-1}(\mathbf{a}_m) =$

$\mathbf{2}_{k-1}$, so $\mathbf{a}_m = 12\ldots 2$. Hence, $\mathbf{b}_m = \hat{\gamma}_k(\phi_k(12\ldots 21)) = \hat{\gamma}_k(\mathbf{2}_{k-1}1) = \mathbf{0}_{k-1}0$. As $\mathbf{b}_1 = \mathbf{1}_{k-1}2$, it follows that $\mathbf{a}_1 = \mathbf{0}_{k-1}$. So, by Claim 3.6 and as $m+1$ is even, $\mathbf{b}_{m+1} = \hat{\gamma}_k(\phi_k(\mathbf{a}_m f_{m+1})) = \hat{\gamma}_k(\phi_k(\underline{12\ldots}2(\gamma(f_m)+1))) = \hat{\gamma}_k(\phi_k(12\ldots 22)) = \hat{\gamma}_k(\mathbf{2}_{k-1}2) = \mathbf{0}_{k-1}2$. So the rotation at $\overline{\mathbf{1}_{k-1}2}$ is either

$$\overline{\mathbf{1}_{k-1}2} : \infty \;\; \mathbf{1}_{k-1}2 \;\; \ldots \;\; \mathbf{0}_{k-1}0 \;\; \mathbf{0}_{k-1}2 \;\; \ldots \text{ or}$$
$$\overline{\mathbf{1}_{k-1}2} : \mathbf{1}_{k-1}2 \;\; \infty \;\; \ldots \;\; \mathbf{0}_{k-1}2 \;\; \mathbf{0}_{k-1}0 \;\; \ldots .$$

The rotation about $\mathbf{0}_{k-1}0$ implies that the facial walk of the face containing the vertices $\mathbf{0}_{k-1}0$, $\mathbf{0}_{k-1}2$ and $\overline{\mathbf{1}_{k-1}2}$ has the cyclic ordering $(\mathbf{0}_{k-1}0, \mathbf{0}_{k-1}2, \overline{\mathbf{1}_{k-1}2})$; hence the first option for the rotation at $\overline{\mathbf{1}_{k-1}2}$ must hold.

The rotation about ∞ implies that the facial walk of the face containing ∞, $\overline{\mathbf{1}_{k-1}2}$ and $\mathbf{1}_{k-1}2$ has the cyclic ordering $(\infty, \overline{\mathbf{1}_{k-1}2}, \mathbf{1}_{k-1}2)$; so the second option for the rotation at $\overline{\mathbf{1}_{k-1}2}$ must hold.

This contradicts Δ^*, so the constructed self-embedding of $\delta(\mathrm{AG}(k,3))$ is in a nonorientable surface. □

Note that the system $\delta(\mathrm{AG}(1,3))$ is isomorphic to the unique STS(7), which has a self-embedding only in an orientable surface (there is no triangulation of K_7 in a nonorientable surface [5] and Figure 1 illustrates a self-embedding on the torus). Following the above construction, with $k = 1$, produces the toroidal biembedding.

References

[1] G.K. Bennett, *Topological embeddings of Steiner triple systems and associated problems in design theory*, Ph.D. thesis, The Open University (2004).

[2] C.J. Colbourn and A. Rosa, *Triple Systems*, Clarendon Press, Oxford (1999).

[3] D.M. Donovan, M.J. Grannell, T.S. Griggs, J.G. Lefevre and T.A. McCourt, Self-Embeddings of Cyclic and Projective Steiner Quasigroups, *J. Combin. Des.* **19** (2011), 16–27.

[4] P.M. Ducrocq and F. Sterboul, On G-triple systems, *Publications du Laboratoire de Calcul de l'Université des Sciences et Techniques de Lille* **103** (1978), 18pp.

[5] P. Franklin, A six color problem, *J. Math. Phys.* **16** (1934), 363–369.

[6] M.J. Grannell and T.S. Griggs, Designs and Topology, in *Surveys in Combinatorics 2007*, A.J.W. Hilton and J. Talbot (Editors), *London Math. Soc. Lecture Note Series* **346**, Cambridge Univ. Press, Cambridge (2007), 121–174.

[7] M.J. Grannell, T.S. Griggs and M. Knor, Biembeddings of Latin squares and Hamilton decompositions, *Glasgow Math. J.* **46** (2005), 443–457.

[8] M.J. Grannell, T.S. Griggs and J. Širáň, Recursive Constructions for Triangulations, *J. Graph Theory* **39** (2002), 87–107.

[9] M.J. Grannell and V.P. Korzhik, Nonorientable biembeddings of Steiner triple systems, *Discrete Math.* **285** (2004), 121–126.

[10] M.J. Grannell and J.Z. Schroeder, Biembeddings of 2-rotational Steiner triple systems, *Electron. J. Combin.* **22** (2015), 16pp.

[11] T.S. Griggs, private communication, April 2015.

[12] T.S. Griggs and T.A. McCourt, Biembeddings of symmetric n-cycle systems, *Graphs Combin.* (in press).

[13] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, John Wiley, New York (1987).

[14] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* **2** (1847), 191–204.

[15] G. Ringel, *Map Color Theorem*, Springer-Verlag, New York (1974).

[16] G. Ringel and J.W.T. Youngs, Das Geschlecht des vollstanändige dreifarbaren Graphen, *Comment. Math. Helv.* **45** (1970), 152–158.

[17] S. Stahl and A.T. White, Genus embeddings for some complete tripartite graphs, *Discrete Math.* **14** (1976), 279–296.

[18] J.W.T. Youngs, The mystery of the Heawood conjecture, in *Graph Theory and its Applications*, Academic Press, New York (1970), 17–50.

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