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On binary codes related to mutually quasi-unbiased weighing matrices

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Dedicated to Hadi Kharaghani on his 70th birthday

Abstract

Some mutually quasi-unbiased weighing matrices are constructed from binary codes satisfying the conditions that the number of non-zero weights of the code is four and the code contains the first order Reed–Muller code. Motivated by this, in this note, we study binary codes satisfying the conditions. The weight distributions of binary codes satisfying the conditions are determined. We also give a classification of binary codes of lengths $8, 16$ and binary maximal codes of length 32 satisfying the conditions. As an application, sets of 8 mutually quasi-unbiased weighing matrices for parameters $(16, 16, 4, 64)$ and 4 mutually quasi-unbiased weighing matrices for parameters $(32, 32, 4, 256)$ are constructed for the first time.

1 Introduction

A weighing matrix of order n and weight k is an $n \times n$ $(1, -1, 0)$-matrix W such that $WW^T = kI_n$, where I_n is the identity matrix of order n and W^T denotes the transpose of W. A weighing matrix of order n and weight n is a Hadamard matrix.

Two weighing matrices W_1, W_2 of order n and weight k are said to be *unbiased* if $(1/\sqrt{k})W_1W_2^T$ is also a weighing matrix of order n and weight k [7] (see also [2]). Unbiased weighing matrices of order n and weight n are unbiased Hadamard matrices (see [7]). Weighing (Hadamard) matrices $W_1, W_2, \ldots, W_f$ are said to be *mutually unbiased* if any distinct two of them are unbiased. Generalizing the above concept, recently the concept "quasi-unbiased" for weighing matrices has been introduced by Nozaki and the second author [9]. Namely, two weighing matrices W_1, W_2 of order n and weight k are said to be *quasi-unbiased for parameters* (n, k, l, a) if $(1/\sqrt{a})W_1W_2^T$ is a weighing matrix of weight l. It follows from the definition that $l = k^2/a$. In addition, weighing matrices $W_1, W_2, \ldots, W_f$ are said to be *mutually quasi-unbiased weighing matrices for parameters* (n, k, l, a) if any distinct two of them are quasi-unbiased for parameters (n, k, l, a). Mutually quasi-unbiased weighing matrices were defined from the viewpoint of a connection with spherical codes [9]. This notion was introduced to show that Conjecture 23 in [2] is true. Only quasi-unbiased weighing matrices are previously known for parameters $(n, n, n/2, 2n)$, where $n = 2^{2k+1}$ and k is a positive integer [2, Section 4] and [9, Section 4], and for parameters $(n, 2, 4, 1)$, where n is an even positive integer with $n \geq 4$ [9, Section 3].

Suppose that $n = 2^m$, where m is an integer with $m \geq 2$. Let C be a binary $[n, k]$ code satisfying the following two conditions:

$$\{i \in \{0, 1, \ldots, n\} \mid A_i(C) \neq 0\} = \{0, n/2 \pm a, n/2, n\}, \quad (1)$$

$$C \text{ contains the first order Reed–Muller code } RM(1, m) \text{ as a subcode}, \quad (2)$$

where $A_i(C)$ denotes the number of codewords of weight i in C, and a is a positive integer with $0 < a < n/2$. Then it follows from [9, Proposition 2.3 and Lemma 4.2] that C constructs a set of 2^{k-m-1} mutually quasi-unbiased weighing matrices for parameters $(n, n, (n/2a)^2, 4a^2)$.

In this note, we study binary $[2^m, k]$ codes satisfying the two conditions (1) and (2). The weight distribution of the above code is determined using an integer a given in (1). We give a classification of binary codes C satisfying the two conditions (1) and (2) for lengths $8, 16$. We also give a classification of binary maximal codes C (with respect to the subspace relation) satisfying the two conditions (1) and (2) for length 32. As an application, sets of 8 mutually quasi-unbiased weighing matrices for parameters $(16, 16, 4, 64)$ and 4 mutually quasi-unbiased weighing matrices for parameters $(32, 32, 4, 256)$ are constructed for the first time. All computer calculations in this note were done by Magma [4].

2 Mutually quasi-unbiased weighing matrices and codes

We begin with definitions on codes used throughout this note. A binary $[n, k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ denotes the finite field of order 2. All codes in this note are binary. A $k \times n$ matrix whose rows form a basis of C is called a *generator matrix* of C. The parameters n and k are called the *length* and the *dimension* of C, respectively. For a vector $x = (x_1, \dots, x_n)$, the set $\{i \mid x_i \neq 0\}$ is called the *support* of x. The *weight* $\mathrm{wt}(x)$ of a vector x is the number of non-zero components of x. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C, which is denoted by $d(C)$. Two codes C and C' are *equivalent* if one can be obtained from the other by permuting the coordinates. A code C is *doubly even* (resp. *triply even*) if all codewords of C have weight divisible by four (resp. eight). The *dual code* $C^\perp$ of a code C of length n is defined as $C^\perp = \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}$, where $x \cdot y$ is the standard inner product. A code C is called *self-orthogonal* (resp. *self-dual*) if $C \subset C^\perp$ (resp. $C = C^\perp$). A *covering radius* $\rho(C)$ of C is $\rho(C) = \max_{x\in\mathbb{F}_2^n} \min_{c\in C} \mathrm{wt}(x-c)$. The *first order Reed–Muller codes* $RM(1, m)$ for all positive integer m are defined recursively by

$$\begin{aligned} &RM(1,1) = \mathbb{F}_2^2, \\ &RM(1,m) = \{(u,u),(u,u+\mathbf{1}) \in \mathbb{F}_2^{2^m} \mid u \in RM(1,m-1)\} \text{ for } m > 1, \end{aligned}$$

where $\mathbf{1}$ denotes the all-one vector of suitable length.

Mutually quasi-unbiased weighing matrices for parameters $(n, n, (n/2a)^2, 4a^2)$ are constructed from $[n, k]$ codes C satisfying the two conditions (1) and (2), where $n = 2^m$ and m is a positive integer as follows [9, Proposition 2.3 and Lemma 4.2]. Define ψ as a map from $\mathbb{F}_2^n$ to $\{\pm 1\}^n$ $(\subset \mathbb{Z}^n)$ by $\psi((x_i)_{i=1}^n) = (\alpha_i)_{i=1}^n$, where $\alpha_i = -1$ if $x_i = 1$ and $\alpha_i = 1$ if $x_i = 0$. Note that $\mathrm{wt}(x - y) = j$ if and only if the standard inner product of $\psi(x)$ and $\psi(y)$ is $n - 2j$ for $x, y \in \mathbb{F}_2^n$. Let $\{u_1, u_2, \dots, u_{2^{k-m-1}}\}$ be a set of complete representatives of $C/RM(1, m)$. Since $\{i \in \{0, 1, \dots, n\} \mid A_i(RM(1, m)) \neq 0\} = \{0, n/2, n\}$, $\psi(u_i + RM(1, m))$ is antipodal, that is, $-\psi(u_i + RM(1, m)) = \psi(u_i+RM(1, m))$. Hence, there exists a subset X_i of $\psi(u_i+RM(1, m))$ such that $X_i \cup (-X_i) = \psi(u_i + RM(1, m))$ and $X_i \cap (-X_i) = \emptyset$. For $1 \leq i \leq 2^{k-m-1}$, define H_i to be an $n \times n$ $(1, -1)$-matrix whose rows consist of the vectors of X_i. Any two different vectors in X_i are orthogonal for $1 \leq i \leq 2^{k-m-1}$, which means that the matrix H_i is a Hadamard matrix for $1 \leq i \leq 2^{k-m-1}$. Let x_i be a vector in X_i. The assumption of (1) implies that $\mathrm{wt}(\psi^{-1}(x_i) - \psi^{-1}(x_j)) = n/2, n/2 \pm a$ $(i \neq j)$, namely, the inner product of x_i and x_j $(i \neq j)$ is $0, \mp 2a$ respectively, where a is the integer given in (1). This shows that for any distinct $i, j \in \{1, 2, \dots, 2^{k-m-1}\}$, $(1/2a)H_iH_j^T$ is a $(1, -1, 0)$-matrix and thus it is a weighing matrix of weight $(n/2a)^2$. Therefore, Hadamard matrices $H_1, H_2, \dots, H_{2^{k-m-1}}$ are mutually quasi-unbiased weighing matrices for parameters $(n, n, (n/2a)^2, 4a^2)$.

Remark 1. Since $n/2a$ must be an integer, a is a divisor of 2^{m-1}.

Proposition 2. *Suppose that* $n = 2^m$*, where* m *is an integer with* $m \geq 2$*. Let* C *be an* $[n, k]$ *code satisfying the two conditions* (1) *and* (2)*. Then the weight distribution of* C *is given by*

$$\begin{aligned} &(A_0(C), A_{n/2-a}(C), A_{n/2}(C), A_{n/2+a}(C), A_n(C)) \\ &\quad = (1, (2^{k-m-1} - 1)l, 2n - 2 + (2^{k-m-1} - 1)(2n - 2l), (2^{k-m-1} - 1)l, 1), \end{aligned}$$

where $l = (n/2a)^2$.

Proof. We denote the set of complete representatives of $C/RM(1,m)$ by $\{u_1, u_2, \ldots, u_{2^{k-m-1}}\}$ described as above, where we assume that $u_1 = \mathbf{0}$. In addition, we denote the mutually quasi-unbiased weighing matrices for parameters $(n, n, (n/2a)^2, 4a^2)$ by $H_1, H_2, \ldots, H_{2^{k-m-1}}$ described as above. Since $(1/2a)H_1H_i^T$ is a weighing matrix of weight l for $1 < i \leq 2^{k-m-1}$, 0 appears $n - l$ times in the first row of $(1/2a)H_1H_i^T$. Since the first row of H_1 is the all-one vector, this implies that the number of codewords of weight $n/2$ in $u_i + RM(1,m)$ for $i > 1$ is $2n - 2l$. Thus, $A_{n/2}(C) = 2n - 2 + (2^{k-m-1} - 1)(2n - 2l)$ holds. Since C contains the all-one vector, we have the desired weight distribution. □

Remark 3. The minimum weight of C determines the weight distribution of C. Indeed, the minimum weight determines a, and thus l. Since k and m are given, the weight distribution is determined.

3 Codes satisfying the conditions (1) and (2)

In this section, we give a classification of codes C of length 2^m satisfying the two conditions (1) and (2) for $m = 3, 4$. We also give a classification of maximal codes C of length 32 satisfying the two conditions (1) and (2).

3.1 Length 8

The case $m = 3$ is somewhat trivial, but we only give the result for the sake of completeness. Note that $RM(1,3)$ is equivalent to the extended Hamming $[8,4,4]$ code e_8. The complete coset weight distribution of e_8 is listed in [8, Example 1.11.7]. From [8, Example 1.11.7], $RM(1,3)$ has seven (nontrivial) cosets of minimum weight 2. In addition, every $[8,5]$ code C satisfying the conditions (1) and (2) can be constructed as $\langle RM(1,3), x\rangle$, where x is a coset leader of the seven cosets. We verified by Magma that there exists a unique $[8,5]$ code $C_{8,5}$ satisfying the conditions (1) and (2). This was done by the Magma function `IsIsomorphic`. Similarly, we verified by Magma that $C_{8,5}$ has three (nontrivial) cosets of minimum weight 2, and there exists a unique $[8,6]$ code $C_{8,6}$ satisfying the conditions (1) and (2). It is trivial that the even weight $[8,7]$ code $C_{8,7}$ is the unique $[8,7]$ code satisfying the conditions (1) and (2). We remark that $\{i \in \{0,1,\ldots,8\} \mid A_i(C) \neq 0\} = \{0, 4 \pm 2, 4, 8\}$ for $C = C_{8,i}$ $(i = 5,6,7)$.

3.2 Length 16

The next case is $m = 4$. First we fix the generator matrix of the first order Reed–Muller $[16, 5, 8]$ code $RM(1, 4)$ as follows:

$$\begin{pmatrix} 1001011001101001 \\ 0101010101010101 \\ 0011001100110011 \\ 0000111100001111 \\ 0000000011111111 \end{pmatrix}.$$

Every $[16, 6]$ code C satisfying the conditions (1) and (2) can be constructed as $\langle RM(1, 4), x\rangle$, where x is an element of a set of complete representatives of $\mathbb{F}_2^{16}/RM(1, 4)$, satisfying that $x + RM(1, 4)$ has minimum weight 4 or 6 (see Remark 1). In this way, we found all $[16, 6]$ code C satisfying the conditions (1) and (2), which must be checked further for equivalences to complete the classification. We verified by MAGMA that any $[16, 6]$ code satisfying the conditions (1) and (2) is equivalent to one of the two inequivalent codes $C_{16,6,1}$ and $C_{16,6,2}$. This was done by the MAGMA function `IsIsomorphic`. The minimum weights $d(C)$ and the constructions of the two codes C are listed in Table 1. This table means that $C_{16,6,1}$ and $C_{16,6,2}$ can be constructed as $\langle RM(1, 4), x_{16,6,1}\rangle$ and $\langle RM(1, 4), x_{16,6,2}\rangle$, respectively, where the supports of $x_{16,6,1}$ and $x_{16,6,2}$ are listed in Table 2.

Table 1: $[16, k]$ codes satisfying (1) and (2)

k	Codes C	$d(C)$	Vectors
6	$C_{16,6,1}$	6	$x_{16,6,1}$
	$C_{16,6,2}$	4	$x_{16,6,2}$
7	$C_{16,7,1}$	6	$x_{16,6,1}$, $x_{16,7,1}$
	$C_{16,7,2}$	4	$x_{16,6,2}$, $x_{16,7,2}$
8	$C_{16,8,1}$	4	$x_{16,6,2}$, $x_{16,7,2}$, $x_{16,18,1}$
	$C_{16,8,2}$	4	$x_{16,6,2}$, $x_{16,7,2}$, $x_{16,18,2}$

Table 2: Vectors in Table 1

	Supports		Supports
$x_{16,6,1}$	$\{1, 8, 12, 14, 15, 16\}$	$x_{16,7,2}$	$\{1, 8, 10, 15\}$
$x_{16,6,2}$	$\{1, 2, 15, 16\}$	$x_{16,8,1}$	$\{2, 3, 13, 16\}$
$x_{16,7,1}$	$\{1, 4, 5, 7, 9, 10\}$	$x_{16,8,2}$	$\{4, 5, 12, 13\}$

Let D be a doubly even $[n, k]$ code satisfying the conditions (1) and (2). Every $[n, k+1]$ code C satisfying the conditions (1) and (2) with $D \subset C$ can be constructed as $\langle D, x\rangle$, where x is an element of a set of complete representatives of $D^{\perp}/D$, satisfying that $0 \neq \mathrm{wt}(x) \in \{i \in \{0,1,\dots,n\} \mid A_i(D) \neq 0\}$, since D is self-orthogonal and $\{i \in \{0,1,\dots,n\} \mid A_i(C) \neq 0\} = \{i \in \{0,1,\dots,n\} \mid A_i(D) \neq 0\}$. This observation reduces the number of codes which need be checked for equivalences. This observation is applied to doubly even codes $C_{16,6,2}$ and $C_{16,7,2}$. Using an approach similar to the previous subsection along with the above observation, we completed the classification of codes satisfying the conditions (1) and (2) for dimensions 7 and 8. In this case, the only results are listed in Tables 1 and 2. We verified by MAGMA that $C_{16,7,1}$ has covering radius 4. This was done by the MAGMA function `CoveringRadius`. Thus, $C_{16,7,1}$ is a maximal code (with respect to the subspace relation). Since $C_{16,8,1}$ and $C_{16,8,2}$ are doubly even self-dual codes, there exists no $[16, k]$ code satisfying the conditions (1) and (2) for $k \geq 9$. Therefore, we have the following:

Proposition 4. *If there exists a* $[16, k]$ *code satisfying the conditions* (1) *and* (2)*, then* $k \in \{6, 7, 8\}$*. Up to equivalence, there exist two* $[16, k]$ *codes satisfying the conditions* (1) *and* (2) *for* $k = 6, 7, 8$*.*

By the construction of quasi-unbiased weighing matrices described in Section 2, we have the following:

Corollary 5. *There exists a set of at least* 8 *mutually quasi-unbiased weighing matrices for parameters* $(16, 16, 4, 64)$*.*

A set of four mutually quasi-unbiased weighing matrices for parameters $(16, 16, 16, 16)$ is also constructed. It is known that the maximum size among sets of mutually quasi-unbiased weighing matrices for the parameters is 8 [5, Proposition 6] and [6, Theorem 5.2].

3.3 Length 32

For the next case $m = 5$, the classification of maximal codes satisfying the conditions (1) and (2) was done by a method similar to that for the cases $(n, k) = (16, 7), (16, 8)$.

Proposition 6. *If there exists a maximal* $[32, k]$ *code satisfying the conditions* (1) *and* (2)*, then* $k \in \{9, 10, 11\}$*. Up to equivalence, there exist* 92 *maximal* $[32, 9]$ *codes satisfying the conditions* (1) *and* (2)*, there exist* 102 *maximal* $[32, 10]$ *codes satisfying the conditions* (1) *and* (2)*, and there exist two maximal* $[32, 11]$ *codes satisfying the conditions* (1) *and* (2)*.*

By the construction of quasi-unbiased weighing matrices described in Section 2, we have the following:

Corollary 7. *There exists a set of at least* 4 *mutually quasi-unbiased weighing matrices for parameters* $(32, 32, 4, 256)$*.*

A set of eight mutually quasi-unbiased weighing matrices for parameters $(32, 32, 16, 64)$ is also constructed. It is known that the maximum size among sets of mutually quasi-unbiased weighing matrices for the parameters is 32 [9, Theorems 4.1, 4.4].

Table 3: Maximal $[32, k]$ codes satisfying (1) and (2)

k	Codes C	$d(C)$
9	$C_{32,9,1}, \dots, C_{32,9,91}$	12
	$C_{32,9,92}$	8
10	$C_{32,10,1}, \dots, C_{32,10,101}$	12
	$C_{32,10,102}$	8
11	$C_{32,11,1}, C_{32,11,2}$	12

We denote the 92 inequivalent maximal $[32, 9]$ codes given in Proposition 6 by $C_{32,9,i}$ $(i = 1, 2, \dots, 92)$. We denote the 102 inequivalent maximal $[32, 10]$ codes given in Proposition 6 by $C_{32,10,i}$ $(i = 1, 2, \dots, 102)$. We denote the two inequivalent maximal $[32, 11]$ codes given in Proposition 6 by $C_{32,11,i}$ $(i = 1, 2)$. The minimum weights of the codes given in Proposition 6 are listed in Table 3.

Table 4: Maximal $[32, 9]$ codes satisfying (1) and (2)

	Codes	Vectors
x_7	$C_{32,9,1}, \dots, C_{32,9,90}$	$x_{32,7,1}$
	$C_{32,9,91}$	$x_{32,7,2}$
	$C_{32,9,92}$	$x_{32,7,3}$
x_8	$C_{32,9,1}, \dots, C_{32,9,15}$	$x_{32,8,1}$
	$C_{32,9,16}, \dots, C_{32,9,22}$	$x_{32,8,2}$
	$C_{32,9,23}, \dots, C_{32,9,51}$	$x_{32,8,3}$
	$C_{32,9,52}, \dots, C_{32,9,76}$	$x_{32,8,4}$
	$C_{32,9,77}, C_{32,9,78}, C_{32,9,79}$	$x_{32,8,5}$
	$C_{32,9,80}, C_{32,9,81}, C_{32,9,82}$	$x_{32,8,6}$
	$C_{32,9,83}, C_{32,9,84}, C_{32,9,85}$	$x_{32,8,7}$
	$C_{32,9,86}, C_{32,9,87}$	$x_{32,8,8}$
	$C_{32,9,88}$	$x_{32,8,9}$
	$C_{32,9,89}, C_{32,9,90}$	$x_{32,8,10}$
	$C_{32,9,91}$	$x_{32,8,11}$
	$C_{32,9,92}$	$x_{32,8,12}$
x_9	$C_{32,9,i}$ $(i = 1, 2, \dots, 92)$	$x_{32,9,i}$

To describe the codes given in Proposition 6, we fix the generator matrix of the

Table 5: Maximal $[32, 10]$ codes satisfying (1) and (2)

	Codes	Vectors	Codes	Vectors
x_7	$C_{32,10,1}, \dots, C_{32,10,101}$	$x_{32,7,1}$	$C_{32,10,102}$	$x_{32,7,3}$
x_8	$C_{32,10,1}, \dots, C_{32,10,30}$	$x_{32,8,1}$	$C_{32,10,31}, \dots, C_{32,10,73}$	$x_{32,8,2}$
	$C_{32,10,74}, \dots, C_{32,10,89}$	$x_{32,8,3}$	$C_{32,10,90}, \dots, C_{32,10,98}$	$x_{32,8,4}$
	$C_{32,10,99}, C_{32,10,100}$	$x_{32,8,5}$	$C_{32,10,101}$	$y_{32,8,1}$
	$C_{32,10,102}$	$x_{32,8,12}$		
x_9	$C_{32,10,1}$	$y_{32,9,1}$	$C_{32,10,2}, C_{32,10,3}$	$y_{32,9,2}$
	$C_{32,10,9}, C_{32,10,10}, C_{32,10,11}$	$y_{32,9,3}$	$C_{32,10,4}$	$y_{32,9,4}$
	$C_{32,10,5}, C_{32,10,6}$	$y_{32,9,5}$	$C_{32,10,7}, C_{32,10,8}$	$y_{32,9,6}$
	$C_{32,10,12}, C_{32,10,13}$	$y_{32,9,7}$	$C_{32,10,14}, C_{32,10,15}, C_{32,10,16}$	$y_{32,9,8}$
	$C_{32,10,17}, C_{32,10,18}$	$y_{32,9,9}$	$C_{32,10,19}, C_{32,10,20}$	$y_{32,9,10}$
	$C_{32,10,21}$	$y_{32,9,11}$	$C_{32,10,22}$	$y_{32,9,12}$
	$C_{32,10,23}, C_{32,10,24}$	$y_{32,9,13}$	$C_{32,10,25}, C_{32,10,26}$	$y_{32,9,14}$
	$C_{32,10,27}$	$y_{32,9,15}$	$C_{32,10,28}$	$y_{32,9,16}$
	$C_{32,10,29}$	$y_{32,9,17}$	$C_{32,10,30}$	$y_{32,9,18}$
	$C_{32,10,31}, C_{32,10,32}$	$y_{32,9,19}$	$C_{32,10,33}$	$y_{32,9,20}$
	$C_{32,10,34}, \dots, C_{32,10,37}$	$y_{32,9,21}$	$C_{32,10,38}$	$y_{32,9,22}$
	$C_{32,10,39}, C_{32,10,40}$	$y_{32,9,23}$	$C_{32,10,41}$	$y_{32,9,24}$
	$C_{32,10,42}, C_{32,10,43}$	$y_{32,9,25}$	$C_{32,10,44}$	$y_{32,9,26}$
	$C_{32,10,45}, C_{32,10,46}, C_{32,10,47}$	$y_{32,9,27}$	$C_{32,10,48}$	$y_{32,9,28}$
	$C_{32,10,49}, C_{32,10,50}$	$y_{32,9,29}$	$C_{32,10,51}, C_{32,10,52}$	$y_{32,9,30}$
	$C_{32,10,53}, \dots, C_{32,10,57}$	$y_{32,9,31}$	$C_{32,10,58}$	$y_{32,9,32}$
	$C_{32,10,59}, C_{32,10,60}$	$y_{32,9,33}$	$C_{32,10,61}$	$y_{32,9,34}$
	$C_{32,10,62}, C_{32,10,63}, C_{32,10,64}$	$y_{32,9,35}$	$C_{32,10,65}$	$y_{32,9,36}$
	$C_{32,10,66}$	$y_{32,9,37}$	$C_{32,10,67}, C_{32,10,99}$	$y_{32,9,38}$
	$C_{32,10,68}$	$y_{32,9,39}$	$C_{32,10,69}$	$y_{32,9,40}$
	$C_{32,10,70}$	$y_{32,9,41}$	$C_{32,10,71}, C_{32,10,72}$	$y_{32,9,42}$
	$C_{32,10,73}$	$y_{32,9,43}$	$C_{32,10,74}$	$y_{32,9,44}$
	$C_{32,10,75}$	$y_{32,9,45}$	$C_{32,10,76}, C_{32,10,77}$	$y_{32,9,46}$
	$C_{32,10,78}, C_{32,10,79}, C_{32,10,80}$	$y_{32,9,47}$	$C_{32,10,81}$	$y_{32,9,48}$
	$C_{32,10,82}$	$y_{32,9,49}$	$C_{32,10,83}$	$y_{32,9,50}$
	$C_{32,10,84}$	$y_{32,9,51}$	$C_{32,10,85}$	$y_{32,9,52}$
	$C_{32,10,86}$	$y_{32,9,53}$	$C_{32,10,87}$	$y_{32,9,54}$
	$C_{32,10,88}$	$y_{32,9,55}$	$C_{32,10,89}$	$y_{32,9,56}$
	$C_{32,10,90}$	$y_{32,9,57}$	$C_{32,10,91}$	$y_{32,9,58}$
	$C_{32,10,92}, C_{32,10,93}$	$y_{32,9,59}$	$C_{32,10,94}$	$y_{32,9,60}$
	$C_{32,10,95}$	$y_{32,9,61}$	$C_{32,10,96}$	$y_{32,9,62}$
	$C_{32,10,97}$	$y_{32,9,63}$	$C_{32,10,98}$	$y_{32,9,64}$
	$C_{32,10,100}$	$y_{32,9,65}$	$C_{32,10,101}$	$y_{32,9,66}$
	$C_{32,10,102}$	$y_{32,9,67}$		

first order Reed–Muller $[32, 6, 16]$ code $RM(1, 5)$ as follows:

$$\begin{pmatrix} 10010110011010010110100110010110 \\ 01010101010101010101010101010101 \\ 00110011001100110011001100110011 \\ 00001111000011110000111100001111 \\ 00000000111111110000000011111111 \\ 00000000000000001111111111111111 \end{pmatrix}.$$

The codes $C_{32,9,i}$ $(i = 1, 2, \dots, 92)$ are constructed as $\langle RM(1, 5), x_7, x_8, x_9\rangle$, where Table 4 indicates x_7, x_8, x_9 and the supports are listed in Table 6. The codes $C_{32,10,i}$ $(i = 1, 2, \dots, 102)$ are constructed as $\langle RM(1, 5), x_7, x_8, x_9, x_{10}\rangle$, where Table 5 indicates x_7, x_8, x_9, x_{10} and the supports are listed in Table 6. The codes $C_{32,11,i}$ $(i = 1, 2)$ are constructed as follows:

$$C_{32,11,1} =\langle RM(1, 5), x_{32,7,2}, z_{32,8,1}, z_{32,9,1}, z_{32,10,1}, z_{32,11,1}\rangle,$$
$$C_{32,11,2} =\langle RM(1, 5), x_{32,7,2}, z_{32,8,1}, z_{32,9,2}, z_{32,10,2}, z_{32,11,2}\rangle,$$

where the supports of the vectors are listed in Table 6.

Finally, we compare our codes with some known codes and we discuss the maximality of our codes. It follows from the weight distributions that $C_{32,9,92}$ (resp. $C_{32,10,102}$) is equivalent to the unique maximal triply even $[32, 9]$ (resp. $[32, 10]$) code, which is given in [3, Table 2]. It follows that $C_{32,9,92}$ and $C_{32,10,102}$ are maximal. We verified by Magma that $C_{32,9,1}, C_{32,9,2}, \dots, C_{32,9,90}$ have covering radius ≤ 11, $C_{32,10,1}, C_{32,10,2}, \dots, C_{32,10,101}$ have covering radius 10 and $C_{32,11,1}, C_{32,11,2}$ have covering radius 8. This shows that these codes are maximal. We verified by Magma that $C_{32,11,2}$ is equivalent to the extended BCH $[32, 11, 12]$ code.

Postscript

After this work, we continued the study of quasi-unbiased weighing matrices obtained from (not necessary linear) codes in [1].

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Table 6: Vectors for $m = 5$

	Supports		Supports
$x_{32,7,1}$	$\{1, 3, 4, 6, 7, 9, 10, 16, 17, 18, 19, 32\}$	$x_{32,7,2}$	$\{1, 2, 3, 5, 6, 9, 10, 16, 17, 18, 19, 32\}$
$x_{32,7,3}$	$\{1, 2, 3, 4, 5, 6, 7, 8\}$		
$x_{32,8,1}$	$\{1, 2, 4, 6, 7, 12, 13, 16, 24, 29, 30, 32\}$	$x_{32,8,2}$	$\{4, 5, 6, 7, 8, 9, 11, 16, 24, 28, 30, 32\}$
$x_{32,8,3}$	$\{2, 4, 6, 7, 8, 9, 10, 11, 24, 28, 31, 32\}$	$x_{32,8,4}$	$\{4, 5, 6, 7, 8, 10, 11, 16, 24, 28, 29, 32\}$
$x_{32,8,5}$	$\{1, 4, 5, 6, 8, 9, 10, 11, 24, 28, 30, 32\}$	$x_{32,8,6}$	$\{1, 2, 4, 6, 8, 9, 10, 16, 24, 28, 29, 31\}$
$x_{32,8,7}$	$\{1, 5, 6, 7, 8, 9, 10, 16, 24, 28, 29, 31\}$	$x_{32,8,8}$	$\{2, 3, 4, 7, 8, 9, 10, 16, 24, 28, 30, 31\}$
$x_{32,8,9}$	$\{2, 3, 6, 7, 8, 11, 13, 16, 24, 28, 29, 31\}$	$x_{32,8,10}$	$\{1, 6, 8, 9, 10, 16, 24, 28, 29, 30, 31, 32\}$
$x_{32,8,11}$	$\{4, 5, 8, 10, 11, 16, 24, 28, 29, 30, 31, 32\}$	$x_{32,8,12}$	$\{1, 2, 3, 4, 9, 10, 11, 12\}$
$x_{32,9,1}$	$\{3, 4, 5, 6, 7, 9, 10, 11, 12, 16, 20, 21, 24, 28, 30, 32\}$	$x_{32,9,2}$	$\{3, 4, 5, 6, 8, 9, 12, 15, 17, 28, 31, 32\}$
$x_{32,9,3}$	$\{4, 6, 7, 8, 9, 10, 11, 16, 26, 28, 29, 30\}$	$x_{32,9,4}$	$\{2, 3, 4, 5, 9, 12, 13, 14, 28, 29, 31, 32\}$
$x_{32,9,5}$	$\{1, 4, 5, 7, 8, 9, 10, 16, 24, 26, 28, 30\}$	$x_{32,9,6}$	$\{3, 4, 7, 8, 9, 10, 12, 16, 20, 21, 26, 28\}$
$x_{32,9,7}$	$\{5, 6, 7, 12, 15, 16, 17, 26, 28, 29, 30, 31\}$	$x_{32,9,8}$	$\{1, 5, 7, 9, 10, 11, 15, 16, 17, 28, 30, 31\}$
$x_{32,9,9}$	$\{2, 3, 5, 6, 9, 10, 12, 16, 28, 30, 31, 32\}$	$x_{32,9,10}$	$\{3, 4, 6, 7, 8, 9, 10, 12, 24, 28, 30, 31\}$
$x_{32,9,11}$	$\{1, 4, 5, 7, 8, 10, 11, 16, 24, 26, 28, 32\}$	$x_{32,9,12}$	$\{6, 7, 8, 9, 11, 13, 14, 16, 24, 26, 28, 30\}$
$x_{32,9,13}$	$\{3, 4, 6, 7, 8, 9, 11, 16, 24, 26, 28, 30\}$	$x_{32,9,14}$	$\{1, 2, 4, 8, 9, 11, 13, 14, 28, 29, 31, 32\}$
$x_{32,9,15}$	$\{1, 2, 3, 5, 7, 9, 10, 16, 24, 28, 30, 31\}$	$x_{32,9,16}$	$\{2, 3, 5, 6, 9, 11, 13, 16, 17, 18, 26, 28\}$
$x_{32,9,17}$	$\{2, 3, 4, 5, 6, 9, 10, 11, 12, 16, 20, 21, 24, 26, 28, 29\}$	$x_{32,9,18}$	$\{1, 4, 6, 9, 10, 12, 13, 16, 24, 26, 29, 30\}$
$x_{32,9,19}$	$\{1, 2, 4, 6, 7, 13, 14, 16, 24, 26, 31, 32\}$	$x_{32,9,20}$	$\{1, 2, 4, 6, 9, 11, 13, 16, 20, 21, 31, 32\}$
$x_{32,9,21}$	$\{1, 7, 11, 12, 15, 16, 17, 24, 26, 28, 30, 31\}$	$x_{32,9,22}$	$\{1, 5, 6, 10, 11, 14, 15, 16, 17, 29, 30, 32\}$
$x_{32,9,23}$	$\{1, 2, 5, 8, 9, 10, 11, 13, 15, 16, 17, 24, 27, 28, 30, 32\}$	$x_{32,9,24}$	$\{1, 4, 7, 8, 9, 12, 14, 16, 27, 28, 30, 32\}$
$x_{32,9,25}$	$\{6, 7, 11, 13, 15, 16, 17, 24, 27, 28, 29, 31\}$	$x_{32,9,26}$	$\{1, 3, 7, 8, 9, 11, 13, 16, 27, 28, 29, 32\}$
$x_{32,9,27}$	$\{2, 8, 9, 10, 12, 13, 15, 16, 17, 24, 28, 30\}$	$x_{32,9,28}$	$\{1, 3, 6, 7, 8, 10, 13, 14, 24, 27, 29, 32\}$
$x_{32,9,29}$	$\{3, 6, 7, 10, 11, 12, 15, 16, 17, 28, 30, 31\}$	$x_{32,9,30}$	$\{1, 3, 4, 7, 10, 11, 12, 15, 17, 24, 28, 30\}$
$x_{32,9,31}$	$\{1, 4, 5, 6, 10, 11, 14, 16, 27, 28, 30, 32\}$	$x_{32,9,32}$	$\{1, 2, 4, 5, 10, 13, 14, 15, 18, 24, 27, 29\}$
$x_{32,9,33}$	$\{1, 2, 4, 5, 6, 8, 10, 11, 13, 16, 17, 18, 28, 29, 30, 32\}$	$x_{32,9,34}$	$\{1, 2, 4, 5, 6, 11, 14, 16, 24, 27, 28, 30\}$
$x_{32,9,35}$	$\{3, 4, 5, 7, 8, 12, 13, 15, 17, 27, 28, 29\}$	$x_{32,9,36}$	$\{1, 2, 3, 4, 8, 10, 13, 14, 24, 27, 29, 31\}$
$x_{32,9,37}$	$\{1, 2, 3, 6, 7, 8, 10, 15, 18, 24, 27, 29\}$	$x_{32,9,38}$	$\{3, 4, 5, 7, 10, 11, 14, 16, 27, 28, 29, 32\}$
$x_{32,9,39}$	$\{1, 3, 4, 7, 9, 13, 14, 15, 17, 24, 27, 30\}$	$x_{32,9,40}$	$\{1, 6, 8, 10, 11, 13, 15, 16, 17, 30, 31, 32\}$
$x_{32,9,41}$	$\{2, 5, 6, 8, 9, 10, 12, 13, 15, 16, 17, 24, 27, 29, 31, 32\}$	$x_{32,9,42}$	$\{1, 2, 3, 4, 5, 8, 9, 10, 11, 13, 17, 18, 28, 29, 30, 32\}$
$x_{32,9,43}$	$\{2, 3, 5, 6, 9, 11, 12, 15, 17, 24, 29, 31\}$	$x_{32,9,44}$	$\{2, 4, 5, 6, 7, 8, 9, 13, 14, 16, 17, 18, 28, 29, 30, 32\}$
$x_{32,9,45}$	$\{1, 4, 5, 7, 8, 9, 13, 14, 15, 16, 17, 27, 28, 29, 30, 31\}$	$x_{32,9,46}$	$\{4, 5, 7, 8, 10, 11, 13, 16, 17, 18, 28, 29\}$
$x_{32,9,47}$	$\{1, 4, 5, 6, 7, 8, 9, 11, 14, 16, 17, 18, 27, 28, 30, 31\}$	$x_{32,9,48}$	$\{1, 3, 5, 6, 7, 8, 9, 15, 17, 24, 30, 31\}$

Table 6: Vectors for $m = 5$ (continued)

	Supports		Supports
$x_{32,9,49}$	$\{2,3,5,6,7,12,13,15,17,27,28,29\}$	$x_{32,9,50}$	$\{2,4,5,6,10,12,13,16,27,28,30,31\}$
$x_{32,9,51}$	$\{2,3,4,5,6,11,14,15,17,28,30,31\}$	$x_{32,9,52}$	$\{1,3,4,7,10,11,13,15,17,24,29,30\}$
$x_{32,9,53}$	$\{3,5,7,10,13,14,17,18,24,29,30,32\}$	$x_{32,9,54}$	$\{1,2,4,5,10,13,14,15,18,24,25,31\}$
$x_{32,9,55}$	$\{1,3,7,9,10,11,14,16,21,22,24,28\}$	$x_{32,9,56}$	$\{3,5,6,7,11,12,21,22,25,29,31,32\}$
$x_{32,9,57}$	$\{2,3,4,5,6,8,9,12,14,16,17,18,28,29,31,32\}$	$x_{32,9,58}$	$\{2,3,4,8,9,10,12,14,17,18,29,30\}$
$x_{32,9,59}$	$\{1,3,5,7,8,12,13,15,17,25,28,30\}$	$x_{32,9,60}$	$\{1,4,5,8,10,12,13,15,18,24,28,30\}$
$x_{32,9,61}$	$\{1,2,6,8,9,15,17,21,22,24,25,28\}$	$x_{32,9,62}$	$\{4,6,7,8,12,13,17,18,25,28,30,32\}$
$x_{32,9,63}$	$\{1,4,5,6,9,10,11,14,25,30,31,32\}$	$x_{32,9,64}$	$\{1,3,4,6,8,9,10,11,14,15,17,28,29,30,31,32\}$
$x_{32,9,65}$	$\{1,2,5,7,9,10,12,14,17,18,28,31\}$	$x_{32,9,66}$	$\{2,4,7,8,9,15,17,24,25,28,29,30\}$
$x_{32,9,67}$	$\{1,2,3,5,7,8,10,11,14,16,17,18,25,29,30,32\}$	$x_{32,9,68}$	$\{1,2,5,6,7,9,11,14,24,25,28,30\}$
$x_{32,9,69}$	$\{6,9,10,12,14,16,21,22,24,25,30,31\}$	$x_{32,9,70}$	$\{2,3,4,5,6,7,15,16,18,21,22,24,25,28,29,31\}$
$x_{32,9,71}$	$\{2,3,4,5,6,8,11,12,13,14,15,16,18,21,22,24,25,28,29,32\}$	$x_{32,9,72}$	$\{1,2,3,8,9,11,13,16,21,22,25,29\}$
$x_{32,9,73}$	$\{2,3,4,5,6,8,10,11,13,15,17,21,22,24,31,32\}$	$x_{32,9,74}$	$\{1,3,7,8,11,14,15,16,18,24,25,28,29,30,31,32\}$
$x_{32,9,75}$	$\{3,4,5,8,10,11,12,13,14,15,17,24,25,28,29,30\}$	$x_{32,9,76}$	$\{2,4,5,7,8,11,13,15,17,21,22,25,28,29,31,32\}$
$x_{32,9,77}$	$\{3,6,7,9,10,11,12,15,17,26,28,30\}$	$x_{32,9,78}$	$\{2,6,8,11,13,16,24,26,28,29,30,31\}$
$x_{32,9,79}$	$\{1,3,7,9,10,11,12,15,17,28,29,30\}$	$x_{32,9,80}$	$\{1,3,4,8,9,10,12,15,18,24,26,30\}$
$x_{32,9,81}$	$\{2,3,8,9,10,12,15,16,18,29,30,32\}$	$x_{32,9,82}$	$\{1,5,6,8,10,12,13,14,28,29,30,32\}$
$x_{32,9,83}$	$\{1,3,5,7,10,11,12,13,17,18,28,31\}$	$x_{32,9,84}$	$\{1,3,4,6,8,11,13,16,24,26,28,32\}$
$x_{32,9,85}$	$\{1,3,4,8,11,13,15,16,17,24,26,32\}$	$x_{32,9,86}$	$\{2,3,5,7,9,11,13,16,17,18,29,30\}$
$x_{32,9,87}$	$\{2,5,6,7,8,10,11,12,13,15,17,21,22,28,29,30\}$	$x_{32,9,88}$	$\{2,4,5,6,9,12,13,14,15,16,17,24,26,29,30,32\}$
$x_{32,9,89}$	$\{1,4,5,7,8,9,12,14,24,27,28,31\}$	$x_{32,9,90}$	$\{1,3,5,10,13,16,20,21,24,27,28,30\}$
$x_{32,9,91}$	$\{2,3,5,8,9,13,14,16,27,30,31,32\}$	$x_{32,9,92}$	$\{1,2,3,4,17,18,19,20\}$
$y_{32,8,1}$	$\{1,3,4,7,11,13,15,16,17,24,28,29\}$		
$y_{32,9,1}$	$\{2,4,5,6,8,9,15,16,17,29,31,32\}$	$y_{32,9,2}$	$\{1,2,3,4,5,9,13,14,24,26,30,31\}$
$y_{32,9,3}$	$\{3,4,5,6,7,11,12,15,17,28,30,32\}$	$y_{32,9,4}$	$\{2,3,6,8,9,12,15,16,17,24,28,32\}$
$y_{32,9,5}$	$\{1,5,6,7,9,10,11,16,26,28,29,30\}$	$y_{32,9,6}$	$\{1,4,5,6,8,9,10,11,15,16,17,26,28,29,31,32\}$
$y_{32,9,7}$	$\{4,5,6,7,8,11,12,15,17,26,28,31\}$	$y_{32,9,8}$	$\{2,3,5,7,9,10,12,16,26,29,31,32\}$
$y_{32,9,9}$	$\{1,2,3,6,10,11,15,16,17,24,30,31\}$	$y_{32,9,10}$	$\{1,3,6,9,10,11,12,16,24,26,28,30\}$
$y_{32,9,11}$	$\{2,7,8,9,10,11,12,16,24,28,30,31\}$	$y_{32,9,12}$	$\{2,5,7,9,10,12,24,26,28,30,31,32\}$
$y_{32,9,13}$	$\{3,5,6,7,9,10,11,12,15,16,17,24,26,28,31,32\}$	$y_{32,9,14}$	$\{1,2,3,4,8,9,13,14,24,28,29,31\}$
$y_{32,9,15}$	$\{2,5,6,8,11,13,14,16,26,28,29,30\}$	$y_{32,9,16}$	$\{2,4,5,7,8,9,10,12,24,26,30,31\}$

Table 6: Vectors for $m = 5$ (continued)

	Supports		Supports
$y_{32,9,17}$	$\{1,4,5,10,13,14,17,18,24,28,30,32\}$	$y_{32,9,18}$	$\{2,6,8,9,10,11,12,16,24,26,29,31\}$
$y_{32,9,19}$	$\{3,4,6,8,9,10,11,13,26,29,31,32\}$	$y_{32,9,20}$	$\{2,4,6,8,9,10,11,12,26,28,29,31\}$
$y_{32,9,21}$	$\{2,3,6,7,8,10,11,12,24,26,29,30\}$	$y_{32,9,22}$	$\{1,7,8,10,11,13,24,26,29,30,31,32\}$
$y_{32,9,23}$	$\{3,5,7,8,9,10,11,13,26,28,29,31\}$	$y_{32,9,24}$	$\{1,2,3,6,8,10,11,13,20,21,24,31\}$
$y_{32,9,25}$	$\{1,2,3,4,7,8,13,14,20,21,28,29\}$	$y_{32,9,26}$	$\{2,3,5,7,8,9,11,12,13,14,24,26,29,30,31,32\}$
$y_{32,9,27}$	$\{1,6,8,10,11,13,20,21,24,26,28,32\}$	$y_{32,9,28}$	$\{1,2,4,8,9,10,11,12,15,16,17,24,26,28,29,30\}$
$y_{32,9,29}$	$\{1,2,5,6,8,12,13,16,24,26,31,32\}$	$y_{32,9,30}$	$\{1,2,3,4,5,7,9,16,20,21,26,28\}$
$y_{32,9,31}$	$\{1,2,4,7,10,12,14,16,20,21,29,32\}$	$y_{32,9,32}$	$\{2,3,4,5,7,10,11,13,24,26,30,31\}$
$y_{32,9,33}$	$\{3,4,5,7,10,15,17,24,26,28,31,32\}$	$y_{32,9,34}$	$\{1,2,3,5,6,8,15,16,17,24,29,32\}$
$y_{32,9,35}$	$\{1,3,5,6,7,8,9,10,20,21,26,30\}$	$y_{32,9,36}$	$\{2,3,4,7,8,9,10,11,12,16,20,21,24,26,28,29\}$
$y_{32,9,37}$	$\{1,2,4,6,9,10,11,12,15,16,17,24,26,28,29,32\}$	$y_{32,9,38}$	$\{1,3,5,6,15,16,17,24,26,30,31,32\}$
$y_{32,9,39}$	$\{1,2,3,4,8,10,20,21,24,26,28,29\}$	$y_{32,9,40}$	$\{1,2,3,5,7,8,11,12,15,16,17,24,26,28,29,32\}$
$y_{32,9,41}$	$\{3,6,7,8,13,14,20,21,26,28,30,32\}$	$y_{32,9,42}$	$\{1,5,7,8,10,13,14,16,20,21,26,29\}$
$y_{32,9,43}$	$\{1,2,3,5,6,7,13,14,20,21,26,28\}$	$y_{32,9,44}$	$\{1,4,5,7,9,10,11,13,15,16,17,24,29,30,31,32\}$
$y_{32,9,45}$	$\{1,5,6,7,9,11,13,16,27,30,31,32\}$	$y_{32,9,46}$	$\{1,2,4,5,6,13,14,15,18,27,28,29\}$
$y_{32,9,47}$	$\{1,6,8,9,15,16,17,28,29,30,31,32\}$	$y_{32,9,48}$	$\{2,3,5,6,10,11,12,15,17,24,30,31\}$
$y_{32,9,49}$	$\{1,2,3,4,6,7,11,13,15,16,17,24,27,28,30,32\}$	$y_{32,9,50}$	$\{2,6,9,12,13,15,17,24,27,28,29,32\}$
$y_{32,9,51}$	$\{5,6,7,10,11,12,13,14,24,27,29,32\}$	$y_{32,9,52}$	$\{1,4,5,6,7,8,9,15,18,20,21,24,28,29,30,31\}$
$y_{32,9,53}$	$\{2,9,10,11,14,16,17,18,24,29,30,32\}$	$y_{32,9,54}$	$\{2,4,6,8,9,12,13,16,17,18,27,28\}$
$y_{32,9,55}$	$\{5,6,7,8,10,11,13,16,27,28,29,30\}$	$y_{32,9,56}$	$\{2,7,15,16,18,20,21,24,28,30,31,32\}$
$y_{32,9,57}$	$\{4,5,6,9,11,12,15,16,17,25,30,32\}$	$y_{32,9,58}$	$\{1,4,5,7,8,9,10,15,18,28,30,31\}$
$y_{32,9,59}$	$\{2,3,6,8,9,10,11,12,25,28,29,31\}$	$y_{32,9,60}$	$\{1,6,9,10,12,14,15,16,18,21,22,24,28,29,30,31\}$
$y_{32,9,61}$	$\{1,2,3,4,6,9,11,13,15,16,17,21,22,25,29,32\}$	$y_{32,9,62}$	$\{3,5,6,7,8,12,14,16,17,18,24,30\}$
$y_{32,9,63}$	$\{1,2,5,7,9,12,14,16,17,18,25,28\}$	$y_{32,9,64}$	$\{3,4,9,10,11,12,13,14,17,18,29,30\}$
$y_{32,9,65}$	$\{1,4,5,6,7,12,13,15,17,26,28,32\}$	$y_{32,9,66}$	$\{2,3,4,5,6,9,11,16,24,25,29,32\}$
$y_{32,9,67}$	$\{1,4,6,7,10,11,13,16\}$		
$z_{32,8,1}$	$\{3,4,8,12,13,14,24,28,29,30,31,32\}$	$z_{32,9,1}$	$\{1,3,6,7,10,12,13,16,27,28,31,32\}$
$z_{32,10,1}$	$\{1,3,4,5,8,9,10,16,22,28,29,31\}$	$z_{32,11,1}$	$\{1,8,9,11,12,15,17,22,28,29,30,32\}$
$z_{32,8,2}$	$\{3,4,8,12,13,14,24,28,29,30,31,32\}$	$z_{32,9,2}$	$\{1,3,6,7,10,12,13,16,27,28,31,32\}$
$z_{32,10,2}$	$\{1,3,4,5,8,9,10,16,22,28,29,31\}$	$z_{32,11,2}$	$\{6,7,8,9,11,12,15,16,17,27,28,30\}$

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