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Pebbling numbers of the Cartesian product of cycles and graphs*

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Abstract

The pebbling number $f(G)$ of a graph G is the least p such that, no matter how p pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. It is conjectured that for all graphs G and H, we have $f(G \times H) \le f(G)f(H)$. If the graph G satisfies the odd 2-pebbling property, we will prove that $f(C_{4k+3} \times G) \le f(C_{4k+3})f(G)$ and $f(M(C_{2n}) \times G) \le f(M(C_{2n}))f(G)$, where C_{4k+3} is the odd cycle of order $4k+3$ and $M(C_{2n})$ is the middle graph of the even cycle C_{2n}.

1 Introduction

Pebbling in graphs was first introduced by Chung ([2]). Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. The pebbling number of a vertex v, the target vertex, in a graph G is the smallest number $f(G, v)$ with the property that, from every placement of $f(G, v)$ pebbles on G, it is possible to move one pebble to v by a sequence of pebbling moves. The *t-pebbling number* of v in G is defined as the smallest number $f_t(G, v)$ such that from every placement of $f_t(G, v)$ pebbles, it is possible to move t pebbles to v. Then the *pebbling number* and the *t-pebbling number* of G are the smallest numbers, $f(G)$ and $f_t(G)$, such that from any placement of $f(G)$ pebbles or $f_t(G)$ pebbles, respectively, it is possible to move one or t pebbles, respectively, to any

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specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ and $f_t(G)$ are the maximum values of $f(G, v)$ and $f_t(G, v)$ over all vertices v.

Chung ([2]) defined the 2-pebbling property of a graph, and Wang ([9]) extended her definition to the odd 2-pebbling property as follows.

Suppose p pebbles are located on G. Let l be the number of occupied vertices (vertices with at least one pebble), and r be the number of vertices with an odd number of pebbles. Then to say G satisfies the 2-*pebbling property* means that two pebbles can be moved to any vertex of G whenever $p > 2f(G) - l$, and the *odd* 2-*pebbling property* means two pebbles can be moved to any vertex of G whenever $p > 2f(G) - r$. It is clear that any graph which satisfies the 2-pebbling property also satisfies the odd 2-pebbling property. It is known that both trees and cycles have the 2-pebbling property ([7, 8]). The graph L, called Lemke graph ([9]), is the minimal graph that does not satisfy the 2-pebbling property; this is shown in Figure 1. It is not hard to see that the pebbling number of this Lemke graph is $f(L) = 8$. If we place 13 pebbles on the vertices of L as shown in Figurer 1, then we have $p + l = 13 + 5 > 16 = 2f(L)$, but we cannot move two pebbles to v_0.

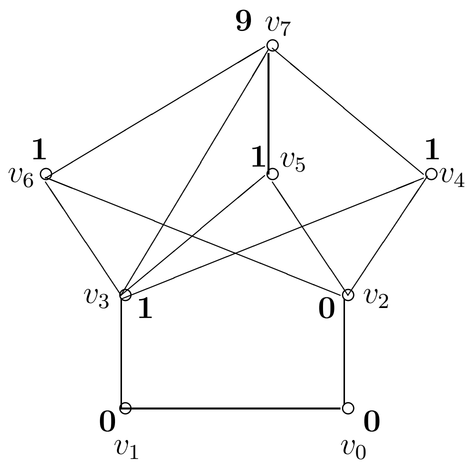

Figure 1: Lemke graph (L)

The *middle graph* of a graph G, denoted by $M(G)$, is obtained from G by inserting a new vertex into each edge of G, and joining the new vertices by an edge if the two corresponding edges share the same vertex of G. For any two graphs G and H, we define the *Cartesian product* $G \times H$ to be the graph with vertex set $V(G) \times V(H)$ and edge set the union of $\{((a, v), (b, v)) | (a, b) \in E(G), v \in V(H)\}$ and $\{((u, x), (u, y)) | u \in V(G), (x, y) \in E(H)\}$.

The following conjecture ([2]), by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture 1.1 (Graham) *For any two graphs G and H, $f(G \times H) \leq f(G)f(H)$.*

While this conjecture is still open, many successful results in support have appeared. It has been proven that $f(G \times H) \leq f(G)f(H)$ for the following cases:

(1) G is a tree and H is a graph with the odd 2-pebbling property ([6]), (and in particlar, H is a tree);

(2) G is an even cycle and H is a graph with the odd 2-pebbling property;

(3) both G and H are cycles ([5]);

(4) G is a complete or complete bipartite graph and H is a graph with the 2-pebbling property ([2, 3]);

(5) both G and H are fan graphs ([4]);

(6) both G and H are wheel graphs ([4]);

(7) G is a thorn graph of the complete graph with every $p_i > 1$ and H is a graph with the 2-pebbling property ([10]);

(8) G is the middle graph of an odd cycle and H is the middle graph of a cycle ([11]).

In Section 2, we show that Graham's conjecture holds for the product of the odd cycle C_{4k+3} with a graph with the odd 2-pebbling property.

In Section 3, we show that Graham's conjecture holds for the product of the middle graph of an even cycle with a graph with the odd 2-pebbling property.

Given a distribution of pebbles on G, let $p(K)$ be the number of pebbles on a subgraph K of G, $p(v)$ be the number of pebbles on vertex v of G and $l(K)$ ($r(K)$) to be the number of vertices of K with at least one pebble (with an odd number of pebbles). Moreover, denote by $\tilde{p}(K)$ and $\tilde{p}(v)$ the number of pebbles on K and v after some sequence of pebbling moves, respectively.

Let T be a tree and let v be a vertex of T. Let $\vec{T_v}$ be the rooted tree obtained from T by directing all edges towards v, which becomes the root. A path-partition is a set of non-overlapping directed paths the union of which is $\vec{T_v}$. The path-size sequence of a path-partition $P_1, \ldots, P_n$, is an n-tuple $(a_1, \ldots, a_n)$, where a_i is the length of P_i (i.e., the number of edges in it), with $a_1 \geq a_2 \geq \ldots \geq a_n$. A path-partition is said to majorize another if the nonincreasing sequence of its path size majorizes that of the other. That is, $(a_1, a_2, \ldots, a_r) > (b_1, b_2, \ldots, b_t)$ if and only if $a_i > b_i$ where $i = \min\{j : a_j \neq b_j\}$. A path-partition of a tree T is said to be maximum if it majorizes all other path-partitions.

The following two lemmas will be the key tools in the next sections.

Lemma 1.2 ([2]) *The pebbling number $f_t(T, v)$ for a vertex v in a tree T is $t2^{a_1} + 2^{a_2} + \cdots + 2^{a_r} - r + 1$, where $a_1, a_2, \ldots, a_r$ is the sequence of the path sizes in a maximum path-partition of $\vec{T_v}$.*

Lemma 1.3 ([6]) *If T is a tree, and G satisfies the odd 2-pebbling property, then $f(T \times G, (x, g)) \leq f(T, x)f(G)$ for every vertex g in G. In particular, if $P_m = x_1 x_2 \ldots x_m$ is a path, then*

$$f(P_m \times G, (x_i, g)) \leq f(P_m, x_i)f(G) = (2^{i-1} + 2^{m-i} - 1)f(G) \leq 2^{m-1}f(G).$$

2 The case $C_{4k+3} \times G$

In 2003, Herscovici [5] proved the following two theorems about cycles.

Theorem 2.1 ([5]) *If G satisfies the odd 2-pebbling property, then*

$$f(C_{2n} \times G) \leq f(C_{2n})f(G) = 2^n f(G).$$

Theorem 2.2 ([5]) *Suppose G is a graph with $m \geq 5$ vertices which satisfies the odd 2-pebbling property and the following inequality*

$$4f_4(G) < 14f(G) - 2(m - 5). \tag{2.1}$$

Then $f(C_{2n+1} \times G) \leq f(C_{2n+1})f(G)$ for $n \geq 3$.

The inequality (2.1) holds for all odd cycles, but does not hold for paths or even cycles. In this section, we show the following theorem.

Theorem 2.3 *If G satisfies the odd 2-pebbling property, then*

$$f(C_{4k+3} \times G) \leq f(C_{4k+3})f(G).$$

Throughout this section, we use the following notation. Let the vertices of C_{4k+3} be $\{v_0, v_1, \ldots, v_{4k+1}, v_{4k+2}\}$ in order. We define the vertex subsets A and B of C_{4k+3} by

$$A = \{v_1, v_2, \ldots, v_{2k}\}, B = \{v_{2k+3}, v_{2k+4}, \ldots, v_{4k+2}\}$$

For simplicity, among $C_{4k+3} \times G$, let $p_i = p(v_i \times G)$, $r_i = r(v_i \times G)$, $p(A) = p(A \times G)$, $p(B) = p(B \times G)$. Thus, the number of pebbles in a distribution on $C_{4k+3} \times G$ is given by $p_0 + p(A) + p(B) + p_{2k+1} + p_{2k+2}$.

Lemma 2.4 ([7]) *The pebbling numbers of the odd cycles C_{4k+1} and C_{4k+3} are*

$$\begin{aligned} f(C_{4k+1}) &= \frac{2^{2k+2} - 1}{3} = 1 + 2^2 + 2^4 + \cdots + 2^{2k}. \\ f(C_{4k+3}) &= \frac{2^{2k+3} + 1}{3} = 1 + 2^1 + 2^3 + \cdots + 2^{2k+1}. \end{aligned}$$

Lemma 2.5 *Let $P_{2k} = x_1x_2 \dots x_{2k}$ be a path with length $2k-1$, and let g be some vertex in a graph G which satisfies the odd 2-pebbling property. Then, from any arrangement of $(2^1 + 2^3 + \cdots + 2^{2k-1})f(G)$ pebbles on $P_{2k} \times G$, it is possible to put a pebble on every (x_i, g) at once, where $i = 1, 3, \dots, 2k-1$.*

Proof. We use induction on k, where the case $k = 1$ is trivial.

Suppose that there are $(2^1 + 2^3 + \cdots + 2^{2k-1})f(G)$ pebbles on $P_{2k} \times G$. Then there are at least $(2^1 + 2^3 + \cdots + 2^{2k-3})f(G)$ pebbles on $\{x_3, x_4, \dots, x_{2k}\} \times G$ (or on $\{x_1, x_2, \dots, x_{2k-2}\} \times G$). By induction, we can use these pebbles to put one pebble to each of these vertices $\{(x_3, g), (x_5, g), \dots, (x_{2k-1}, g)\}$ (or $\{(x_1, g), (x_3, g), \dots, (x_{2k-3}, g)\}$). By Lemma 1.2, $f(P_{2k}, x_1) = 2^{2k-1}$, $f(P_{2k}, x_{2k-1}) = 2^{2k-2} + 1 \leq 2^{2k-1}$. By Lemma 1.3, with the remaining $2^{2k-1}f(G)$ pebbles, one pebble can be moved to (x_1, g)(or (x_{2k-1}, g)), and we are done. ∎

Similarly, we can obtain the following lemma.

Lemma 2.6 *Let $P_{2k+1} = x_1x_2 \dots x_{2k+1}$ be a path with length $2k$, and let g be some vertex in a graph G which satisfies the odd 2-pebbling property. Then, from any arrangement of $(2^2 + 2^4 + \cdots + 2^{2k})f(G)$ pebbles on $P_{2k+1} \times G$, it is possible to put a pebble on every (x_i, g) at once, where $i = 1, 3, \dots, 2k-1$.*

From the proof of Theorem 3.2 in [5], it follows that:

Lemma 2.7 ([5]) *If $p(A) \geq 2^{2k-1}f(G)$, then with $f(C_{4k+3})f(G)$ pebbles on $C_{4k+3} \times G$, one pebble can be moved to (v_0, g).*

Proof of Theorem 2.3:
Suppose that there are $f(C_{4k+3})f(G)$ pebbles located on $C_{4k+3} \times G$. Then

$$p_0 + p_{2k+1} + p_{2k+2} + p(A) + p(B) = (1 + 2^1 + 2^3 + \cdots + 2^{2k+1})f(G). \tag{2.2}$$

Without loss of generality, we may assume that $p(A) \geq p(B)$ and the target vertex is (v_0, g). The case $k = 0$ is trivial, so we assume that $k \geq 1$.

Note that the vertices of $B \cup \{v_{2k+2}\} \cup \{v_0\}$ form a path isomorphic to P_{2k+2}. It follows from Lemma 1.3 that if we move as many pebbles as possible from $v_{2k+1} \times G$ to $v_{2k+2} \times G$, then one pebble could be moved to (v_0, g) unless

$$\frac{p_{2k+1} - r_{2k+1}}{2} + p_{2k+2} + p(B) + p_0 < 2^{2k+1}f(G). \tag{2.3}$$

From Lemma 2.7, we could move one pebble to (v_0, g) unless

$$p(A) < 2^{2k-1}f(G). \tag{2.4}$$

If (2.2) and (2.3) hold, then

$$\frac{p_{2k+1} + r_{2k+1}}{2} + p(A) > (1 + 2^1 + 2^3 + \cdots + 2^{2k-1})f(G). \tag{2.5}$$

From (2.4) and (2.5), we obtain $p_{2k+1} + r_{2k+1} > 2f(G)$, and

$$\frac{p_{2k+1} - (2f(G) - r_{2k+1} + 2)}{2} + p(A) \ \geq \ (2^1 + 2^3 + \cdots + 2^{2k-1})f(G).$$

This implies that we can move enough pebbles from $v_{2k+1} \times G$ to $A \times G$ so that the number of the pebbles on $A \times G$ will reach $(2^1 + 2^3 + \cdots + 2^{2k-1})f(G)$, and at the same time, h_{2k+1} pebbles are kept on $v_{2k+1} \times G$, where

$$h_{2k+1} = \begin{cases} 2f(G) - r_{2k+1} + 2, & \text{if } r_{2k+1} \geq 2, \\ 2f(G), & \text{if } r_{2k+1} \leq 1. \end{cases}$$

Assume that $2x$ pebbles are taken away from $v_{2k+1} \times G$ such that there are x pebbles that reach $A \times G$, i.e.,

$$x + p(A) = (2^1 + 2^3 + \cdots + 2^{2k-1})f(G). \tag{2.6}$$

Step 1. With the h_{2k+1} pebbles on $v_{2k+1} \times G$, we can move one pebble to (v_{2k}, g).

Now there are at least $p_{2k+1} - 2x - h_{2k+1}$ pebbles on $v_{2k+1} \times G$, that is,

$$\begin{aligned} \tilde{p}_{2k+1} &= p_{2k+1} - 2x - h_{2k+1} \\ &= p_{2k+1} + 2p(A) - (2^2 + \cdots + 2^{2k})f(G) - h_{2k+1}. \end{aligned}$$

So the remaining pebbles on $\{v_0, v_{2k+1}, v_{2k+2}\} \times G$ are

$$\begin{aligned} & p_0 + p_{2k+2} + \tilde{p}_{2k+1} \\ & = p_0 + p_{2k+2} + p_{2k+1} + 2p(A) - (2^2 + \cdots + 2^{2k})f(G) - h_{2k+1} \\ & \geq p_0 + p_{2k+2} + p_{2k+1} + p(A) + p(B) - (2^2 + \cdots + 2^{2k})f(G) - h_{2k+1} \\ & \geq (1 + 2^2 + 2^4 + \cdots + 2^{2k})f(G). \end{aligned}$$

Now $p_0 < f(G)$ (otherwise one pebble can be moved to (v_0, g), and we are done), so

$$p_{2k+2} + \tilde{p}_{2k+1} \geq (2^2 + 2^4 + \cdots + 2^{2k})f(G). \tag{2.7}$$

Step 2. It follows from (2.6) and Lemma 2.5 that with $(2^1+2^3+\cdots+2^{2k-1})f(G)$ pebbles on $A \times G$, we can put one pebble to each vertex of $\{(v_1, g), (v_3, g), \ldots, (v_{2k-1}, g)\}$.

Step 3. From the inequality (2.7) and Lemma 2.6, it follows that, with $(2^2 + 2^4 + \cdots + 2^{2k})f(G)$ pebbles on $\{v_{2k+1}, v_{2k+2}\} \times G$, we can put one pebble to each vertex of $\{(v_2, g), (v_4, g), \ldots, (v_{2k}, g)\}$.

The above three steps imply that at least one pebble can be moved to (v_0, g). ∎

3 The case $M(C_{2n}) \times G$

Throughout this section, we will use the following notation (see Figure 2).

Let $C_{2n} = v_0 v_1 \dots v_{2n-1} v_0$. The middle graph of C_{2n}, denoted by $M(C_{2n})$, is obtained from C_{2n} by inserting u_i into $v_i v_{(i+1) \bmod (2n)}$, and connecting $u_i u_{(i+1) \bmod (2n)}$ $(0 \le i \le 2n-1)$. The graph $M^*(C_{2n})$ is obtained from $M(C_{2n})$ by removing the edges $v_i u_i$ for $1 \le i \le n-1$, $u_{n-1}u_n$, $u_j v_{j+1}$ for $n \le j \le 2n-2$ and $u_0 u_{2n-1}$.

We define the vertex subsets A and B of $V(M^*(C_{2n}))$ by

$$A = \{v_1, v_2, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\},$$

$$B = \{v_{n+1}, v_{n+2}, \dots, v_{2n-1}, u_n, u_{n+1}, \dots, u_{2n-1}\}.$$

For simplicity, among $M(C_{2n}) \times G$ (or $M^*(C_{2n}) \times G$), let $p_i = p(v_i \times G)$, $r_i = r(v_i \times G)$, $q_i = p(u_i \times G)$, $s_i = r(u_i \times G)$, $p(A) = p(A \times G)$, $p(B) = p(B \times G)$.

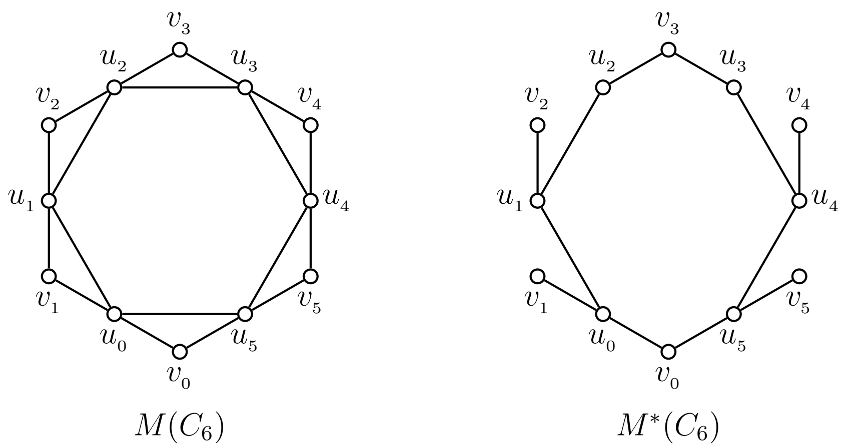

Figure 2: The graphs $M(C_6)$ and $M^*(C_6)$.

Lemma 3.1 ([6]) *Trees satisfy the* 2*-pebbling property.*

Lemma 3.2 ([11]) *If* $n \ge 2$*, then* $f(M(C_{2n})) = 2^{n+1} + 2n - 2$.

From Lemma 1.2 and the proof of Lemma 3.2, it is not hard to obtain the following.

Lemma 3.3 *If* $n \ge 2$*, then* $f(M^*(C_{2n}), v_0) = 2^{n+1} + 2n - 2$.

Proposition 3.4 $M(C_{2n})$ *satisfies the* 2*-pebbling property.*

Proof. By symmetry, it is clear that

$$f(M(C_{2n})) = \max\{f(M(C_{2n}), v_0), f(M(C_{2n}), u_0)\}.$$

Assume that the target vertex is v_0, and $p + l \geq 2f(M(C_{2n})) + 1$. Since $l \leq 4n \leq f(M(C_{2n}))$, we have $p \geq f(M(C_{2n})) + 1$. Thus if there is one pebble located on v_0, then with the remaining $f(M(C_{2n}))$ pebbles, a second pebble can be moved to v_0.

Now, suppose that $p(v_0) = 0$. We will prove that with the same arrangement of pebbles on $M^*(C_{2n})$, two pebbles can be moved to v_0.

Let $H = M^*(C_{2n})$, $C = H[A \setminus v_1]$, and $D = H[B \setminus v_{2n-1}]$. Then by Lemma 1.2,

$$f(C) = f(D) = 1^{n-1} + n - 2,$$

$$f(C \cup \{v_0\}) = f(D \cup \{v_0\}) = 2^n + n - 2,$$
$$f(C \cup \{v_n\}) = f(D \cup \{v_n\}) = 2^n + n - 2.$$

We consider the worst case, which is $p(v_1) = l(v_1) = p(v_{2n-1}) = l(v_{2n-1}) = 1$ (where $l(v_i) = 1$ if there is at least one pebble located on v_i and 0 otherwise), then

$$p(C) + l(C) + p(D) + l(D) + p_n + l_n + 4 \geq 2^{n+2} + 4n - 3,$$

where $p_n = p(v_n), l_n = l(v_n)$.

If $p(C)+l(C) > 2^{n+1}+2n-4$, then by Lemma 3.1, two pebbles can be moved to v_0. Thus we may assume that $p(C)+l(C) \leq 2^{n+1}+2n-4$ and $p(D)+l(D) \leq 2^{n+1}+2n-4$. We will show that both u_0 and u_{2n-1} will get at least two pebbles by a sequece of pebbling moves.

Let $p'_n = 2^{n+1} + 2n - 4 - p(C) - l(C) \geq 0$, and paint all the pebbles on C red along with the p'_n pebbles on v_n. Similarly, paint the pebbles on D black, along with $p''_n = 2^{n+1} + 2n - 4 - p(D) - l(D)$ pebbles on v_n. Since $p_n \geq p'_n + p''_n$, there are enough pebbles on v_n to do this.

Now either $p(C) + l(C) = 2^{n+1} + 2n - 4$ or there are red pebbles on v_n. If equality holds, then $p(C) \geq 2^n + n - 2$, then two red pebbles can be moved to u_0. If there are red pebbles on v_n, then $l'_n = 1$, and the red pebbles satisfy

$$p(C) + l(C) + p'_n + l'_n = 2^{n+1} + 2n - 3,$$

and again two red pebbles can be moved to u_0. Similarly, two black pebbles can be moved to u_{2n-1}, so we can move one red pebble and one black pebble to v_0.

If the target vertex is u_0, then a similar argument can show that there are at least two pebbles which can be moved to u_0. ■

Lemma 3.5 ([5]) *Let $P_k = x_1x_2 \ldots x_k$ be a path, and let g be some vertex in a graph G which satisfies the odd 2-pebbling property. Then, from any arrangement of $(2^k - 1)f(G)$ pebbles on $P_k \times G$, it is possible to put a pebble on every (x_i, g) at once $(1 \leq i \leq k)$.*

Lemma 3.6 *Let $P_k = x_1x_2 \dots x_k$ be a path ($k \geq 2$), and g be some vertex in a graph G which satisfies the odd 2-pebbling property. Then from any arrangement of $(2^k - 2)f(G)$ pebbles on $x_k \times G$, it is possible to put a pebble on every (x_i, g) at once ($1 \leq i \leq k-1$).*

Proof. We use induction on k, where the case $k = 2$ is trivial. If it is true for $k-1$, suppose there are $(2^k - 2)f(G)$ pebbles on $x_k \times G$, we use $(2^{k-1} - 2)f(G)$ pebbles to put a pebble on every (x_i, g) at once ($2 \leq i \leq k-1$), and with the remaining $2^{k-1}f(G)$ pebbles we can put one pebble on (x_1, g). ∎

Lemma 3.7 *Let T_k be the graph obtained from P_k by joining x_i to a new vertex y_i ($1 \leq i \leq k-1$), where $P_k = x_1x_2 \dots x_k$ is a path ($k \geq 2$). Let g be some vertex in a graph G which satisfies the odd 2-pebbling property. Then for any arrangement of $(2^k + k - 3)f(G)$ pebbles on $T_k \times G$, one of the following will occur*

(1) we can put a pebble on every (x_i, g) at once ($1 \leq i \leq k-1$);

(2) we can put two pebbles on (x_1, g).

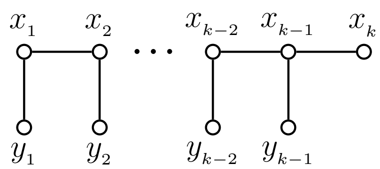

Figure 3: The graph T_k in Lemma 3.7.

Proof. When $k = 2$, by Lemma 1.2 and Lemma 1.3, with $3f(G)$ pebbles on $T_2 \times G$, one pebble can be moved to the vertex (x_1, g).

Suppose that there are $(2^k + k - 3)f(G)$ pebbles on $T_k \times G$ for $k \geq 3$. Let $T'_k = T_k \setminus \{x_1, y_1\}$. Clearly, $T'_k \cong T_{k-1}$.

If $p(T'_k \times G) < (2^{k-1} + k - 4)f(G)$, then $p(P_{x_1y_1} \times G) \geq (2^{k-1} + 1)f(G) \geq 5f(G)$. Then clearly, we can move two pebbles to (x_1, g).

Suppose $p(T'_k \times G) \geq (2^{k-1} + k - 4)f(G)$, and $p_k \geq (2^{k-1} - 2)f(G)$. By Lemma 3.6, using $(2^{k-1} - 2)f(G)$ pebbles on $x_k \times G$, we can put a pebble on every (x_i, g) for $2 \leq i \leq k-1$. With the remaining $(2^{k-1} + k - 1)f(G)$ pebbles, by Lemma 1.2 and Lemma 1.3, we can put one pebble on (x_1, g) for $f(T_k \times G, (x_1, g)) \leq f(T_k, x_1)f(G) = (2^{k-1} + k - 1)f(G)$.

Suppose $p(T'_k \times G) \geq (2^{k-1} + k - 4)f(G)$, and $p_k < (2^{k-1} - 2)f(G)$. We use induction in this case, while the case $k = 2$ holds.

Let r_y be the number of vertices with an odd number of pebbles in $\{y_2, y_3, \dots, y_{k-1}\} \times G$. We only need to take off r_y pebbles from $\{y_2, y_3, \dots, y_{k-1}\} \times G$ so that each vertex in it has an even number of pebbles. It is clear that $r_y \leq (k-2)|V(G)| \leq (k-2)f(G)$, so $r_y + p_k < (2^{k-1} + k - 4)f(G)$. So we can choose $(2^{k-1} + k - 4)f(G)$ pebbles from $T'_k \times G$ which contains all pebbles on $x_k \times G$, so that the number of the remaining pebbles on each vertex of $\{y_2, y_3, \ldots, y_{k-1}\} \times G$ is even except at most one vertex. By induction, with these $(2^{k-1} + k - 4)f(G)$ pebbles we can put one pebble on every (x_i, g) at once for $2 \leq i \leq k-1$ or move two pebbles to (x_2, g) and then at least one pebble can be moved to (x_1, g).

Now we prove that with the remaining $(2^{k-1} + 1)f(G)$ pebbles, one pebble can be moved to (x_1, g).

Let $\tilde{p}_y = \sum_{i=2}^{k-1} \tilde{p}(y_i \times G)$. Let P' denote the path $y_1x_1x_2 \ldots x_{k-1}$, and P'' denote the path $x_1x_2 \ldots x_{k-1}$.

Since the number of the remaining pebbles on each vertex of $\{y_2, y_3, \ldots, y_{k-1}\} \times G$ is even except at most one vertex, then we can move $\lfloor \frac{1}{2}\tilde{p}_y \rfloor$ pebbles from the vertices of $\{y_2, y_3, \ldots, y_{k-1}\} \times G$ to $\{x_2, x_3, \ldots, x_{k-1}\} \times G$.

Case 1. $\tilde{p}_y \leq 2^{k-1}f(G) - 1$. Then

$$\tilde{p}(P' \times G) = (2^{k-1} + 1)f(G) - \tilde{p}_y + \left\lfloor \frac{1}{2}\tilde{p}_y \right\rfloor \geq (2^{k-2} + 1)f(G).$$

By Lemma 1.3, $f(P' \times G, (x_1, g)) \leq f(P', x_1)f(G) = (2^{k-2} + 1)f(G)$, so one pebble can be moved to (x_1, g).

Case 2. $\tilde{p}_y \geq 2^{k-1}f(G)$. Then

$$\tilde{p}(P'' \times G) \geq \left\lfloor \frac{1}{2}\tilde{p}_y \right\rfloor \geq 2^{k-2}f(G).$$

By Lemma 1.3, $f(P'' \times G, (x_1, g)) \leq f(P'', x_1)f(G) = 2^{k-2}f(G)$, so one pebble can be moved to (x_1, g). ∎

Theorem 3.8 *If G satisfies the odd 2-pebbling property, then*

$$f(M(C_{2n}) \times G) \leq f(M(C_{2n}))f(G) = (2^{n+1} + 2n - 2)f(G).$$

Proof. Suppose that there are $(2^{n+1} + 2n - 2)f(G)$ pebbles placed on the vertices of $M(C_{2n}) \times G$. We will show that at least one pebble can be moved to the target vertex.

By symmetry, it is clear that

$$f(M(C_{2n}) \times G) = \max\{f(M(C_{2n}) \times G, (v_0, g)), f(M(C_{2n}) \times G, (u_0, g))\}.$$

So we only need to distinguish two cases.

Case 1. The target vertex is (v_0, g).

Subcase 1.1. $p_n + r_n \leq 2f(G)$.

We remove all the pebbles off $v_n \times G$ such that

$$\begin{aligned}\tilde{p}((M^*(C_{2n}) \setminus v_n) \times G) &= \frac{p_n - r_n}{2} + p(A) + p(B) + p_0 \\ &= -\frac{1}{2}(p_n + r_n) + p_n + p(A) + p(B) + p_0 \\ &\geq (2^{n+1} + 2n - 3)f(G).\end{aligned}$$

By Lemma 1.2, $f(M^*(C_{2n}) \setminus v_n, v_0) = 2^{n+1} + 2n - 3$. According to Lemma 1.3, one pebble can be moved to (v_0, g).

Subcase 1.2. $p_n + r_n > 2f(G)$.

Then we can move two pebbles to (v_n, g). Note that p_n and r_n are of the same-parity, we keep $2f(G) - r_n + 2$ pebbles on $v_n \times G$ so that at least two pebbles still can be moved to (v_n, g), and move the rest pebbles to $A \times G$. So

$$\tilde{p}(A \times G) = \frac{1}{2}(p_n - (2f(G) - r_n + 2)) + p(A) = \frac{p_n + r_n}{2} - f(G) - 1 + p(A). \quad (3.1)$$

By Lemma 1.2, $f(M^*(C_{2n})[B, v_0], v_0) = 2^n + n - 1$, so if we move as many pebbles as possible from $v_n \times G$ to $B \times G$, then one pebble can be moved to (v_0, g) unless

$$\frac{p_n - r_n}{2} + p(B) + p_0 \leq (2^n + n - 1)f(G) - 1. \quad (3.2)$$

If (3.2) holds, then

$$\begin{aligned} \tilde{p}(A \times G) &= \tfrac{1}{2}(p_n + r_n) - f(G) - 1 + p(A) \\ &\geq p_n + p(A) + p(B) + p_0 - f(G) - (2^n + n - 1)f(G) \\ &= (2^{n+1} + 2n - 2)f(G) - (2^n + n)f(G) \\ &= (2^n + n - 2)f(G). \end{aligned} \quad (3.3)$$

It follows from Lemma 1.2 that

$$f(M^*(C_{2n}) \setminus \{v_n, u_{n-1}, v_{n-1}\}, v_0) = 3 \cdot 2^{n-1} + 2n - 4.$$

Thus if we move as many pebbles as possible from $v_n \times G$ to $u_n \times G$, and from $v_{n-1} \times G$ to $u_{n-2} \times G$, then one pebble can be moved to (v_0, g) unless

$$\begin{aligned} &\tfrac{1}{2}(p_n - r_n) + \tfrac{1}{2}(p_{n-1} - r_{n-1}) + p(B) + p_0 + (p(A) - p_{n-1} - q_{n-1}) \\ &\qquad \leq \ (3 \cdot 2^{n-1} + 2n - 4)f(G) - 1. \end{aligned} \quad (3.4)$$

If (3.4) holds, then $\frac{1}{2}(p_n + r_n) + \frac{1}{2}(p_{n-1} + r_{n-1}) + q_{n-1} \geq (2^{n-1} + 2)f(G) + 1$. Thus

$$\left(\frac{1}{2}(p_n + r_n) - f(G) - 1 + q_{n-1}\right) + \left(\frac{1}{2}(p_{n-1} + r_{n-1}) - f(G) - 1\right) \geq 2^{n-1}f(G) - 1. \quad (3.5)$$

Subcase 1.2.1. $\frac{1}{2}(p_n + r_n) - f(G) - 1 + q_{n-1} \geq f(G)$.

Then from (3.3) it follows that, with $f(G)$ pebbles on $u_{n-1} \times G$, one pebble can be moved to (u_{n-1}, g); and from Lemma 3.7 it follows that, with the remaining $(2^n+n-3)f(G)$ pebbles, we can put one pebble to each (u_i, g) for $0 \leq i \leq n-2$ or put two pebbles to (u_0, g), we can move one more pebble to (u_{n-1}, g) with $2f(G) - r_n + 2$ pebbles on $v_n \times G$, so one pebble can be moved to (v_0, g).

Subcase 1.2.2. $\frac{1}{2}(p_n + r_n) - f(G) - 1 + q_{n-1} < f(G)$.

Then from (3.5), we have

$$\frac{p_{n-1}+r_{n-1}}{2}-f(G)-1\geq(2^{n-1}-1)f(G). \tag{3.6}$$

So we can keep $2f(G)-r_{n-1}+2$ pebbles on $v_{n-1}\times G$, so that one pebble can be moved to (u_{n-2},g), and moving no less than $(2^{n-1}-1)f(G)$ pebbles to $u_{n-2}\times G$. With these pebbles, by Lemma 3.5, we can put one pebble to every (u_i,g) at once $(0\leq i\leq n-2)$. So one pebble can be moved to (v_0,g).

Case 2. The target vertex is (u_0,g).

Let $M'(C_{2n})$ be the graph obtained from $M(C_{2n})$ by removing the edges u_iv_{i+1} for $0\leq i\leq n-2$ and u_jv_j for $n+2\leq j\leq 2n-1$ and u_nv_n, u_nv_{n+1}, u_0v_0.

Let $A'=\{u_1,u_2,\ldots,u_{n-1},v_1,v_2,\ldots,v_n\}$ and $B'=\{u_{n+1},u_{n+2},\ldots,u_{2n-1},v_{n+1},v_{n+2},\ldots,v_{2n-1},v_0\}$.

It is clear that $M'(C_{2n})[A']\cong M^*(C_{2n})[A]$, and $M'(C_{2n})[B',u_0]\cong M^*(C_{2n})[B,v_0]$. We only need to prove that one pebble can be moved from $M'(C_{2n})\times G$ to (u_0,g).

Subcase 2.1. $q_n+s_n\leq 2f(G)$. By a similar process as before, one pebble can be moved to (u_0,g).

Subcase 2.2. $q_n+s_n>2f(G)$.

Then by a similar process as before, we can keep $2f(G)-s_n+2$ pebbles on $u_n\times G$ so that two pebbles can be moved to (u_n,g), and move the remaining pebbles to $A'\times G$. So

$$\begin{aligned}\tilde{p}(A'\times G) &= \frac{q_n-(2f(G)-s_n+2)}{2}+p(A')\\ &= \frac{q_n+s_n}{2}-f(G)-1+p(A').\end{aligned}$$

Similarly, if we move as many as possible pebbles from $u_n\times G$ to $B'\times G$, then one pebble can be moved from $B'\times G$ to (u_0,g), unless

$$\tilde{p}(A'\times G) \geq (2^n+n-2)f(G).$$

According to Lemma 3.7, we can put one pebble on (u_i,g) at once for $1\leq i\leq n-1$ or put two pebbles on (u_1,g). With $2f(G)-s_n+2$ pebbles on $u_n\times G$, one more pebble can be moved to (u_{n-1},g). So one pebble can be moved to (u_0,g). ∎

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