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On 4-chromatic subgraphs of $G(\mathbb{Q}^3, d)$

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Abstract

We address the chromatic number of a class of Euclidean distance graphs having vertex set $\mathbb{Q}^3$, the 3-dimensional rational space. It is shown that if s is any positive integer with no prime factors congruent to $2 \pmod 3$, and G is the graph with vertex set $\mathbb{Q}^3$ where any two vertices are adjacent if and only if they are distance $\sqrt{2s}$ apart, then G has chromatic number 4. Along the way, we obtain a few results on the chromatic numbers of certain Euclidean distance graphs having vertex set $\mathbb{Z}^3$, the 3-dimensional integer space. We conclude by constructing an example (possibly the first) of a triangle-free, 4-chromatic distance graph in $\mathbb{Q}^3$.

1 Definitions

Suppose $(\mathbf{X}, \rho)$ is a metric space and $d > 0$. Let $G(\mathbf{X}, d)$ be the graph with vertex set $\mathbf{X}$ where any two vertices are adjacent if and only if they are distant d apart. Define $\chi(\mathbf{X}, d)$ to be the chromatic number of $G(\mathbf{X}, d)$ – that is to say, the minimum number of colors needed to color $\mathbf{X}$ such that any two vertices distance d apart receive different colors. As is customary, we will use $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ to denote the rings of real numbers, rational numbers, and integers, respectively. Throughout this paper, if $\mathbf{X} \subseteq \mathbb{R}^n$, the distance function used will be the usual Euclidean distance metric. Also throughout this paper, we will make use of two non-standard bits of notation. We designate by S the set of all positive, square-free integers whose prime factorization consists solely of factors congruent to $1 \pmod 3$. For any vector $v = \langle a, b, c\rangle$ with $a, b, c \in \mathbb{Q}$, we write Φ_v to denote the set of all vectors formed by permuting the entries of v along with replacing any number of those entries with their negatives. Just to make sure the previous notation is clear, as an example if $v = \langle 1, 0, 0\rangle$, then $\Phi_v = \{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle, \langle -1, 0, 0\rangle, \langle 0, -1, 0\rangle, \langle 0, 0, -1\rangle\}$. If $v = \langle 1, 2, 3\rangle$, then Φ_v consists of forty-eight vectors which we decline to write out.

2 Introduction and Preliminaries

Although coloring $\mathbb{Q}^n$ does not get the same publicity as coloring $\mathbb{R}^n$ (see [8], [12], or [14] for a historical perspective), the subject has been around for quite a while. Woodall introduced the matter as a secondary result in a 1973 paper [15], offhandedly showing that $\chi(\mathbb{Q}^2, 1) = 2$. Benda and Perles also focused on the unit distance in [1], showing that, among other results, $\chi(\mathbb{Q}^3, 1) = 2$ as well. The following question remained unaddressed for some time however, and is in fact still open today:

Given an arbitrary distance d, what is $\chi(\mathbb{Q}^3, d)$?

It is readily seen that for any distance d and any $q \in \mathbb{Q}^+$, the graphs $G(\mathbb{Q}^3, d)$ and $G(\mathbb{Q}^3, qd)$ are isomorphic. This is a straightforward observation, but it leads to the following lemma which we state for easy reference in later sections of the paper.

Lemma 2.1 *For any $d_1, d_2 > 0$, $\chi(\mathbb{Q}^3, d_1) = \chi(\mathbb{Q}^3, d_2)$ if d_1 and d_2 are rational multiples of each other.*

Any distance realized in $\mathbb{Q}^3$ is of the form $\sqrt{q}$ for some $q \in \mathbb{Q}^+$. So to completely resolve the above question, it suffices to determine $\chi(\mathbb{Q}^3, \sqrt{z})$ for every positive, square-free integer z. Fortunately, a lot of the work has already been done. In [9] Johnson showed that for any odd integer p such that $\sqrt{p}$ is a distance actually realized in $\mathbb{Q}^3$, $\chi(\mathbb{Q}^3, \sqrt{p}) = 2$. In [3], Chow showed that for any odd, positive integer p, $\chi(\mathbb{Q}^3, \sqrt{2p}) \geq 3$. And in [7], Johnson, Schneider, and Tiemeyer provide an upper bound for the chromatic numbers in question, demonstrating that for any $d > 0$, $\chi(\mathbb{Q}^3, d) \leq 4$. Putting these facts together, the original question is narrowed down to the following:

Given any odd, positive, square-free integer p,
does $\chi(\mathbb{Q}^3, \sqrt{2p}) = 3$ or does $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$?

The main result of this paper is that for all such values of p whose prime factorization contains no factors congruent to $2 \pmod 3$, $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$. We obtain this conclusion by proving two slightly stronger results concerning colorings of $\mathbb{Z}^3$. Although we find a large class of distances d such that $\chi(\mathbb{Z}^3, d) = 3$ (see Theorem 3.2), it is ultimately still an open question whether there exists d such that $\chi(\mathbb{Q}^3, d) = 3$.

Our arguments will make frequent use of the following lemma.

Lemma 2.2 *Let $v = \langle a, b, c\rangle$ where $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b, c) = 1$ and $a + b + c \equiv 0 \pmod 2$. Let V be the group of vectors generated by those in Φ_v under the usual vector addition. Then $V = \{\langle x, y, z\rangle : x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2\}$.*

Proof Clearly $V \subseteq \{\langle x, y, z\rangle : x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2\}$. Since $\gcd(a, b, c) = 1$ and $a + b + c \equiv 0 \pmod 2$, it must be the case that exactly two of a, b, c are odd. Note that if $\langle v_1, v_2, v_3\rangle \in V$ and σ is any permutation of the sequence (v_1, v_2, v_3), then $\langle \pm\sigma(v_1), \pm\sigma(v_2), \pm\sigma(v_3)\rangle \in V$. Note also that for any even integers

m, n, and p, $\langle ma, 0, 0\rangle$, $\langle nb, 0, 0\rangle$, $\langle pc, 0, 0\rangle \in V$. Since $\gcd(a, b, c) = 1$, there exist integers r, s, t such that $ra + sb + tc = 1$. But this gives us $\langle 2ra, 0, 0\rangle + \langle 2sb, 0, 0\rangle + \langle 2tc, 0, 0\rangle = \langle 2, 0, 0\rangle$. Thus $\langle \pm 2, 0, 0\rangle$, $\langle 0, \pm 2, 0\rangle$, $\langle 0, 0, \pm 2\rangle \in V$. So given any vector $\langle x, y, z\rangle$ satisfying $x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2$, we can select $\langle 0, 0, 0\rangle$ or an appropriate vector from those used to generate V and add to it some combination of multiples of $\langle \pm 2, 0, 0\rangle$, $\langle 0, \pm 2, 0\rangle$, $\langle 0, 0, \pm 2\rangle$ to construct $\langle x, y, z\rangle$. □

In a series of recent papers (see [5] and [6]), Ionascu gives a complete characterization of equilateral triangles whose vertices are points of $\mathbb{Z}^3$. Central to his work in [5] is the following lemma which will feature prominently here as well.

Lemma 2.3 *A square-free positive integer n can be represented as $n = a^2 + ab + b^2$ for some $a, b \in \mathbb{Z}$ if and only if n has no prime factor congruent to $2 \pmod 3$.*

Ionascu attributes this result to Euler. However, the author's own efforts to track down its genesis were ultimately unsuccessful. In any case, this lemma appears to be a fairly well-known fact and can be gleaned from the material presented in many introductory number theory texts or any work focusing on representations of integers using quadratic forms ([2], for example). For our purposes, the following consequence of Lemma 2.3 will suffice.

Lemma 2.4 *Let n be a square-free positive integer which contains no prime factor congruent to $2 \pmod 3$. Then there exist $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = 2n$ and $a + b + c = 0$.*

Proof Given some n as described above, by Lemma 2.3 there exist $a, b \in \mathbb{Z}$ such that $a^2 + ab + b^2 = n$. We then have $2a^2 + 2ab + 2b^2 = 2n$ which implies that $a^2 + b^2 + (-a - b)^2 = 2n$. Now letting $c = -a - b$ we have that $a^2 + b^2 + c^2 = 2n$ and $a + b + c = 0$. □

We feel that before venturing forward it is worthwhile to mention that in the case of $p = 1$, it is trivial to find a 4-chromatic subgraph of $G(\mathbb{Q}^3, \sqrt{2p})$. It is well-known that the points $(0, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 1)$ constitute the vertices of a regular tetrahedron of edge-length $\sqrt{2}$, or in other words, a copy of the complete graph K_4 appearing as a subgraph of $G(\mathbb{Q}^3, \sqrt{2})$. It is shown in [5], however, that there is no other square-free integer p where this occurs.

3 Results

Theorem 3.1 *For every $s \in S$, $\chi(\mathbb{Z}^3, \sqrt{2s}) = 4$.*

Proof Let $s \in S$. As $\chi(\mathbb{Q}^3, \sqrt{2s}) \leq 4$ [7], clearly $\chi(\mathbb{Z}^3, \sqrt{2s}) \leq 4$ as well. By Lemma 2.4 there exists $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = 2s$ and $a + b + c = 0$. Thus the vectors $\langle a, b, c\rangle$, $\langle b, c, a\rangle$, and $\langle c, a, b\rangle$ each have length $\sqrt{2s}$ and together sum to the zero vector. We can use these vectors to create the pair of equilateral triangles in Figure 1, each with side length $\sqrt{2s}$ and vertices in $\mathbb{Z}^3$.

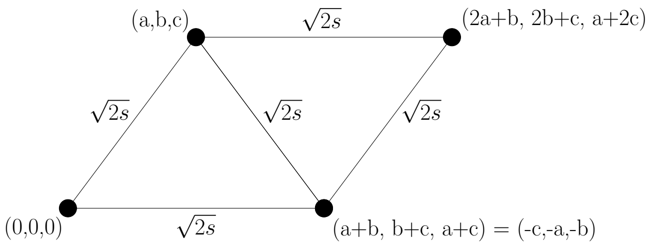

Figure 1

We will now assume there exists some proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$ and obtain a contradiction. The diagram in Figure 1 shows that in any proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$, the vertices $(0, 0, 0)$ and $(2a+b, 2b+c, a+2c)$ must receive the same color. Furthermore, any arrow in $\mathbb{Z}^3$ representing the vector $v = \langle 2a+b, 2b+c, a+2c\rangle$ must have initial and terminal point colored the same color. By permuting the coordinates and changing the sign of some of the coordinates in the above construction, it can be seen that each of the vectors of Φ_v must also have initial and terminal point colored the same color. Let V be the group of $\mathbb{Z}^3$ vectors generated under vector addition by those of Φ_v. In any proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$, any vector in V must have initial and terminal point colored the same color. Note that $(2a + b)^2 + (2b + c)^2 + (a + 2c)^2 = 6s$ and since $6s \equiv 2 \pmod 4$, it must be the case that exactly two of $(2a + b), (2b + c), (a + 2c)$ are odd. Also, since $6s$ is square-free, $\gcd(2a + b, 2b + c, a + 2c) = 1$. Then by Lemma 2.2, $V = \{\langle x, y, z\rangle : x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2\}$. This means that $\langle a, b, c\rangle \in V$ and thus that $(0, 0, 0)$ and (a, b, c) must be colored the same color which is our desired contradiction. Hence $\chi(\mathbb{Z}^3, \sqrt{2s}) = 4$. □

It would appear that the next logical step in uncovering 4-chromatic subgraphs of $\mathbb{Q}^3$ would be focusing on the graph $G(\mathbb{Z}^3, \sqrt{6s})$. However, it just so happens to be the case that $\chi(\mathbb{Z}^3, \sqrt{6s}) \neq 4$. The proof of this fact we include below as a matter of secondary interest.

Theorem 3.2 *For every $s \in S$, $\chi(\mathbb{Z}^3, \sqrt{6s}) = 3$.*

Proof Let $s \in S$. By Lemma 2.4, there exist $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = 6s$ and $a + b + c = 0$. Just as in the proof of Theorem 3.1, we can use the vectors $\langle a, b, c\rangle$, $\langle b, c, a\rangle$, and $\langle c, a, b\rangle$ to create a 3-cycle in $G(\mathbb{Z}^3, \sqrt{6s})$, thus ensuring that $\chi(\mathbb{Z}^3, \sqrt{6s}) \geq 3$. Let $X, Y \in \mathbb{Z}^3$ where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. Let $\Delta_i = x_i - y_i$ for $i \in \{1, 2, 3\}$ and suppose that X and Y are adjacent in $G(\mathbb{Z}^3, \sqrt{6s})$. Then $(\Delta_1)^2+(\Delta_2)^2+(\Delta_3)^2 = 6s$. This means that $(\Delta_1)^2+(\Delta_2)^2+(\Delta_3)^2 \equiv 0 \pmod 3$ but $(\Delta_1)^2 + (\Delta_2)^2 + (\Delta_3)^2 \not\equiv 0 \pmod 9$, which in turn implies that $\Delta_i \not\equiv 0 \pmod 3$ for $i \in \{1, 2, 3\}$. Note here that we are using the fact that for any integer n such that $n \not\equiv 0 \pmod 3$, $n^2 \equiv 1 \pmod 3$. So in order to properly 3-color $G(\mathbb{Z}^3, \sqrt{6s})$, we need only use $\varphi : \mathbb{Z}^3 \to \{0, 1, 2\}$ where for every $X = (x_1, x_2, x_3)$, $\varphi(X) = x_1 \pmod 3$. □

Theorem 3.3 *For every $s \in S$, $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 4$.*

Proof Let $s \in S$. Again, by the results of [7] we have that $\chi(\mathbb{Z}^3, 3\sqrt{6s}) \leq 4$. By Lemmas 2.3 and 2.4, there exist $a, b, c \in \mathbb{Z}$ such that $a^2 + ab + b^2 = 3s$, $a^2 + b^2 + c^2 = 6s$, and $a + b + c = 0$. From this we obtain two facts. First notice that $(3a)^2 + (3b)^2 + (3c)^2 = 54s$ implying that the vertices $(0, 0, 0)$ and $(3a, 3b, 3c)$ are adjacent in $G(\mathbb{Z}^3, 3\sqrt{6s})$. Secondly, the points in Figure 2 define an equilateral triangle with side length $3\sqrt{6s}$ and vertices in $\mathbb{Z}^3$. We note that this triangle parameterization (and others like it) is given in [6].

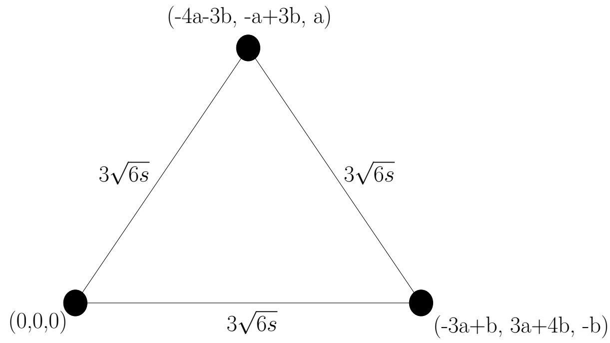

Figure 2

We can extend Figure 2 into three equilateral triangles each with side length $3\sqrt{6s}$ and vertices in $\mathbb{Z}^3$ as is done in Figure 3.

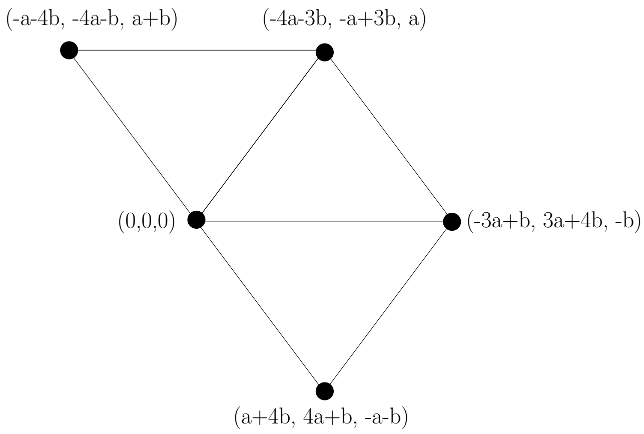

Figure 3

Let V_1 be the vector with initial point $(a + 4b, 4a + b, -a - b)$ and terminal point $(-4a - 3b, -a + 3b, a)$. Let V_2 be the vector with initial point $(-3a + b, 3a + 4b, -b)$ and terminal point $(-a - 4b, -4a - b, a + b)$. Using the same strategy as that in the proof of Theorem 3.1, we will again suppose that $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 3$ and obtain a contradiction. In any proper 3-coloring of $G(\mathbb{Z}^3, 3\sqrt{6s})$ the initial and terminal point of V_1 must be colored the same color and the initial and terminal point of V_2 must be colored the same color. Writing V_1 and V_2 in component form we have that

$$V_1 = \langle -5a - 7b, -5a + 2b, 2a + b\rangle$$
$$V_2 = \langle 2a - 5b, -7a - 5b, a + 2b\rangle$$

We know that $a^2 + ab + b^2 = 3s$. This implies that $\gcd(a, b) = 1$ and either a and b are both congruent to $1 \pmod 3$ or a and b are both congruent to $2 \pmod 3$. Also note that since $a \equiv b \pmod 3$, the individual entries of V_1 and the individual entries of V_2 are each congruent to $0 \pmod 3$. We now desire to show that at least one of V_1 and V_2 has all three entries not congruent to $0 \pmod 9$. This can be done through simple inspection. The chart in Figure 4 lists all possible congruences modulo 9 for a and b along with a vector V_1 or V_2 whose individual entries are each not congruent to $0 \pmod 9$.

With this information in mind, and also considering that V_1 and V_2 both have length $9\sqrt{2s}$, it must be the case that at least one of V_1, V_2 can be written as $\langle 3x, 3y, 3z\rangle$ for some $x, y, z \in \mathbb{Z}$ where $x + y + z \equiv 0 \pmod 2$ and $\gcd(x, y, z) = 1$. Letting $v = \langle 3x, 3y, 3z\rangle$ and again using the same ideas as in the proof of Theorem 3.1, we have that in any proper 3-coloring of $G(\mathbb{Z}^3, 3\sqrt{6s})$ each vector of Φ_v must have initial and terminal point colored the same color.

a	b	Vector whose entries are each not congruent to $0 \pmod 9$
$1 \pmod 9$	$1 \pmod 9$	V_2
$1 \pmod 9$	$4 \pmod 9$	V_1
$1 \pmod 9$	$7 \pmod 9$	V_2
$4 \pmod 9$	$1 \pmod 9$	V_2
$4 \pmod 9$	$4 \pmod 9$	V_1
$4 \pmod 9$	$7 \pmod 9$	V_1
$7 \pmod 9$	$1 \pmod 9$	V_1
$7 \pmod 9$	$4 \pmod 9$	V_2
$7 \pmod 9$	$7 \pmod 9$	V_1
$2 \pmod 9$	$2 \pmod 9$	V_1
$2 \pmod 9$	$5 \pmod 9$	V_2
$2 \pmod 9$	$8 \pmod 9$	V_1
$5 \pmod 9$	$2 \pmod 9$	V_1
$5 \pmod 9$	$5 \pmod 9$	V_1
$5 \pmod 9$	$8 \pmod 9$	V_2
$8 \pmod 9$	$2 \pmod 9$	V_2
$8 \pmod 9$	$5 \pmod 9$	V_1
$8 \pmod 9$	$8 \pmod 9$	V_1

Figure 4

Let V be the group of vectors generated by those of Φ_v under vector addition. By applying Lemma 2.2, we have that $V = \{\langle 3m, 3n, 3p\rangle : m, n, p \in \mathbb{Z}$ and $m+n+p \equiv 0 \pmod 2\}$. Any vector in V must have initial and terminal point colored the same color. But $\langle 3a, 3b, 3c\rangle \in V$ and $(0, 0, 0)$ is adjacent to $(3a, 3b, 3c)$ in $G(\mathbb{Z}^3, 3\sqrt{6s})$. This contradiction implies that $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 4$. □

To now establish the main result of this section, we need only do a little bookkeeping with previous theorems.

Theorem 3.4 *Let p be any positive integer which contains no prime factor congruent to* $2 \pmod 3$*. Then* $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$.

Proof If p is not divisible by 3, we have

$$\begin{aligned} 4 &\geq \chi(\mathbb{Q}^3, \sqrt{2p}) && ([7]) \\ &\geq \chi(\mathbb{Z}^3, \sqrt{2p}) \\ &= 4. && (\text{Theorem 3.1}) \end{aligned}$$

If p is divisible by 3, we have

$$\begin{aligned} 4 &\geq \chi(\mathbb{Q}^3, \sqrt{2p}) && ([7]) \\ &= \chi(\mathbb{Q}^3, 3\sqrt{2p}) && (\text{Lemma 2.1}) \\ &\geq \chi(\mathbb{Z}^3, 3\sqrt{2p}) \\ &= 4. && (\text{Theorem 3.3}) \end{aligned}$$

□

4 Extensions

In this section we denote by T the set of all odd, positive, square-free integers whose prime factorization contains at least one factor congruent to $2 \pmod 3$. We now return to the question presented in Section 2, that of determining $\chi(\mathbb{Q}^3, d)$ for arbitrary distance d. In light of Theorem 3.4, it suffices to determine $\chi(\mathbb{Q}^3, \sqrt{2t})$ for each $t \in T$.

As a consequence of [5], we have that each graph $G(\mathbb{Q}^3, \sqrt{2t})$ is triangle-free. This fact in itself has no bearing on $\chi(\mathbb{Q}^3, \sqrt{2t})$ as there exist triangle-free graphs of arbitrarily large chromatic number (see [10]). It does ensure that no direct application of the technique used in Theorems 3.1 and 3.3 is possible. We can however, offer a modified version of this technique that is shown to be successful in the specific case of $G(\mathbb{Q}^3, \sqrt{10})$.

Theorem 4.1 $\chi(\mathbb{Q}^3, \sqrt{10}) = 4$.

Proof Consider the graph G given in Figure 5. It is known alternately as the Grötzsch graph or as the Mycielskian graph, and is shown in [4] to be the unique triangle-free, 4-chromatic graph with minimum vertex set.

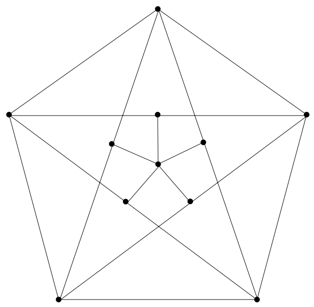

Figure 5

Whether or not G appears as a subgraph of $G(\mathbb{Q}^3, \sqrt{10})$ (or of any other $G(\mathbb{Q}^3, \sqrt{2t})$) we cannot say. We were, however, able to embed a particular subgraph of G in $G(\mathbb{Q}^3, \sqrt{10})$. We denote this graph G', and it is given in Figure 6 with the aforementioned representation.

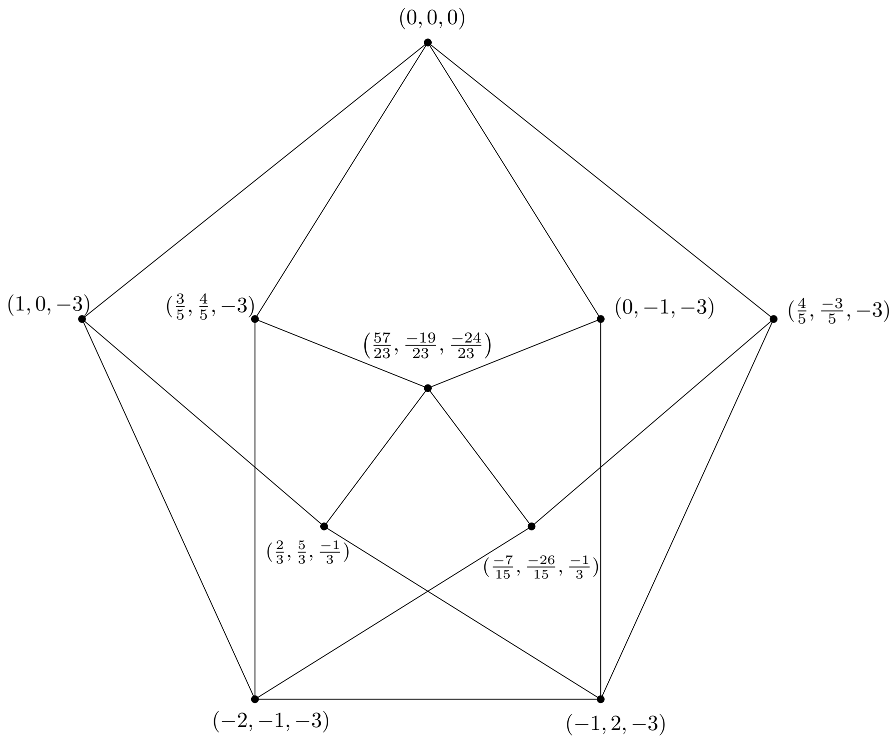

Figure 6

Since G' contains an odd cycle and is a proper subgraph of G, we have that $\chi(G') = 3$. Any proper 3-coloring of G' has the property that the vertices $u = (0, 0, 0)$

and $v = (\frac{57}{23}, \frac{-19}{23}, \frac{-24}{23})$ must be colored the same color. To see this, suppose that G' has been 3-colored, say with the colors red, blue, and green, and suppose that u and v have been colored red and blue, respectively. It follows that $(\frac{3}{5}, \frac{4}{5}, -3)$ and $(0, -1, -3)$ must each be colored green. Vertices $(\frac{2}{3}, \frac{5}{3}, \frac{-1}{3})$ and $(\frac{-7}{15}, \frac{-26}{15}, \frac{-1}{3})$ cannot then both be colored red as that would force the adjacent vertices $(-2, -1, -3)$ and $(-1, 2, -3)$ to both be colored blue. Similarly, the vertices $(\frac{2}{3}, \frac{5}{3}, \frac{-1}{3})$ and $(\frac{-7}{15}, \frac{-26}{15}, \frac{-1}{3})$ cannot both be colored green as that would force each of the vertices $(1, 0, -3)$ and $(\frac{4}{5}, \frac{-3}{5}, -3)$ to be colored blue, which would in turn force $(-2, -1, -3)$ and $(-1, 2, -3)$ to both be colored red. Regarding the symmetry of G', we may assume that $(\frac{2}{3}, \frac{5}{3}, \frac{-1}{3})$ is colored green and $(\frac{-7}{15}, \frac{-26}{15}, \frac{-1}{3})$ is colored red. We are then forced to color $(-2, -1, -3)$ blue, which results in $(1, 0, -3)$ being adjacent to vertices colored red, blue, and green, and requiring the use of a fourth color.

We now assume $\chi(\mathbb{Q}^3, \sqrt{10}) = 3$ and use the same strategy as that used in the proofs of Theorems 3.1 and 3.3. Let w be the vector with initial point u and terminal point v, and note that $w = \langle \frac{57}{23}, \frac{-19}{23}, \frac{-24}{23} \rangle$. The preceding arguments show that in any 3-coloring of $G(\mathbb{Q}^3, \sqrt{10})$, any arrow representing the vector w must have initial and terminal point colored the same color. It follows that the vector $w' = \langle 57, -19, -24 \rangle$ must also have initial and terminal point colored the same color, and furthermore that any vector in $\Phi_{w'}$ must have initial and terminal point colored the same color. Letting W denote the group of vectors generated by those of $\Phi_{w'}$ under vector addition, and observing that $|w'| = \sqrt{4186} = \sqrt{2 \cdot 7 \cdot 13 \cdot 23}$, we may use Lemma 2.2 to conclude that $\langle 3, 1, 0 \rangle \in W$. This contradiction, along with the previously mentioned results of [7], gives us that $\chi(\mathbb{Q}^3, \sqrt{10}) = 4$. $\square$

The proof of Theorem 4.1 is satisfactory in showing that $\chi(\mathbb{Q}^3, \sqrt{10}) = 4$, and we include it as is due to the fact that it uses the machinery already put in place by the proofs of Theorems 3.1 and 3.3. However, it is a bit unsatisfying in the fact that no 4-chromatic subgraph of $G(\mathbb{Q}^3, \sqrt{10})$ is explicitly produced. Fortunately, this issue can be addressed with an application of Rodrigues' well-known rotation formula (originally appearing in [13]) which is usually presented in the following form. For vectors $v_1, v_2 \in \mathbb{R}^3$ with $|v_1| = |v_2|$, a rotational matrix R mapping v_1 to v_2 is given by

$$R = I + (\sin\alpha)K + (1 - \cos\alpha)K^2$$

where α is the angle between v_1 and v_2 and

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

with $k = \langle k_1, k_2, k_3 \rangle$ where either $k = \frac{1}{|v_1 \times v_2|}(v_1 \times v_2)$ or $k = \frac{1}{|v_1 \times v_2|}(v_2 \times v_1)$, depending on the orientation of the vectors v_1 and v_2.

Observing the basic identities $\cos\alpha = \frac{v_1 \cdot v_2}{|v_1||v_2|}$ and $\sin\alpha = \frac{|v_1 \times v_2|}{|v_1||v_2|}$, we have that for $v_1, v_2 \in \mathbb{Q}^3$, the entries of the matrix R are rational. In other words, for $v_1, v_2 \in \mathbb{Q}^3$ with $|v_1| = |v_2|$, there is a rotation mapping v_1 to v_2 which is a bijection on $\mathbb{Q}^3$.

Now returning to the proof of Theorem 4.1, it was shown that if there was a proper 3-coloring of $G(\mathbb{Q}^3, \sqrt{10})$, in such a coloring the vector $w = \langle \frac{57}{23}, \frac{-19}{23}, \frac{-24}{23} \rangle$ must have initial point and terminal point colored the same color. The above observations concerning isometries of $\mathbb{Q}^3$ extend this claim, showing that in a proper 3-coloring of $G(\mathbb{Q}^3, \sqrt{10})$, any $\mathbb{Q}^3$ vector of length $|w|$ must have initial and terminal point colored the same color. As we desire to construct an explicit 4-chromatic subgraph of $G(\mathbb{Q}^3, \sqrt{10})$, our next question is to ask how many vectors of length $|w| = \sqrt{\frac{182}{23}}$ are needed to create a vector of length $\sqrt{10}$. It turns out that three vectors are required. Letting $w_1 = \langle \frac{64}{23}, \frac{9}{23}, \frac{3}{23} \rangle$, $w_2 = \langle \frac{-57}{23}, \frac{19}{23}, \frac{24}{23} \rangle$, and $w_3 = \langle \frac{8}{23}, \frac{39}{23}, \frac{-51}{23} \rangle$, we have that $|w_1| = |w_2| = |w_3| = \sqrt{\frac{182}{23}}$ and $w_1+w_2+w_3 = \langle \frac{15}{23}, \frac{67}{23}, \frac{-24}{23} \rangle$ with $|w_1+w_2+w_3| = \sqrt{10}$. Incidentally, it is impossible to have two $\mathbb{Q}^3$ vectors of length $\sqrt{\frac{182}{23}}$ which together sum to a vector of length $\sqrt{10}$. Proof of this fact is omitted, but we remark that it can be shown by applying the results found in Chapter 5 of [11].

We now return to the graph G' in Figure 6 and consider two rotations of $\mathbb{Q}^3$. R_1 rotates about the origin u and maps v to the point $v' = (\frac{8}{23}, \frac{39}{23}, \frac{-51}{23})$. R_2 rotates about the point v and maps u to the point $u'' = (\frac{-7}{23}, \frac{-28}{23}, \frac{-27}{23})$. Note that the vector with initial point u'' and terminal point v is equal to w_1, the vector with initial point v and terminal point u is equal to w_2, and the vector with initial point u and terminal point v' is equal to w_3. Thus an induced subgraph of $G(\mathbb{Q}^3, \sqrt{10})$ whose vertices are those of G' along with their images under the mappings R_1 and R_2 must have chromatic number 4.

The vertex set and edge set of the subgraph we have produced are given below. Any vertex of the form x' or x'' indicates the image of vertex x under the mapping R_1 or R_2, respectively. Note that there is no vertex labeled u' or v'' as R_1 fixes u and R_2 fixes v.

Vertex Set of a 4-chromatic Subgraph of $G(\mathbb{Q}^3, \sqrt{10})$		
$u = (0,0,0)$		$u'' = (-\frac{7}{23}, -\frac{28}{23}, -\frac{27}{23})$
$v = (\frac{57}{23}, -\frac{19}{23}, -\frac{24}{23})$	$v' = (\frac{8}{23}, \frac{39}{23}, -\frac{51}{23})$	
$a = (1, 0, -3)$	$a' = (-\frac{937497}{429065}, \frac{41983}{33005}, -\frac{162996}{85813})$	$a'' = (\frac{3231632}{1634633}, -\frac{5419943}{1634633}, -\frac{38895}{71071})$
$b = (-2, -1, -3)$	$b' = (-\frac{11339}{3731}, -\frac{494}{287}, -\frac{5007}{3731})$	$b'' = (-\frac{179936}{1634633}, -\frac{7725314}{1634633}, -\frac{4019760}{1634633})$
$c = (-1, 2, -3)$	$c' = (-\frac{13523}{3731}, \frac{227}{287}, \frac{1818}{3731})$	$c'' = (-\frac{481328}{1634633}, -\frac{7296249}{1634633}, \frac{1122741}{1634633})$
$d = (\frac{4}{5}, -\frac{3}{5}, -3)$	$d' = (-\frac{177453}{85813}, \frac{5080}{6601}, -\frac{194391}{85813})$	$d'' = (\frac{16459552}{8173165}, -\frac{5505756}{1634633}, -\frac{418062}{355355})$
$e = (\frac{3}{5}, \frac{4}{5}, -3)$	$e' = (-\frac{2455}{943}, \frac{1323}{943}, -\frac{1056}{943})$	$e'' = (\frac{124904}{89815}, -\frac{64243}{17963}, \frac{6813}{89815})$
$f = (\frac{2}{3}, \frac{5}{3}, -\frac{1}{3})$	$f' = (-\frac{731099}{1287195}, \frac{152186}{99015}, \frac{207293}{257439})$	$f'' = (-\frac{751768}{4903899}, -\frac{6398174}{4903899}, \frac{3155528}{4903899})$
$g = (-\frac{7}{15}, -\frac{26}{15}, -\frac{1}{3})$	$g' = (\frac{24569}{257439}, -\frac{25945}{19803}, -\frac{326422}{257439})$	$g'' = (\frac{1364824}{24519495}, -\frac{7856995}{4903899}, -\frac{71644877}{24519495})$
$h = (0, -1, -3)$	$h' = (-\frac{259}{115}, -\frac{12}{115}, -\frac{51}{23})$	$h'' = (\frac{26968}{17963}, -\frac{67072}{17963}, -\frac{32544}{17963})$

Explicitly stated, the edge set of the subgraph we have constructed is given by

$E = \{ab, af, au, bc, be, bg, cd, cf, ch, dg, du, eu, ev, fv, gv, hu, hv, a'b', a'f', a'u, b'c', b'e', b'g', c'd', c'f', c'h', d'g', d'u, e'u, e'v', f'v', g'v', h'u, h'v', a''b'', a''f'', a''u'', b''c'', b''e'', b''g'', c''d'', c''f'', c''h'', d''g'', d''u'', e''u'', e''v, f''v, g''v, h''u'', h''v, u''v'\}.$

We close by remarking that to our knowledge this is the first example given of a triangle-free, 4-chromatic distance graph in $\mathbb{Q}^3$.

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