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Bijective proofs for some results on the descent polytope

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Abstract

For a word $\mathbf{v}$ in variables $\mathbf{x}$ and $\mathbf{y}$, Chebikin and Ehrenborg found that the number of faces of the descent polytope $\mathrm{DP}_{\mathbf{v}}$ equals the number of factorizations of $\mathbf{v}$ using subfactors of the form $\mathbf{x}^i\mathbf{y}$ and $\mathbf{y}^i\mathbf{x}$ with some additional constraints. They also showed the number of faces of $\mathrm{DP}_{\mathbf{v}}$ equals the number of alternating subwords of $\mathbf{v}$ and raised the problem of finding a bijective proof between these two enumerative results. In this paper, we provide an algorithmically defined combinatorial proof, which also gives a correspondence between factorizations of an $\mathbf{xy}$-word and its reverse. For the alternating descent polytope, we show the faces of the descent polytope are in bijection with certain weighted compositions of n and a class of lattice paths of length $n+1$ contained in the region $-2 \leq y \leq 2$.

1 Introduction

Given a set $S \subseteq [n-1] := \{1, 2, \ldots, n-1\}$, we define the *descent polytope* DP_S as the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \in [0,1]$ such that $x_i \geq x_{i+1}$ if $i \in S$ and $x_i \leq x_{i+1}$ otherwise. Thus DP_S is the order polytope of the ribbon poset; for more details see [5]. We denote by f_i the number of i-dimensional faces in the polytope, and the f-polynomial of an n-dimensional descent polytope DP_S is defined as

$$F_S(t) := \sum_{i=0}^{n} f_i \cdot t^i.$$

For two letters $\mathbf{x}$ and $\mathbf{y}$, let $\mathbf{v}_S = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_{n-1}$ with $\mathbf{v}_i = \mathbf{x}$ if $i \notin S$ and $\mathbf{v}_i = \mathbf{y}$ otherwise. Let $\mathbf{v}^* = \mathbf{v}_{n-1} \cdots \mathbf{v}_2\mathbf{v}_1$ denote the reverse of $\mathbf{v}$, and denote by $\mathbf{v}^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$ for the subset $T = \{j_1, j_2, \ldots, j_k\} \subseteq [n-1]$ with $j_1 < j_2 < \cdots < j_k$.

In particular, we denote the empty word by 1 and $\mathbf{v}^T = 1$ for $T = \emptyset$. For more information on the combinatorial properties of words, see [2, 3].

We further denote $F_S(t)$ (respectively, DP_S) by $F_{\mathbf{v}}(t)$ (respectively, $\mathrm{DP}_{\mathbf{v}}$) since $\mathbf{v} = \mathbf{v}_S$ encodes both $S \subseteq [n-1]$ and the dimension. Chebikin and Ehrenborg [1, Theorem 2.2] expressed the f-polynomial $F_{\mathbf{v}}(t)$ as

$$F_{\mathbf{v}}(t) = 1 + \sum_{T \subseteq [n-1]} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}, \tag{1.1}$$

where $\kappa(1) = 1$ for $\mathbf{v} = 1$; otherwise, $\kappa(\mathbf{v}) = 2 + |\{i : \mathbf{v}_i \neq \mathbf{v}_{i+1}\}|$.

An $\mathbf{xy}$-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_k$ is said to be *alternating* if $\mathbf{v}_i \neq \mathbf{v}_{i+1}$ for all $1 \leq i \leq k-1$. From Eq. (1.1), Chebikin and Ehrenborg [1, Corollary 2.3] noticed that for an $\mathbf{xy}$-word $\mathbf{v}$ of length $n-1$, the number of vertices of the descent polytope $\mathrm{DP}_{\mathbf{v}}$ is one greater than the number of subsets $T \subseteq [n-1]$ for which the word $\mathbf{v}^T$ is alternating.

Chebikin and Ehrenborg [1, Theorem 3.2] further studied the non-commutative rational generating function $\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v}$ for the f-polynomial $F_{\mathbf{v}}$. They then specialized it to obtain a more concise expression [1, Corollary 3.3] for $F_{\mathbf{v}}$, that is

$$F_{\mathbf{v}}(t) = 1 + \sum_{(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k)} (t+1)^k, \tag{1.2}$$

where the sum ranges over factorizations of the $\mathbf{xy}$-word $\mathbf{v} = \mathbf{u}_1 \cdots \mathbf{u}_{k-1} \cdot \mathbf{u}_k$ such that the factors $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ are of the form $\mathbf{x}^i\mathbf{y}$ or $\mathbf{y}^i\mathbf{x}$ for $i \geq 0$, and where the last factor $\mathbf{u}_k$ does not have any constraint. Note that $\mathbf{u}_k = 1$ is allowed for the last factor. By combining Eqs. (1.1) and (1.2), Chebikin and Ehrenborg observed the following.

Proposition 1.1. *The number of factorizations of an* $\mathbf{xy}$*-word* $\mathbf{v}$ *is equal to the number of alternating subwords of* $\mathbf{v}$*.*

Let $\mathbf{z}_n$ be the alternating word of length n with initial letter $\mathbf{x}$, i.e., $\mathbf{z}_n = \mathbf{xyxyx} \cdots$. Substituting $\mathbf{z}_{n-1}$ for $\mathbf{v}$ in formula (1.2), Chebikin and Ehrenborg [1, Corollary 3.5] found that

$$F_{\mathbf{z}_{n-1}}(t) = 1 + \sum_{(c_1, c_2, \ldots, c_k)} (t+1)^k, \tag{1.3}$$

where the sum ranges over all compositions $(c_1, c_2, \ldots, c_k)$ of n such that $c_1, \ldots, c_{k-1} \in \{1, 2\}$. By setting $t = 1$ in formula (1.3), Chebikin and Ehrenborg derived the sequence

$$\{F_{\mathbf{z}_{n-1}}(1)\}_{n \geq 1} = \{3, 7, 19, 51, 139, \ldots\}.$$

A different combinatorial interpretation for this sequence is given in [4] (A052948), as the number of paths from $(0,0)$ to $(n+1, 0)$ with allowed steps $(1,1), (1,0)$ and $(1,-1)$ contained within the region $-2 \leq y \leq 2$. Therefore, Chebikin and Ehrenborg obtained the following.

Proposition 1.2. *The sequence A052948 in [4] counts both the number of faces of the descent polytope* $\mathrm{DP}_{\mathbf{z}_{n-1}}$ *and the number of those paths described above.*

Chebikin and Ehrenborg [1] derived Propositions 1.1 and 1.2 from the generating function point of view and left the open problems whether there exist combinatorial proofs. This paper aims to present relevant proofs. Our bijection in the proof of Proposition 1.1 leads to an algorithmic correspondence between the factorizations of $\mathbf{v}$ and $\mathbf{v}^*$ as well.

2 Bijective proof of Proposition 1.1

In this section, we will construct a bijection between $\mathcal{F}(\mathbf{v})$, the set of factorizations of $\mathbf{v}$, and $\mathcal{A}(\mathbf{v})$, the set of alternating subwords of $\mathbf{v}$. We begin with some notations for simplicity. For a word $w = w_1 w_2 \cdots w_k$, we denote by $\mathbb{F}(w) := w_1$ and $\mathbb{L}(w) := w_k$ the first and last letter of w, respectively, and let $w + w_{k+1} := w_1 w_2 \cdots w_k w_{k+1}$.

Theorem 2.1. *For an* $\mathbf{xy}$*-word* $\mathbf{v}$*, the following map* η *is a bijection between the set* $\mathcal{F}(\mathbf{v})$ *and set* $\mathcal{A}(\mathbf{v})$*.*

Proof. Let $\mathbf{v} = \mathbf{u}_1 \cdot \mathbf{u}_2 \cdots \mathbf{u}_{k-1} \cdot \mathbf{u}_k$ be a factorization of $\mathbf{v}$ with k factors. For $k = 1$, we define $\eta(\mathbf{v}) = 1$, i.e., an empty word. For $k \geq 2$, we will construct a subword w as follows:

1) We initialize $w := \mathbb{L}(\mathbf{u}_{k-1})$ and $j := k - 2$;

2) This process stops for $j = 0$; otherwise, let $a := \mathbb{L}(w)$ and $b := \mathbb{L}(\mathbf{u}_j)$;

3) For $a \neq b$, we set $w := w + b$ and $j := j - 1$, and go to step 2);

4) For $a = b$, we consider the following two cases:

   i) For $|\mathbf{u}_j| > 1$, $\mathbf{u}_j$ is of the form either $\mathbf{y}^i\mathbf{x}$ or $\mathbf{x}^i\mathbf{y}$ $(i \geq 1)$. We set $w := w + \mathbb{F}(\mathbf{u}_j)$, $j := j - 1$ and go to step 2);

   ii) For $|\mathbf{u}_j| = 1$, we stop if $|\mathbf{u}_1| = \cdots = |\mathbf{u}_j| = 1$; otherwise, we find the largest l such that $l < j$ and $|\mathbf{u}_l| > 1$. Set $j := l$ and then go to step 2).

Therefore, we get an alternating subword w of $\mathbf{v}$, and finally set $\eta(\mathbf{v}) = w^*$.

For the converse, we could get the map from an alternating subword $\mathbf{v}^T$ to one factorization of $\mathbf{v}$ by reversing the above procedure. Given a word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_n$ and an alternating subword $\mathbf{v}^T$, if $\mathbf{v}^T = 1$, then $\mathbf{v}$ itself is a factor; if $\mathbf{v}^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$, then $\mathbf{v}_{j_k+1} \cdots \mathbf{v}_n$ is the last factor of $\mathbf{v}$, since there is no constraint on the last factor. We can recover the factorization of $\mathbf{v}$ from right to left by the above rules and using the requirements that other factors are in the form $\mathbf{y}^i\mathbf{x}$ or $\mathbf{x}^i\mathbf{y}$ $(i \geq 0)$. Thus, the map η is a bijection. ∎

Let us give an example to illustrate the bijection η. For $\mathbf{v} = \mathbf{yyx} \cdot \mathbf{yx} \cdot \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{yyyx} \cdot \mathbf{yx} \cdot 1$, we label the letters $\mathbf{x}$ and $\mathbf{y}$ from left to right in natural order so as to distinguish them. Say $\mathbf{v} = \mathbf{u}_1 \cdot \mathbf{u}_2 \cdot \mathbf{u}_3 \cdot \mathbf{u}_4 \cdot \mathbf{u}_5 \cdot \mathbf{u}_6 \cdot \mathbf{u}_7$, where $\mathbf{u}_1 = \mathbf{y}_1\mathbf{y}_2\mathbf{x}_1$, $\mathbf{u}_2 = \mathbf{y}_3\mathbf{x}_2$, $\mathbf{u}_3 = \mathbf{x}_3$, $\mathbf{u}_4 = \mathbf{x}_4$, $\mathbf{u}_5 = \mathbf{y}_4\mathbf{y}_5\mathbf{y}_6\mathbf{x}_5$, $\mathbf{u}_6 = \mathbf{y}_7\mathbf{x}_6$, $\mathbf{u}_7 = 1$.

According to the bijection η, the corresponding alternating subword is determined by the following steps. We initialize $w := \mathbb{L}(\mathbf{u}_6) = \mathbf{x}_6$. For $j = 5$, we have $\mathbb{L}(\mathbf{u}_5) = \mathbb{L}(w)$ and $w := w + \mathbb{F}(\mathbf{u}_5) = \mathbf{x}_6\mathbf{y}_4$ since $|\mathbf{u}_5| > 1$. For $j = 4$, we have $\mathbb{L}(\mathbf{u}_4) \neq \mathbb{L}(w)$, and we set $w := \mathbf{x}_6\mathbf{y}_4\mathbf{x}_4$. For $j = 3$, we have $\mathbb{L}(\mathbf{u}_3) = \mathbb{L}(w)$, and skip this factor from $|\mathbf{u}_3| = 1$. Similar analysis can be applied to determine the remaining elements of w, and finally we have $w := \mathbf{x}_6\mathbf{y}_4\mathbf{x}_4\mathbf{y}_3\mathbf{x}_1$, thereby $\eta(\mathbf{v}) = \mathbf{x}_1\mathbf{y}_3\mathbf{x}_4\mathbf{y}_4\mathbf{x}_6$.

Table 1 gives an illustration of the correspondence between all the eighteen factorizations of $\mathbf{v} = \mathbf{yxxyx}$ and the alternating subwords of $\mathbf{v}$.

$\mathbf{v}$	$\mathbf{v}^T$	$\mathbf{v}$	$\mathbf{v}^T$
$\mathbf{y}_1\mathbf{x}_1\mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$	1	$\mathbf{y}_1 \cdot \mathbf{x}_1\mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{y}_2\mathbf{x}_3$
$\mathbf{y}_1 \cdot \mathbf{x}_1\mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$	$\mathbf{y}_1$	$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_3$	$\mathbf{y}_1\mathbf{x}_2\mathbf{y}_2$
$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$	$\mathbf{x}_1$	$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{x}_1\mathbf{y}_2\mathbf{x}_3$
$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$	$\mathbf{y}_1\mathbf{x}_1$	$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2\mathbf{x}_3 \cdot 1$	$\mathbf{y}_1\mathbf{x}_3$
$\mathbf{y}_1 \cdot \mathbf{x}_1\mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3$	$\mathbf{y}_2$	$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_3$	$\mathbf{x}_2\mathbf{y}_2$
$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2\mathbf{x}_3$	$\mathbf{y}_1\mathbf{x}_2$	$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{y}_1\mathbf{x}_1\mathbf{y}_2\mathbf{x}_3$
$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3$	$\mathbf{x}_1\mathbf{y}_2$	$\mathbf{y}_1\mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{y}_1\mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$
$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2\mathbf{x}_3 \cdot 1$	$\mathbf{x}_3$
$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2\mathbf{y}_2 \cdot \mathbf{x}_3$	$\mathbf{y}_1\mathbf{x}_1\mathbf{y}_2$	$\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{x}_2\mathbf{y}_2\mathbf{x}_3$

Table 1: The correspondence between $\mathcal{F}(\mathbf{v})$ and $\mathcal{A}(\mathbf{v})$ for $\mathbf{v} = \mathbf{yxxyx}$.

It is noteworthy that $\mathrm{DP}_{\mathbf{v}}$ and $\mathrm{DP}_{\mathbf{v}^*}$ yield the same descent polytope up to an affine transformation, and therefore their f-polynomials agree, i.e., $F_{\mathbf{v}} = F_{\mathbf{v}^*}$. However, their f-polynomials are quite different from Eq. (1.2), and Chebikin and Ehrenborg [1] asked for a bijection between the factorizations of $\mathbf{v}$ and $\mathbf{v}^*$. We could easily obtain the desired bijection by using our bijection η.

Corollary 2.1. *For an* $\mathbf{xy}$*-word* $\mathbf{v}$*, there is a bijection* θ *between the factorizations of* $\mathbf{v}$ *and* $\mathbf{v}^*$*.*

Proof. Given a factorization of the word $\mathbf{v}$, define $\theta(\mathbf{v}) := \eta^{-1}(\eta(\mathbf{v})^*)$. Since $\eta(\mathbf{v})$ is an alternating subword of $\mathbf{v}$, we find that $\eta(\mathbf{v})^*$ is an alternating subword of $\mathbf{v}^*$. Therefore, θ is the desired bijection from the properties of η. ∎

An illustrative example is given in Table 2, which shows how the bijection θ works between the factorizations of $\mathbf{v} = \mathbf{xyxx}$ and $\mathbf{v}^* = \mathbf{xxyx}$.

3 Bijective proof of Proposition 1.2

Let $\mathcal{L}_n$ denote the set of paths from $(0, 0)$ to $(n + 1, 0)$ with allowable up steps $u = (1, 1)$, down steps $d = (1, -1)$ and horizontal steps $h = (1, 0)$ contained within

$\mathbf{v}$	$\mathbf{v}^T$	$(\mathbf{v}^T)^*$	$\mathbf{v}^*$
$\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2\mathbf{x}_3$	1	1	$\mathbf{x}_3\mathbf{x}_2\mathbf{y}_1\mathbf{x}_1$
$\mathbf{x}_1 \cdot \mathbf{y}_1\mathbf{x}_2\mathbf{x}_3$	$\mathbf{x}_1$	$\mathbf{x}_1$	$\mathbf{x}_3 \cdot \mathbf{x}_2 \cdot \mathbf{y}_1\mathbf{x}_1 \cdot 1$
$\mathbf{x}_1\mathbf{y}_1 \cdot \mathbf{x}_2\mathbf{x}_3$	$\mathbf{y}_1$	$\mathbf{y}_1$	$\mathbf{x}_3\mathbf{x}_2\mathbf{y}_1 \cdot \mathbf{x}_1$
$\mathbf{x}_1 \cdot \mathbf{y}_1\mathbf{x}_2 \cdot \mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3 \cdot \mathbf{x}_2 \cdot \mathbf{y}_1\mathbf{x}_1$
$\mathbf{x}_1 \cdot \mathbf{y}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_3 \cdot \mathbf{x}_2\mathbf{y}_1\mathbf{x}_1$
$\mathbf{x}_1 \cdot \mathbf{y}_1 \cdot \mathbf{x}_2\mathbf{x}_3$	$\mathbf{x}_1\mathbf{y}_1$	$\mathbf{y}_1\mathbf{x}_1$	$\mathbf{x}_3\mathbf{x}_2\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot 1$
$\mathbf{x}_1\mathbf{y}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_3$	$\mathbf{y}_1\mathbf{x}_2$	$\mathbf{x}_2\mathbf{y}_1$	$\mathbf{x}_3 \cdot \mathbf{x}_2 \cdot \mathbf{y}_1 \cdot \mathbf{x}_1$
$\mathbf{x}_1\mathbf{y}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{y}_1\mathbf{x}_3$	$\mathbf{x}_3\mathbf{y}_1$	$\mathbf{x}_3 \cdot \mathbf{x}_2\mathbf{y}_1 \cdot \mathbf{x}_1$
$\mathbf{x}_1 \cdot \mathbf{y}_1 \cdot \mathbf{x}_2 \cdot \mathbf{x}_3$	$\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2$	$\mathbf{x}_2\mathbf{y}_1\mathbf{x}_1$	$\mathbf{x}_3 \cdot \mathbf{x}_2 \cdot \mathbf{y}_1 \cdot \mathbf{x}_1 \cdot 1$
$\mathbf{x}_1 \cdot \mathbf{y}_1\mathbf{x}_2 \cdot \mathbf{x}_3 \cdot 1$	$\mathbf{x}_1\mathbf{y}_1\mathbf{x}_3$	$\mathbf{x}_3\mathbf{y}_1\mathbf{x}_1$	$\mathbf{x}_3 \cdot \mathbf{x}_2\mathbf{y}_1 \cdot \mathbf{x}_1 \cdot 1$

Table 2: The correspondence between factorizations of $\mathbf{v}$ and $\mathbf{v}^*$ for $\mathbf{v} = \mathbf{xyxx}$.

the region $-2 \le y \le 2$. A path in $\mathcal{L}_n$ is called a *positive* (respectively, *negative*) path if all the steps of the path are on or above (respectively, on or below) the x-axis. A point of a path with y-coordinate k is said to be at level k. The level of a step (or a path) is the level of its endpoint. By a return step we mean a d step or a u step at level 0. A *primitive* path is a path with just one return step, and we define a positive (respectively, negative) primitive path similarly. Denote by $\mathcal{P}(n)$ the set of primitive paths in $\mathcal{L}_n$, and we have

Lemma 3.1. *For $k \ge 1$, $|\mathcal{P}(k)| = 2^k$.*

Proof. Given a positive primitive path in $\mathcal{P}(k)$, there is only one down step from level 1 to level 0, namely the last step. Assume we leave the x-axis at position $x = j$, and note that $0 \le j \le k-1$. For $j \ne k-1$, we claim the remaining $k-j-2$ steps each having two choices. Namely, if you are currently at level 1 then your next step is either a horizontal step h or an up step u, whereas if you are currently at level 2 then your next step is either a horizontal step h or a down step d. Hence for $0 \le j < k-1$, we have 2^{k-j-2} possible paths. When $j = k-1$, there is only one path, namely the path consisting of the initial $k-2$ horizontal steps followed by one up step u and one down step d. The total number of positive primitive paths in $\mathcal{P}(k)$ is thus $1 + \sum_{j=0}^{k-2} 2^{k-j-2} = 2^{k-1}$.

We can obtain all the negative primitive paths in $\mathcal{P}(k)$ by reflecting the positive primitive paths in $\mathcal{P}(k)$ over the x-axis. Thus the total number of primitive paths in $\mathcal{P}(k)$ is $|\mathcal{P}(k)| = 2 \cdot 2^{k-1} = 2^k$ as claimed. ∎

A *composition* of a positive integer n is a finite sequence of positive integers $c_1, c_2, \ldots, c_k$ such that $c_1 + c_2 + \cdots + c_k = n$, and we say that this composition has k parts if there are exactly k summands appeared in it. A composition of n can also be represented as a line of n dots in which there is at most one vertical bar in between each of the $n-1$ spaces determined by these points. For example, the composition $10 = 4 + 1 + 3 + 2$ can be represented as

$$\bullet\ \ \bullet\ \ \bullet\ \ \bullet \,\big|\, \bullet \,\big|\, \bullet\ \ \bullet\ \ \bullet \,\big|\, \bullet\ \ \bullet$$

A *weighted composition* is one where each part is assigned a given weight. Let $\mathcal{W}_1(n)$ be the set of weighted compositions of n such that each part has one dot and the weight of each part is given by 1 or t. Let $\mathcal{W}_2(n)$ be the set of weighted compositions of n such that there are at most two dots in each part except the last one, and each part is given by the weight 1 or t.

From Eq. (1.3), we can identify a face of the descent polytope $\mathrm{DP}_{\mathbf{z}_{n-1}}$ as some weighted composition. In order to give a correspondence between paths in $\mathcal{L}_n$ and faces in descent polytope $\mathrm{DP}_{\mathbf{z}_{n-1}}$, it suffices to provide a bijection between paths in $\mathcal{L}_n$ and weighted compositions in $\mathcal{W}_2(n)$.

It is easy to see that the number of weighted compositions in $\mathcal{W}_1(n)$ is 2^n since every part has two different weight assignment, and this gives $|\mathcal{P}(n)| = |\mathcal{W}_1(n)|$ from Lemma 3.1. It motivates us to first consider a bijection between primitive paths of length $n+1$ and the set $\mathcal{W}_1(n)$.

Lemma 3.2. *There is a bijection φ between the set $\mathcal{P}(n)$ and set $\mathcal{W}_1(n)$.*

Proof. Given a weighted composition in $\mathcal{W}_1(n)$, it is uniquely determined by the weight of the n parts, and thus we encode such a composition by a sequence $c = c_1c_2\cdots c_n$, where c_i is the weight of the i-th part of the composition with $c_i = 1$ or t.

Let $P = p_1p_2\cdots p_{n+1}$ be a primitive path in $\mathcal{P}(n)$, where p_i $(1 \le i \le n+1)$ equals u, d or h. To establish the correspondence $\varphi(P)$, it suffices to determine a sequence $c = c_1c_2\cdots c_n$. We first define $c_n = t$ for positive path P; otherwise $c_n = 1$, and this distinguishes positive primitive paths from negative primitive paths. We claim that $c_1, c_2, \ldots, c_{n-1}$ are determined by the first $n-1$ steps $p_1, p_2, \ldots, p_{n-1}$ of P as follows.

For $1 \le i \le n-1$ and $c_n = t$, if p_i is a first up step of P or an up step at level 2 or a horizontal step at level 2, then $c_i = t$; otherwise $c_i = 1$. For $1 \le i \le n-1$ and $c_n = 1$, if p_i is a first down step of P or a down step at level -2 or a horizontal step at level -2, then $c_i = t$; otherwise $c_i = 1$. Apparently, $\varphi(P)$ is an element of $\mathcal{W}_1(n)$.

To show that φ is a bijection, it remains to construct the inverse map φ^{-1} from $\mathcal{W}_1(n)$ to $\mathcal{P}(n)$. Let $c = c_1c_2\cdots c_n$ be a weighted composition in $\mathcal{W}_1(n)$, and suppose $P = \varphi^{-1}(c)$. If $c_n = t$, then P is positive, and if $c_n = 1$, then P is negative. Without loss of generality, we assume that $c_n = t$, and the path P can be generated recursively as follows:

1. Set $P_0 = \emptyset$ be the empty path, and the level of P_0 is 0.
2. For $1 \le i \le n-1$, the path P_i is obtained from P_{i-1} by adding one step as follows:
   (i) For the path P_{i-1} at level 0 or 1, we add a horizontal step after P_{i-1} if $c_i = 1$; otherwise, we add an up step after P_{i-1}.
   (ii) For the path P_{i-1} at level 2, we add a down step after P_{i-1} if $c_i = 1$; otherwise, we add a horizontal step after P_{i-1}.

3. The path P is obtained from P_{n-1} by setting $P := P_{n-1}ud$ if the path P_{n-1} is at level 0; $P := P_{n-1}hd$ if the path P_{n-1} is at level 1; and $P := P_{n-1}dd$ if the path P_{n-1} is at level 2.

From the previous construction, we can easily see that P is primitive and lies in the region $0 \leq y \leq 2$, and the negative path P can be constructed similarly for $c_n = 1$. Thus we obtain the desired bijection between the set $\mathcal{P}(n)$ and set $\mathcal{W}_1(n)$. ∎

Now we are in the position to give a combinatorial proof of Proposition 1.2.

Theorem 3.1. *There exists a bijection ψ between the set $\mathcal{L}_n$ and set $\mathcal{W}_2(n)$.*

Proof. For a path $P \in \mathcal{L}_n$, we first decompose it into the form $P = P_1P_2\cdots P_kh^m$ $(m \geq 0)$, where each P_i $(1 \leq i \leq k)$ is a primitive path. The weighted composition $\psi(P)$ can be defined as follows.

For each primitive path P_i $(1 \leq i \leq k-1)$, let C_i be the weighted composition obtained from $\varphi(P_i)$ by adding one dot in the last of its part. For the last primitive path P_k, let C_k be the weighted composition obtained from $\varphi(P_k)$ by adding m dots in the last of its part if there are m horizontal steps after the path P_k in P. Then the desired composition $\psi(P)$ is $C_1C_2\cdots C_k$. Since the length of the composition C_i $(1 \leq i \leq k-1)$ equals the length of the path P_i, and the length of the composition C_k equals the length of the path P_k plus $m-1$. Thus, $\psi(P)$ is a composition of $\mathcal{W}_2(n)$.

It is routine to verify that for any weighted composition of n, one may reverse every step of the map ψ to obtain a path in $\mathcal{L}_n$. Thus the map ψ is the desired bijection. ∎

In Figure 1, we give an example for a path in $\mathcal{L}_{19}$, where the vertical lines are used to indicate the primitive path decomposition and the labels on the parts of weight 1 are omitted.

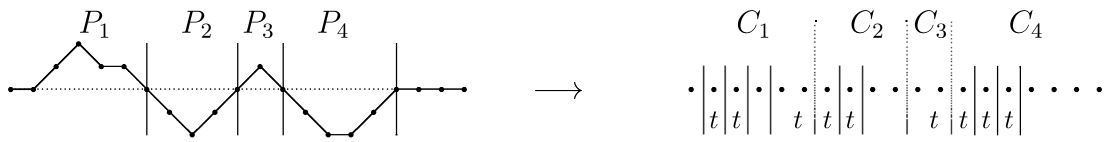

Figure 1: An illustration of the bijection ψ on a path in $\mathcal{L}_{19}$.

For completeness, Figure 2 shows the correspondence between all the paths in $\mathcal{L}_n$ and weighted compositions in $\mathcal{W}_2(n)$ for $n = 3$.

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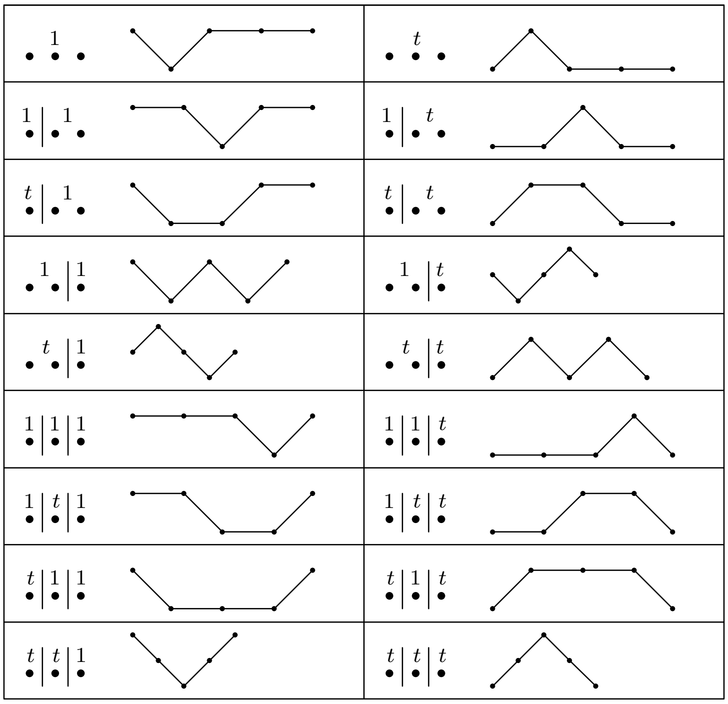

Figure 2: The correspondence between $\mathcal{L}_3$ and $\mathcal{W}_2(3)$.

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