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Covering a subset with two cycles

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Abstract

Let G be a graph of order n. Let W be a subset of $V(G)$ with $|W| \geq 6$. We show that if $d(x) \geq 2n/3$ for each $x \in W$, then for any partition $|W| = n_1 + n_2$ with $n_1 \geq 3$ and $n_2 \geq 3$, G contains two vertex-disjoint cycles C_1 and C_2 such that C_1 contains n_1 vertices of W and C_2 contains n_2 vertices of W.

1 Introduction

Let G be a graph of order n. A set of subgraphs of G is said to be independent if no two of them have any common vertex in G. Corrádi and Hajnal [3] investigated the maximum number of independent cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k independent cycles. In particular, when the order of G is exactly $3k$, then G contains k independent triangles. El-Zahar [4] proved that if G is a graph of order $n_1 + n_2$ with $n_1 \geq 3$ and $n_2 \geq 3$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil$, then G contains two independent cycles of orders n_1 and n_2, respectively. Sauer and Spencer in their work [5] conjectured that if the minimum degree of G is at least $2n/3$ then G contains every graph of order n with maximum degree of at most 2. This conjecture was proved by Aigner and Brandt [1]. In [7], we proposed the following conjecture:

Conjecture [7] *Let G be a graph of order $n \geq 3$. Let W be a subset of $V(G)$ with $|W| \geq 3k$ where k is a positive integer. Suppose that $d(x) \geq 2n/3$ for each $x \in W$. Then for any integer partition $|W| = n_1 + \cdots + n_k$ with $n_i \geq 3 (1 \leq i \leq k)$, G contains k independent cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap W| = n_i$ for all $1 \leq i \leq k$.*

This conjecture is supported by the following theorem:

Theorem A [7] *Let G be a graph of order $n \geq 3$. Let W be a subset of $V(G)$ with $|W| \geq 3k$ where k is a positive integer. Suppose that $d(x) \geq 2n/3$ for each $x \in W$. Then G contains k independent cycles such that each of the k cycles contains at least three vertices of W.*

Our work is also motivated by the work of Ronghua Shi [6], who showed that if G is 2-connected and $d(x) \geq n/2$ for each $x \in U$ then G contains a cycle passing through all the vertices of U, where U is a subset of $V(G)$.

In this paper, we prove the following:

Theorem B *Let G be a graph of order n. Let W be a subset of $V(G)$ with $|W| \geq 6$. If $d(x) \geq 2n/3$ for each $x \in W$, then for any partition $|W| = n_1 + n_2$ with $n_1 \geq 3$ and $n_2 \geq 3$, G contains two independent cycles C_1 and C_2 such that C_1 contains n_1 vertices of W and C_2 contains n_2 of W.*

We discuss only finite simple graphs and use standard terminology and notation from [2] except as indicated. Let G be a graph and u be a vertex of G. If H is a subgraph of G or a subset of $V(G)$ or a sequence of vertices of G, we define $N(u, H)$ to be the set of neighbors of u contained in H. Let $e(u, H) = |N(u, H)|$. Thus $e(u, G)$ is the degree of u in G. If each of $X_1, \dots, X_k$ is a subgraph of G or a subset of $V(G)$ or a sequence of vertices of G, we use $[X_1, X_2, \dots, X_k]$ to denote the subgraph of G induced by the set of all the vertices x i that belongs to some of $X_1, X_2, \dots, X_k$. If each of X and Y is a subgraph of G or a subset of $V(G)$ or a sequence of vertices of G, we define $e(X, Y) = \sum_x e(x, Y)$ where x runs over X. The length of a cycle or a path L is denoted by $l(L)$. If W is a subset of $V(G)$, then the W-length of L is the number of vertices of L that are contained in W. We denote the W-length of L by $l_W(L)$. i If we list $V(L) = \{u_1, u_2, \dots, u_k\}$, then operations in the subscripts of u_i's will be taken modulo k in $\{1, 2, \dots, k\}$.

A chord of a cycle C in G is an edge of $G - E(C)$ that joins two vertices of C. If we write $C = x_1 x_2 \dots x_m x_1$, we assume that an orientation of C is given such that x_2 is the successor of x_1. Moreover, we use x_i^+ and x_i^- to denote the successor and predecessor of x_i, respectively. We use $C[x_i, x_j]$ to represent the path of C from x_i to x_j along the orientation of C. We adopt the notation $C(x_i, x_j] = C[x_i, x_j] - x_i$, $C[x_i, x_j) = C[x_i, x_j] - x_j$ and $C(x_i, x_j) = C[x_i, x_j] - x_i - x_j$. We use C^- to denote the cycle C with its opposite orientation.

If x and y are two vertices of G and H is a subgraph of G or a subset of V, we define $I(xy, H) = N(x, H) \cap N(y, H)$. Let $i(xy, H) = |I(xy, H)|$. For a subset W of V, let $\delta_W(G) = \min\{e(x, G) | x \in W\}$.

2 Lemmas

Let $G = (V, E)$ be a graph of order n and W a subset of V. Lemma 2.1 is an easy observation.

Lemma 2.1 *If $P = x_1 \dots x_k$ is a path of G and u is a vertex in $V - V(P)$ such that $e(u, P) \geq (k+1)/2$, then $[P, u]$ has a hamiltonian path from x_1 to x_k or k is odd and $N(u, P) = \{x_1, x_3, x_5, \dots, x_k\}$.*

Lemma 2.2 *If $P = x_1 \dots x_k$ is a path of G and u is a vertex in $V - V(P)$ such that $e(ux_k, P) \geq k + 1$, then $[P, u]$ has a hamiltonian path from x_1 to u.*

Proof. The condition implies that for some $i \in \{1, \ldots, k-1\}$, $\{x_k x_i, u x_{i+1}\} \subseteq E$ and so $x_1 \ldots x_i x_k x_{k-1} \ldots x_{i+1} u$ is a required path. ∎

Lemma 2.3 *If $P = x_1 \ldots x_k$ is a path of G and u and v are two vertices in $V - V(P)$ such that $e(uv, P) \geq k+2$, then $[P, u, v]$ has a hamiltonian path from x_1 to x_k or $e(uv, P) = k+2$ and $e(uv, x_1 x_k) = 4$.*

Proof. Let $X = \{x_{i+1} | ux_i \in E, 1 \leq i \leq k\}$ and $Y = \{x_{i-1} | ux_i \in E, 1 \leq i \leq k\}$, where $x_{k+1} = x_1$ and $x_0 = x_k$. Then $|X| = e(u, P)$. Thus $e(uv, P) = |X| + e(v, P) \geq k+2$. Therefore $N(v, P) \cap X$ contains at least two distinct vertices x_{i+1} and x_{j+1} with $i < j$. Let x_{i+1} and x_{j+1} be chosen with j minimal. If $j < k$, then $x_1 \ldots x_i u x_j x_{j-1} \ldots x_{i+1} v x_{j+1} \ldots x_k$ is a required path. If $j = k$, then $|N(v, P) \cap X| = 2$, $e(uv, P) = k+2$ and $\{ux_k, vx_1\} \subseteq E$. Applying a similar argument with Y in place of X, we obtain $\{ux_1, vx_k\} \subseteq E$. ∎

Lemma 2.4 *Let C be a cycle of order k in G with a given direction and $V(C) \supseteq W$. Let x and y be two vertices on C. Let x' be the first vertex of W that succeeds x and y' the first vertex of W that succeeds y. If $e(x'y', C) \geq k+1$, then $[C]$ contains an x-y path P such that $W \subseteq V(P)$.*

Proof. The condition implies that either there exists u on $C[x', y')$ such that $\{y'u^-, x'u\} \subseteq E$ or there exists v on $C[y', x')$ such that $\{x'v^-, y'v\} \subseteq E$. If $e(x', C(y, y']) > 0$ or $e(y', C(x, x']) > 0$, then we readily see that there is a required path. So assume that $e(x', C(y, y']) = 0$ and $e(y', C(x, x']) = 0$. Thus either u is on $C(x', y]$ or v is on $C(y', x]$. Say without loss of generality that the former holds. Then $xC^-[x, y']u^-C^-[u^-, x']uC[u, y]y$ is a required path. ∎

Lemma 2.5 *Let C be a cycle of order k in G with a given direction and $V(C) \supseteq W$. Let λ be a nonnegative integer. Suppose that for each pair x and y of vertices in W, if $[C]$ has an x-y path containing W then $e(xy, C) \geq k + \lambda$. Then $e(uv, C) \geq k + \lambda$ for all $\{u, v\} \subseteq W$ with $u \neq v$.*

Proof. On the contrary, say $e(uv, C) \leq k + \lambda - 1$ for some $\{u, v\} \subseteq W$ with $u \neq v$. Let x be the first vertex of W that succeeds u and y the first vertex of W succeeds v. Then $e(xu, C) \geq k+\lambda$ and $e(yv, C) \geq k+\lambda$. Thus $e(xy, C) \geq 2(k+\lambda) - (k+\lambda-1) = k + \lambda + 1$. By Lemma 2.4, $[C]$ has a u-v path containing W and so $e(uv, C) \geq k + \lambda$, a contradiction. ∎

Lemma 2.6 *Let W be a subset of V with $|W| \geq 3$. If $e(x, G) \geq n/2$ for all $x \in W$, then G has a cycle C such that $V(C) \supseteq W$.*

Proof. Let P be a path with its two endvertices in W such that $l_W(P)$ is as large as possible. Say $P = x_1 \ldots x_k$. If there exists $y \in W - V(P)$, then $e(yx_k, G - V(P)) \leq n - k - 1$. This would yield that $e(yx_k, P) \geq n - (n-k-1) = k+1$ and so $[P, y]$ contains a hamiltonian path from x_1 to y by Lemma 2.2, contradicting the maximality of P. Thus $V(P) \supseteq W$. The lemma holds if $I(x_1 x_k, G - V(P)) \neq \emptyset$. If $I(x_1 x_k, G - V(P)) = \emptyset$ then $e(x_1 x_k, G - V(P)) \leq n - k$ and so $e(x_1 x_k, P) \geq k$ and consequently, $[P]$ is hamiltonian. ∎

3 Proof of the Theorem

Let $G = (V, E)$ be a graph of order n. Let W be a subset of V such that $|W| \geq 6$ and $e(x, G) \geq 2n/3$ for each $x \in W$. Suppose, for a contradiction, that G does not contain two independent cycles of W-lengths n_1 and n_2, respectively for some partition $|W| = n_1 + n_2$ with $n_1 \geq 3$ and $n_2 \geq 3$. Then $n_1 + n_2 < n$ by El-Zahar's result mentioned in the introduction and $n_1 + n_2 \geq 7$ by Theorem A. Thus $n \geq 8$. The degree condition is still maintained when the edges of $G - W$ are removed from G. So we may assume that $G - W$ has no edges.

We need some special terminology and notation. A W-path of G is a path with its endvertices in W. Let $\mathcal{H}$ denote the set of all the subgraphs H such that H has a cycle C with $V(C) \supseteq V(H) \cap W$. Let $\mathcal{P}$ denote the set of all the subgraphs H such that H has a path P with $V(P) \supseteq V(H) \cap W$.

By Lemma 2.6, $G \in \mathcal{H}$ and so G conatins two independent W-paths P_1 and P_2 such that

$$l_W(P_1) = n_1 \text{ and } l_W(P_2) = n_2. \tag{1}$$

Subject to (1), we choose P_1 and P_2 in G such that

$$l(P_1) + l(P_2) \text{ is minimal.} \tag{2}$$

Let $G_1 = [P_1]$ and $G_2 = [P_2]$. Subject to (1) and (2), choose P_1 and P_2 such that

$$e(G_1) + e(G_2) \text{ is maximal.} \tag{3}$$

Say $R = V(G) - V(G_1 \cup G_2)$, $P_1 = x_1x_2 \dots x_s$, $P_2 = y_1y_2 \dots y_t$ and $|R| = r$. Thus R is an independent set of G and $n = r + s + t$. Note that $\lceil 2n/3 \rceil \geq \lceil n/2 \rceil + 1$.

Lemma 3.1 *Either* $I(x_1x_s, R) = \emptyset$ *or* $I(y_1y_t, R) = \emptyset$.

Proof. On the contrary, say $I(x_1x_s, R) \neq \emptyset$ and $I(y_1y_t, R) \neq \emptyset$. As G does not contain two required cycles, there exists $u \in R$ such that $I(x_1x_s, R) = I(y_1y_t, R) = \{u\}$. Moreover, $G_1 \notin \mathcal{H}$ and $G_2 \notin \mathcal{H}$. It follows that $e(x_1x_s, R) \leq r + 1$, $e(y_1y_t, R) \leq r+1$, $e(x_1x_s, G_1) \leq s-1$, and $e(y_1y_t, G_2) \leq t-1$. Thus $e(x_1x_s, G_2) \geq 4n/3-(r+s) = t + n/3 > t + 2$ and $e(y_1y_t, G_1) \geq 4n/3 - (r + t) = s + n/3 > s + 2$. By Lemma 2.3, $G_1 - x_1 - x_s + y_1 + y_t \in \mathcal{P}$ and $G_2 - y_1 - y_t + x_1 + x_s \in \mathcal{P}$. In the meantime, we have

$$\begin{aligned}
&e(G_1 - x_1 - x_s + y_1 + y_t) + e(G_2 - y_1 - y_t + x_1 + x_s) \\
&\quad = e(G_1) - e(x_1x_s, G_1) + e(y_1y_t, G_1) + e(G_2) - e(y_1y_t, G_2) + e(x_1x_s, G_2) \\
&\qquad\qquad -2e(x_1x_s, y_1y_t) \\
&\quad \geq e(G_1) - (s - 1) + (s + 3) + e(G_2) - (t - 1) + (t + 3) - 2e(x_1x_s, y_1y_t) \\
&\quad = e(G_1) + e(G_2) + 8 - 2e(x_1x_s, y_1y_t) \geq e(G_1) + e(G_2).
\end{aligned}$$

By (1), (2) and (3), we see the equality must holds in these inequalities and $e(x_1x_s, y_1y_t) = 4$. On the other hand, we see that

$$\begin{aligned} &e(x_1, G_2) + e(y_1, G_1) - e(x_1, G_1) - e(y_1, G_2) \\ &\quad + e(x_s, G_2) + e(y_t, G_1) - e(x_s, G_1) - e(y_t, G_2) \geq 8. \end{aligned}$$

Thus either $e(x_1, G_2) + e(y_1, G_1) - e(x_1, G_1) - e(y_1, G_2) \geq 4$ or $e(x_s, G_2) + e(y_t, G_1) - e(x_s, G_1) - e(y_t, G_2) \geq 4$. Say without loss of generality that the former holds. Then

$$\begin{aligned} &e(G_1 - x_1 + y_1) + e(G_2 - y_1 + x_1) \\ &= e(G_1) - e(x_1, G_1) + e(y_1, G_1) + e(G_2) - e(y_1, G_2) + e(x_1, G_2) - 2e(x_1, y_1) \\ &\geq e(G_1) + e(G_2) + 4 - 2e(x_1, y_1) \geq e(G_1) + e(G_2) + 2. \end{aligned}$$

This contradicts (3) since $G_1 - x_1 + y_1 \in \mathcal{P}$ and $G_2 - y_1 + x_1 \in \mathcal{P}$. ∎

Lemma 3.2 *Either $G_1 \notin \mathcal{H}$ and $I(x_1x_s, R) = \emptyset$ or $G_2 \notin \mathcal{H}$ and $I(y_1y_t, R) = \emptyset$.*

Proof. Since either $G_1 \notin \mathcal{H}$ or $G_2 \notin \mathcal{H}$, say without loss of generality $G_1 \notin \mathcal{H}$. If $I(x_1x_s, R) = \emptyset$, we are done. Otherwise, $I(x_1x_s, R) \neq \emptyset$, and so $G_2 \notin \mathcal{H}$. Moreover, by Lemma 3.1, $I(y_1y_t, R) = \emptyset$. ∎

By Lemma 3.2, we may assume without loss of generality that $G_1 \notin \mathcal{H}$ and $I(x_1x_s, R) = \emptyset$. Thus

$$e(x_1x_s, G_1 + R) \leq s - 1 + r. \quad (4)$$

Therefore $2t \geq e(x_1x_s, G_2) \geq 4n/3 - (r + s - 1) = t + n/3 + 1$ and this implies

$$t \geq \lceil n/3 \rceil + 1. \quad (5)$$

We shall divide our proof of the theorem into two parts: $r \leq \lceil n/3 \rceil - 1$ or $r \geq \lceil n/3 \rceil$.

Part I: $r \leq \lceil n/3 \rceil - 1$

Let $H = G - R$ and $p = |V(H)|$. Then $\delta_W(H) \geq \lceil 2n/3 \rceil - r = \lceil p/2 + (p - 2r)/6 \rceil \geq (p + 1)/2$. As $e(x_1x_s, G_1) \leq s - 1$, we may assume that $e(x_1, G_1) \leq e(x_s, G_1)$. Thus $e(x_1, G_1) \leq (s - 1)/2$ and so $e(x_1, G_2) \geq \lceil (p + 1)/2 \rceil - \lfloor (s - 1)/2 \rfloor \geq t/2 + 1$. We claim that if u is an endvertex of a hamiltonian path of G_2, then either $e(u, G_2) \geq (t + 1)/2$ or $e(u, G_2) = t/2$ and $x_1u \in E$. To see this, say without loss of generality that $e(u, G_2) \leq t/2$. Then $e(u, G_1) \geq \lceil (p + 1)/2 \rceil - \lfloor t/2 \rfloor \geq (s + 1)/2$. By Lemma 2.1, $G_1 - x_1 + u \in \mathcal{P}$ and $G_2 - u + x_1 \in \mathcal{P}$. By (3), we have

$$\begin{aligned} &e(G_1) + e(G_2) \\ &\geq e(G_1 - x_1 + u) + e(G_2 - u + x_1) \\ &\geq e(G_1) - (s - 1)/2 + (s + 1)/2 + e(G_2) - t/2 + t/2 + 1 - 2e(x_1, u) \\ &= e(G_1) + e(G_2) + 2 - 2e(x_1, u) \\ &\geq e(G_1) + e(G_2). \end{aligned}$$

This implies that $e(u, G_2) = t/2$ and $x_1u \in E$. Therefore the claim holds. Thus $G_2 \in \mathcal{H}$ and so $G_2 + x_1 \in \mathcal{H}$ by Lemma 2.1. By (2), $n_2 = t$. Say without loss of generality that $y_1y_2 \ldots y_ty_1$ is a hamiltonian cycle of G_2. For each y_i, if $G_2 - y_i \in \mathcal{H}$, then $G_2 - y_i + x_1 \in \mathcal{H}$ because $e(x_1, G_2 - y_i) \geq t/2$, and if $G_2 - y_i \notin \mathcal{H}$ then $e(y_{i-1}, G_2) = e(y_{i+1}, G_2) = t/2$ and so $G_2 - y_i + x_1 \in \mathcal{H}$ since $e(x_1, y_{i-1}y_{i+1}) = 2$ in this situation.

Say $H_1 = G_1 - x_1$ and $H_2 = G_2 + x_1$. Then $H_1 + R + v \notin \mathcal{H}$ for all $v \in V(H_2)$. Thus for any x-y W-path P of H_1 with $P \in \mathcal{P}$, $e(v, xy) \leq 1$ for all $v \in V(H_2)$ and so $e(xy, H_1) \geq p + 1 - e(xy, H_2) \geq p + 1 - (t + 1) = (s - 1) + 1$. It follows that $H_1 \in \mathcal{H}$. Let C be a cycle of H_1 such that if H_1 is hamiltonian then C is a hamiltonian cycle of H_1 and otherwise $x_2 \notin W$, $x_3 \in W$ and C is a hamiltonian cycle of $H_1 - x_2$. Let u and v be any two vertices in $V(H_1) \cap W$. We claim that H_1 has a u-v path containing $V(H_1) \cap W$ and $e(uv, H_1) \geq (s - 1) + 1$. To see this, let x be the first vertex of W that succeeds u and y the first vertex of W that succeeds v on C. If $I(xy, H_1 - V(C)) \neq \emptyset$, we readily see that H_1 has u-v path $P \in \mathcal{P}$ and so $e(uv, H_1) \geq (s - 1) + 1$. So assume $I(xy, H_1 - V(C)) = \emptyset$. By Lemma 2.4, we may also assume that $e(xy, C) \leq |V(C)|$. Thus $e(xy, H_1) \leq s - 1$ and so H_1 does not have an x-y path containing $V(H_1) \cap W$. This implies that $e(uv, C) \leq |V(C)|$ by Lemma 2.4 and $I(uv, H_1 - V(C)) = \emptyset$. Thus $e(uv, H_1) \leq s - 1$. Since $e(ux, H_1) \geq (s - 1) + 1$ and $e(vy, H_1) \geq (s - 1) + 1$, either $e(xy, H_1) \geq (s - 1) + 1$ or $e(uv, H_1) \geq (s - 1) + 1$, a contradiction. Therefore the claim holds.

Label $C = c_1c_2 \ldots c_lc_1$ with $l = |V(C)|$ such that if C is a hamiltonian cycle of H_1 then $c_1 = x_2$ and otherwise C is a hamiltonian cycle of $H_1 - x_2$ with $x_2 \notin W$ and we let $c_1 = x_3$. Then G_1 has an x_1-c_2 hamiltonian path and an x_1-c_l hamiltonian path. By (2), we see that $\{c_2, c_l\} \subseteq W$. Suppose that there exists $i \in \{3, \ldots, l - 1\}$ such that $c_i \notin W$. Let c_i be chosen with i maximal. Then $e(c_2c_{i+1}, H_1) \geq (s - 1) + 1$. Notice that if C is not a hamiltonian cycle of H_1 then $c_2x_2 \notin E$ and $c_lx_2 \notin E$. By Lemma 2.4, $[C]$ contains a c_1-c_i path containing $V(C) \cap W$. Thus G_1 has x_1-c_i path P' containing $V(G_1) \cap W$. By (2), $c_i \in W$, a contradiction. Therefore $\{c_2, \ldots, c_l\} \subseteq W$. Thus either $n_1 = s$ or $n_1 = s - 1$ with $x_2 \notin W$. Since $G_1 \notin \mathcal{H}$, we also see, from this argument, that $e(x_1, C) \leq 1$ and so $e(x_1, H_1) \leq 2$.

As $e(c_2c_l, H_2) \leq t + 1$, we may assume without loss of generality that $e(c_l, H_2) \leq (t + 1)/2$. Clearly, $I(x_1c_l, R) = \emptyset$. Let a be a rational number such that $e(x_1, R) = r/2 + a$. Then $e(c_l, R) \leq r/2 - a$. Clearly, $t \geq e(x_1, G_2) \geq 2n/3 - r/2 - a - 2 = t/2 + s/2 + n/6 - a - 2$ and $s - 2 \geq e(c_l, H_1) \geq 2n/3 - (t + 1)/2 - r/2 + a = s/2 + n/6 - 1/2 + a$. It follows that $t \geq s + n/3 - 2a - 4$ and $s \geq n/3 + 2a + 3$. Consequently, $n = s + t + r \geq s + 2n/3 - 1 + r \geq n + 2a + 2 + r \geq n + 2$, a contradiction.

Part II: $r \geq \lceil n/3 \rceil$

Since $n \geq 8$, $r \geq 3$. By (5), $n = s + r + t \geq 3 + \lceil n/3 \rceil + \lceil n/3 \rceil + 1$ and it follows that $n \geq 12$. We claim

$$n_1 \geq 4, n_2 \geq 4 \text{ and } n \geq 15. \tag{6}$$

If this is not true, say $\min\{n_1, n_2\} = 3$. Let C be a cycle of G containing at least three vertices of W with $l_W(C)$ as small as possible and subject to this, we choose

C with $l(C)$ as small as possible. Suppose that $l_W(C) \geq 4$. Then $e(x,C) = 2$ for all $x \in V(C) \cap W$. Thus $e(xy, G - V(C)) \geq 4n/3 - 4 = n - l(C) + n/3 + l(C) - 4$ and so $i(xy, G - V(C)) \geq n/3 + l(C) - 4$ for all $\{x, y\} \subseteq V(C) \cap W$ with $x \neq y$. By the mimimality of C, we see that $l_W(C) = l(C) = 4$. Say $C = w_1w_2w_3w_4w_1$. Then $I(w_iw_{i+1}, G - V(C)) \cap W = \emptyset$ and $I(w_iw_{i+2}, G - V(C)) \subseteq W$ by the minmality of $l_W(C)$ for all $i \in \{1, 2, 3, 4\}$. Clearly, $i(w_1w_2, G - V(C)) + e(w_3, G - V(C)) \geq n/3 + 2n/3 - 2 > n - l(C)$ and so $I(w_iw_{i+1}, G - V(C)) \cap N(w_3, G - V(C)) \neq \emptyset$, a contradiction. Therefore $l_W(C) = 3$ and so $G - V(C) \notin \mathcal{H}$. Moreover, we see $e(x, C) \leq 3$ for all $x \in W - V(C)$ by the minimality of $l(C)$ and so $e(x, G - V(C)) \geq 2n/3 - 3 > (n - l(C))/2$ for all $x \in W - V(C)$. Consequently, $G - V(C) \in \mathcal{H}$, a contradiction. So $n_1 \geq 4$ and $n_2 \geq 4$. Since $n = s + r + t \geq 4 + \lceil n/3 \rceil + \lceil n/3 \rceil + 1$, it follows that $n \geq 15$. Hence (6) holds.

We claim that for each $y \in V(G_2) \cap W$, $e(y, G_2 + R) \geq (r + t + 1)/2$. If this is not true, say $e(y, G_2 + R) \leq (r + t)/2$ for some $y \in V(G_2) \cap W$. Then $s \geq e(y, G_1) \geq 2n/3 - (r + t)/2 = s/2 + n/6$. Thus $s \geq n/3$. With (5), we obtain $n = r + s + t \geq n/3 + n/3 + n/3 + 1 = n + 1$, a contradiction. Hence the claim holds. Thus either G_2 is hamiltonian and so $V(G_2) \subseteq W$ by (2) or $G_2 + u$ is hamiltonian for some $u \in R$. Let C be a hamiltonian cycle of G_2 if G_2 is hamiltonian and otherwise let C be a hamiltonian cycle of $G_2 + y_0$ for some $y_0 \in I(y_1y_t, R)$. Clearly, $l(C) = t$ or $l(C) = t+1$. Rename the vertices of $V(C) \cap W$ as $b_1, b_2, \ldots, b_{n_2}$ along the direction of C. Moreover, we may assume that if $l(C) = t + 1$ then $b^+_{n_2} = y_0$. Let $b_{n_2+1} = b_1$ and $b_0 = b_{n_2}$. Let $Z_i = C[b_i, b_{i+1})$ for all $i \in \{1, \ldots, n_2\}$. As $V(G) - W$ is an independent set, Z_i has at most two vertices for all $i \in \{1, \ldots, n_2\}$. Set $R' = R - V(C)$. Clearly, either $R' = R$ or $R' = R - \{y_0\}$. We may assume without loss of generality that $e(x_1, G_1 + R) \leq e(x_s, G_1 + R)$. Thus by (4),

$$e(x_1, G_2) \geq 2n/3 - (r + s - 1)/2 = t/2 + n/6 + 1/2. \qquad (7)$$

Lemma 3.3 *For each $i \in \{1, 2, \ldots, n_2\}$ there exists a cycle L_i with $W \cap V(C) - \{b_i\} \subseteq V(L_i)$ such that either $V(L_i) \subseteq V(C) - V(Z_i)$ and $L_i + x_1 \in \mathcal{H}$ or $V(L_i) \subseteq (V(C) - V(Z_i)) \cup \{v_i\}$ for some $v_i \in R'$ and $L_i + x_1 \in \mathcal{H}$.*

Proof. Let $i \in \{1, 2, \ldots, n_2\}$. By (7), we have

$$e(x_1, G_2 - V(Z_i)) \geq \lceil t/2 + n/6 + 1/2 \rceil - e(x_1, Z_i). \qquad (8)$$

First, assume that $b_{i+1} = b^+_i$. Then $Z_i = b_i$. If $b^+_{i-1} = b_i$, then $e(b_{i-1}b_{i+1}, G_2 + R - b_i) \geq r + t + 1 - 2 = t + r - 1$. Thus either $[V(C - b_i)]$ is hamiltonian or there exists $v_i \in R'$ such that $e(v_i, b_{i-1}b_{i+1}) = 2$. Thus either there is a hamiltonian cycle L_i of $[V(C - b_i)]$ or $L_i = C - b_i + v_ib_{i-1} + v_ib_{i+1}$ is a hamiltonian cycle of $[V(C - b_i) \cup \{v_i\}]$ for some $v_i \in R'$. By (7), $e(x_1, L_i) \geq \lceil (t + 1)/2 \rceil + 1$ and so $L_i + x_1$ is hamiltonian.

Next, assume that $b_{i+1} = b^+_i$ and $b^{++}_{i-1} = b_i$. By (2) and the assumption on C, $b_ib_{i-1} \notin E$. If $[V(C - b_i - b^+_{i-1})]$ is hamiltonian, then there is a hamiltonian cycle L_i of $[V(C - b_i - b^+_{i-1})]$ and $e(x_1, L_i) \geq e(x_1, G_2) - e(x_1, b_ib^+_{i-1}) > (t - 1)/2$ and so $L_i + x_1$ is hamiltonian. So assume that $[V(C - b_i - b^+_{i-1})]$ is not hamiltonian. Then $b_{i+1}b_{i-1} \notin E$.

Similarly, we may assume that $[V(C - b_i)]$ is not hamiltonian and so $b_{i+1}b_{i-1}^+ \notin E$. Then $e(b_{i-1}b_{i+1}, b_ib_{i-1}^+) \leq 2$ as $b_ib_{i-1} \notin E$. Hence $e(b_{i-1}b_{i+1}, G_2 + R - b_i - b_{i-1}^+) \geq r+t+1-2 = t+r-1$. Thus $I(b_{i-1}b_{i+1}, R') \neq \emptyset$. Let $L_i = C - \{b_{i-1}^+, b_i\} + v_ib_{i-1} + v_ib_{i+1}$ with $v_i \in R'$. Clearly, $|V(L_i)| \leq t$. For the proof, we may assume that x_1 is not adjacent to two consecutive vertices of L_i. Then $e(x_1, L_i) \leq t/2$ by Lemma 2.1. By (7), we obtain that $2 \geq e(x_1, b_{i-1}^+b_i) \geq t/2 + n/6 + 1/2 - e(x_1, L_i) \geq n/6 + 1/2 \geq 3$, a contradiction.

Next, assume that $b_i^{++} = b_{i+1}$ and $b_{i-1}^+ = b_i$. Then $Z_i = b_ib_i^+$. The proof is similar as above.

Finally, assume that $b_i^{++} = b_{i+1}$ and $b_{i-1}^{++} = b_i$. Then $Z_i = b_ib_i^+$. As above, we may assume that none of $b_{i+1}b_i$, $b_{i+1}b_i^-$ and $b_{i-1}b_i$ is an edge of G. Moreover, $[V(C) - \{b_i, b_i^+, b_i^-\}]$ is not hamitonian. Thus $I(b_{i-1}b_{i+1}, R') \neq \emptyset$. Let $L_i = C - \{b_i^-, b_i, b_i^+\} + v_ib_{i-1} + v_ib_{i+1}$ with $v_i \in R'$. For the proof, we may assume that x_1 is not adjacent to two consecutive vertices of L_i. Thus $e(x_1, C - \{b_i^-, b_i, b_i^+\}) \leq (t-1)/2$. Then by (7), $3 \geq e(x_1, b_i^-b_ib_i^+) \geq e(x_1, G_2) - \lfloor (t-1)/2 \rfloor \geq n/6 + 1 \geq 21/6$, a contradiction. ∎

By Lemma 3.3,

$$G_1 - x_1 + V(Z_i) \notin \mathcal{H} \text{ for all } i \in \{1, \ldots, n_2\}. \tag{9}$$

Let $H = G_1 - x_1$. Let $x^* = x_2$ if $x_2 \in W$ and otherwise $x_2 \notin W$ and $x^* = x_3$ with $x_3 \in W$. By (9), $e(x^*x_s, Z_i) \leq |V(Z_i)|$ for all $i \in \{1, 2, \ldots, n_2\}$. Thus $e(x^*x_s, C) \leq l(C)$ and so

$$e(x^*x_s, G_1 + R') \geq 2\lceil 2n/3 \rceil - l(C) \geq s + |R'| + \lceil n/3 \rceil. \tag{10}$$

Thus if $H \notin \mathcal{H}$ then $e(x^*x_s, H) \leq s-2$ and so $e(x^*x_s, R') \geq s+|R'|+\lceil n/3 \rceil-(s-1) = |R'| + \lceil n/3 \rceil + 1$. Consequently, $|R'| \geq \lceil n/3 \rceil + 1$ and $i(x^*x_s, R') \geq \lceil n/3 \rceil + 1 \geq 6$.

If H is a hamiltonian, let Q be a hamiltonian cycle of H. If H is not hamiltonian but $H - x_2$ is hamiltonian with $x_2 \notin W$, let Q be a hamiltonian cycle of $H - x_2$. Otherwise let $Q = wP_1[x^*, x_s]w$ with $w \in I(x^*x_s, R')$. Fix a direction of Q and rename the vertices of $V(Q) \cap W$ as $a_1, a_2, \ldots, a_{n_1-1}$ along the direction of Q. Let $a_{n_1} = a_1$. Note that we have at least $\lceil n/3 \rceil + 1$ different candidates for w since $i(x^*x_s, R') \geq \lceil n/3 \rceil + 1 \geq 6$.

Lemma 3.4 *For each $j \in \{1, \ldots, n_1 - 1\}$, we have $e(a_ja_{j+1}, C) \leq l(C)$.*

Proof. On the contrary, say $e(a_ja_{j+1}, C) \geq l(C) + 1$ for some $j \in \{1, \ldots, n_1 - 1\}$. Then $e(a_ja_{j+1}, Z_i) \geq |V(Z_i)| + 1$ for some $i \in \{1, \ldots, n_2\}$. Thus $[Q, V(Z_i)] \in \mathcal{H}$. If Q is a cycle of H, then we have two required cycles by Lemma 3.3. If Q is not a cycle of H, we may choose w so that $w \notin V(L_i)$, where L_i is as described in Lemma 3.3, and so there are two required cycles. ∎

With Lemmas 2.4 and 3.4, we now generalize Lemma 3.4 to Lemma 3.5 in the following.

Lemma 3.5 *For all* $\{j,k\} \subseteq \{1,\ldots,n_1-1\}$ *with* $j<k$, *we have* $e(a_j a_k, G_1+R') \geq s+|R'|+n/3$.

Proof. By Lemma 3.4, we see that $e(a_j a_{j+1}, G_1+R') \geq 2\lceil 2n/3\rceil - l(C) \geq s+|R'|+n/3$ for all $j \in \{1,\ldots,n_1-1\}$. For the proof, assume that $e(a_j a_k, G_1+R') \leq s+|R'|+\lceil n/3\rceil - 1$ for some $j<k$. Then $e(a_j a_k, C) \geq l(C)+1$. Thus $e(a_j a_k, Z_i) \geq |V(Z_i)|+1$ for some $i \in \{1,\ldots,n_2\}$. Since $e(a_j a_{j+1}, G_1+R') \geq s+|R'|+n/3$ and $e(a_k a_{k+1}, G_1+R') \geq s+|R'|+n/3$, it follows that

$$\begin{aligned} e(a_{j+1}a_{k+1}, G_1+R') &\geq 2(s+|R'|+\lceil n/3\rceil) - (s+|R'|+\lceil n/3\rceil - 1) \\ &= s+|R'|+\lceil n/3\rceil + 1. \end{aligned}$$

If Q contains a vertex of R', i.e. w, we choose w so that $w \notin V(L_i)$, where L_i is as described in Lemma 3.3. If $e(a_{j+1}a_{k+1}, Q) \geq l(Q)+1$, then $[Q]$ contains a path P from a_j to a_k with $l_W(P) = n_1 - 1$ by Lemma 2.4, and so $[Q, Z_i] \in \mathcal{H}$ as $e(a_j a_k, Z_i) \geq |V(Z_i)|+1$, a contradiction since $L_1 + x_1 \in \mathcal{H}$ by Lemma 3.3. Hence $e(a_{j+1}a_{k+1}, Q) \leq l(Q)$. If $I(a_{j+1}a_{k+1}, G_1+R') - V(Q)$ contains a vertex u not belonging to $V(L_i) \cup \{x_1, w\}$, then $Q+u$ contains a path P' from a_j to a_k and $V(Q) \cap W \subseteq V(P')$ and so $[P', Z_i] \in \mathcal{H}$, again a contradiction since $L_1 + x_1 \in \mathcal{H}$. Therefore $I(a_{j+1}a_{k+1}, G_1+R') - V(Q)$ does not contain a vertex not belonging to $V(L_i) \cup \{x_1, w\}$. From Lemma 3.3, we see that $|V(L_i) \cap R'| \leq 1$. Therefore $|I(a_{j+1}a_{k+1}, G_1+R') - V(Q)| \leq 3$ and $e(a_{j+1}a_{k+1}, G_1+R') \leq s+|R'|+3$, a contradiction. ∎

Lemma 3.6 *For any* $\{v,v'\} \subseteq R'$ *and any* $\{x,y\} \subseteq V(Q)-R'$ *with* $x \neq y$, $[H, R'-\{v,v'\}]$ *has an* x-y *path* P *such that* $V(P) \cap V(H) \subseteq V(Q)$, $\{a_1, a_2, \ldots, a_{n_1-1}\} \subseteq V(P)$ *and* $|V(P) \cap R'| \leq 2$.

Proof. Let a_j be the first vertex of W that succeeds x and a_k the first vertex of W that succeeds y on Q. Then $e(a_j a_k, G_1+R') \geq s+|R'|+n/3$ by Lemma 3.5. If Q contains a vertex of R', i.e., w, we choose w so that $w \notin \{v,v'\}$. By Lemma 2.4, if $e(a_j a_k, Q) \geq l(Q)+1$, then $[Q]$ contains an x-y path P with $V(P) \supseteq V(Q) \cap W$ and we are done. So we may assume that $e(a_j a_k, Q) \leq l(Q)$. Then $I(a_j a_k, G_1+R'-V(Q)) \geq n/3 \geq 5$. Therefore $I(a_j a_k, G_1+R'-V(Q))$ contains a vertex u of $R' - \{v,v',w\}$ and so $Q+u$ contains a required x-y path. ∎

By Lemma 3.3 and Lemma 3.6, we see that $e(a_j a_k, Z_i) \leq |V(Z_i)|$ for all $i \in \{1,\ldots,n_2\}$ and $\{j,k\} \subseteq \{1,\ldots,n_1-1\}$ with $j \neq k$, for otherwise G contains two required cycles. Thus $e(a_j a_k, C) \leq l(C)$ for all $\{j,k\} \subseteq \{1,\ldots,n_1-1\}$ with $j \neq k$. Let v and v' be two given arbitrary vertices of R'. Choose w so that $w \notin \{v,v'\}$. As $n_1 \geq 4$ and by Lemma 3.6, there exists $\{j,k\} \subseteq \{1,\ldots,n_1-1\}$ with $j \neq k$ such that $G_1 + R' - \{v,v'\}$ has an x_1-a_j path P' and an x_1-a_k path P'' such that $l_W(P') = n_1$ and $l_W(P'') = n_1$. As $e(a_j a_k, C) \leq l(C)$, we may assume that $e(a_k, C) \leq l(C)/2$.

We claim that $I(x_1 a_k, R' - V(Q)) = \emptyset$. If this is not true, we choose $v' \in I(x_1 a_k, R' - V(Q))$. If $x_2 \in V(Q)$, we apply Lemma 3.6 with x_2 and a_k in place of x and y and see that $G_1 + R' \in \mathcal{H}$, a contradiction. Hence $x_2 \notin V(Q)$ and

$x_2 \notin W$. Then apply Lemma 3.6 with x_3 and a_k in place of x and y and see that $G_1 + R' \in \mathcal{H}$, a contradiction. Hence $i(x_1a_k, R' - V(Q)) = \emptyset$. By Lemma 3.6, $e(x_1, H) \leq 2$ for otherwise $G_1 + R' \in \mathcal{H}$. Let $r' = |R'|$ and c a rational number such that $e(x_1, R') = r'/2 + c$. Then $e(a_k, R') \leq r' - (r'/2 + c) + 1 = r'/2 - c + 1$. Note that $x_1a_k \notin E$. Thus

$$\begin{aligned} l(C) \geq e(x_1, C) &\geq \lceil 2n/3 \rceil - r'/2 - c - e(x_1, H) \\ &\geq l(C)/2 + n/6 + s/2 - c - 2; \qquad (11) \\ s - 2 \geq e(a_k, G_1) &\geq \lceil 2n/3 \rceil - (r' + l(C))/2 + c - 1 = s/2 + n/6 + c - 1. \qquad (12) \end{aligned}$$

By (11), $l(C) \geq n/3 + s - 2c - 4$. By (12), $s \geq n/3 + 2c + 2$ and so $l(C) \geq 2n/3 - 2$. Since $r' \geq \lceil n/3 \rceil - 1$ and $n \geq 15$, we obtain that $n = s + l(C) + r' \geq n + 2c + r' > n$, a contradiction. This proves the theorem. ∎

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