AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(1)** (2016), Pages 166–193

Q-analogues of convolutions of Fibonacci numbers

Jeffrey B. Remmel   Janine LoBue Tiefenbruck

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112
U.S.A.

jremmel@ucsd.edu   jlobue@ucsd.edu

Abstract

Let $NR([k])$ denote the set of words over the alphabet $[k] = \{1, \ldots, k\}$ with no consecutive repeated letters. Given a word $w = w_1 \ldots w_n \in NR([k])$, or more generally in $[k]^*$, we say that a pair $\langle w_i, w_j \rangle$ matches the μ pattern if $i < j$, $w_i < w_j$, and there is no $i < k < j$ such that $w_i \leq w_k \leq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ-match if $w_i + 1 = w_j$ and a nontrivial μ-match if $w_i + 1 < w_j$. For each word w in $NR([k])$, let the weight of w be given by $t^{|w|}p^{ntriv(w)}q^{triv(w)}$, where $|w|$ is the length of w, $ntriv(w)$ is the number of nontrivial μ-matches in w and $triv(w)$ is the number of trivial μ-matches in w. We study the generating functions $N^{i,j}(p, q, t)$ that sum the weights of all words w in $NR([3])$ starting with the letter i and ending with the letter j. In particular, we show that $N^{3,1}(p, 1, t) = \sum_{r \geq 0} \frac{t^{2r+2}p^r}{(1 - t - t^2)^{r+1}}$ so that the number of words in $NR([3])$ starting with 3, ending with 1, and having r nontrivial μ-matches is counted by the convolutions of $r + 1$ copies of the Fibonacci numbers. It follows that the coefficient of p^r in $N^{3,1}(p, q, t)$ is a q-analogue of the generating function of the convolution of $r + 1$ copies of the Fibonacci numbers. The main goals of this paper are to compute the generating functions $N^{i,j}(p, q, t)$ and prove a number of combinatorial properties of their coefficients.

1 Introduction

Mesh patterns were introduced in [2] by Brändén and Claesson, and they were studied in a series of papers (e.g. see [5] by Kitaev and Liese, and references therein). A particular class of mesh patterns is *boxed patterns* introduced in [1] by Avgustinovich, Kitaev, and Valyuzhenich, who later suggested to call this type of pattern *frame patterns*. The simplest frame pattern, which is called the μ pattern, is defined as follows. Let S_n denote the set of all permutations of $\{1, \ldots, n\}$. Given $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, we say that a pair $\langle \sigma_i, \sigma_j \rangle$ matches the μ pattern or is a μ-match in σ if $i < j$, $\sigma_i < \sigma_j$, and there is no $i < k < j$ such that $\sigma_i < \sigma_k < \sigma_j$. Similarly, we say that the pair $\langle \sigma_i, \sigma_j \rangle$ matches the μ' pattern or is a μ'-match in σ if $i < j$, $\sigma_i > \sigma_j$, and there is no $i < k < j$ such that $\sigma_i > \sigma_k > \sigma_j$.

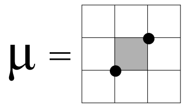

Figure 1: The simplest frame pattern, μ.

The μ pattern is shown in Figure 1 using the notation of [2]. This means that if we graph a permutation σ as a set of dots with coordinates (i, σ_i), then a μ-match is a pair of increasing dots such that the rectangle they define contains no other dots. For example, Figure 2 shows the permutation 6741325 with the μ-match $\langle 3, 5 \rangle$ highlighted.

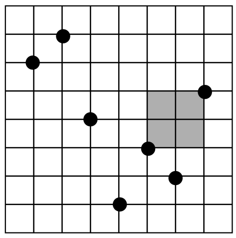

Figure 2: The graph of the permutation 6741325 with the occurrence $\langle 3, 5 \rangle$ highlighted.

Avgustinovich, Kitaev, and Valyuzhenich [1] were the first to study the avoidance of frame patterns including μ and μ' in permutations in the symmetric group S_n. The distribution of μ-matches has also been studied in another setting, namely, Jones, Kitaev, and Remmel [4] studied *cycle-occurrences* of the μ pattern in the cycle structure of permutations.

The concept of a μ-match can easily be extended from permutations to words. The authors began the study of μ-patterns in words in [6]. For any positive integer k, we let $[k] = \{1, \ldots, k\}$. We let $[k]^*$ denote the set of all words over the alphabet $[k]$. We let ϵ denote the empty word and we say ϵ has length 0. If $u = u_1 \ldots u_s$ and $v = v_1 \ldots v_t$ are words in $[k]^*$, we let $uv = u_1 \ldots u_s v_1 \ldots v_t$ denote the concatenation of u and v. We say that a word $u = u_1 \ldots u_j$ is a prefix of w if $j \geq 1$ and there is word v such that $uv = w$, we say that $v = v_1 \ldots v_j$ is a suffix of w if $j \geq 1$ and there is a word u such that $uv = w$, and we say that $f = f_1 \ldots f_j$ is a factor of w if $j \geq 1$ and there are words u and v such that $ufv = w$. We let $NR([k])$ denote the set of all words $w \in [k]^*$ such that w has no repeated letters, i.e., such that w has no factor of the form ii for $i \in [k]$. Such words are sometimes called Smirnov words [3]. Now suppose that $n \geq 1$ and $w = w_1 \ldots w_n \in [k]^*$. Then we let $|w| = n$ denote the length of w. We say that a pair $\langle w_i, w_j \rangle$ is a μ-match in w if $i < j$, $w_i < w_j$, and there is no $i < k < j$ such that $w_i \leq w_k \leq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ-match if $w_i + 1 = w_j$ and is a nontrivial μ-match if $w_i + 1 < w_j$. We then let $\mathrm{triv}_\mu(w)$ denote the number of trivial μ-matches in w and $\mathrm{ntriv}_\mu(w)$ denote the number of nontrivial μ-matches in w. For example, if $w = 123121242416$, then $\mathrm{triv}_\mu(w) = 5$ since w has three $\langle 1, 2 \rangle$-matches, one $\langle 2, 3 \rangle$-match, and one $\langle 3, 4 \rangle$-match. Also, $\mathrm{ntriv}_\mu(w) = 4$ as w has one $\langle 1, 6 \rangle$-match, two $\langle 2, 4 \rangle$-matches, and one $\langle 4, 6 \rangle$-match. Similarly, we say that a pair $\langle w_i, w_j \rangle$ is a μ'-match in w if $i < j$, $w_i > w_j$, and there is no $i < k < j$ such that $w_i \geq w_k \geq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ'-match if $w_i = w_j + 1$ and is a nontrivial μ'-match if $w_i > w_j + 1$. As is the case with permutations, the correspondence that sends $w = w_1 \dots w_n \in [k]^*$ to its reverse $w^r = w_n \dots w_1$ or to its complement $w^c = (k + 1 - w_1) \dots (k + 1 - w_n)$ shows that the problem of studying μ-matches in words is equivalent to the problem of studying μ'-matches in words.

In [6], the authors studied the generating functions

$$A_\mu^{(k)}(p, q, t) = 1 + \sum_{n \geq 1} A_{n,\mu}^{(k)}(p, q) t^n \quad \text{and}$$

$$NR_\mu^{(k)}(p, q, t) = 1 + \sum_{n \geq 1} NR_{n,\mu}^{(k)}(p, q) t^n,$$

where

$$A_{n,\mu}^{(k)}(p, q) = \sum_{w \in [k]^n} q^{\mathrm{triv}_\mu(w)} p^{\mathrm{ntriv}_\mu(w)} \quad \text{and}$$

$$NR_{n,\mu}^{(k)}(p, q) = \sum_{w \in NR([k]), |w| = n} q^{\mathrm{triv}_\mu(w)} p^{\mathrm{ntriv}_\mu(w)}.$$

Given a word $w \in [k]^*$, we can write $w = w_1^{j_1} w_2^{j_2} \dots w_s^{j_s}$ where $w_1 \dots w_s$ has no repeated letters. In such a situation, we say that $w_1 \dots w_s$ is the contraction of w and write $\mathrm{cont}(w) = w_1 \dots w_s$. For example, if $w = 112221123333222444$, then $\mathrm{cont}(w) = 1212324$. It is easy to see that for any $w \in [k]^*$, $\mathrm{triv}_\mu(w) = \mathrm{triv}_\mu(\mathrm{cont}(w))$ and $\mathrm{ntriv}_\mu(w) = \mathrm{ntriv}_\mu(\mathrm{cont}(w))$. Moreover, if $w_1 \dots w_s \in NR([k])$ and $n \geq s$, then the number of $u \in [k]^n$ such that $\mathrm{cont}(u) = w$ equals the number of solutions to $j_1 + \dots + j_s = n$ where each $j_i \geq 1$, which is the composition number $\binom{n-1}{s-1}$. Thus it follows that for all $n \geq 1$,

$$A_{n,\mu}^{(k)}(p, q) = \sum_{s=1}^{n} \binom{n-1}{s-1} NR_{s,\mu}^{(k)}(p, q) \quad \text{and}$$

$$A_\mu^{(k)}(p, q, t) = NR_\mu^{(k)}(p, q, \frac{t}{1-t}).$$

For this reason, [6] focused on computing the generating functions $NR_\mu^{(k)}(p, q, t)$.

It will be useful to consider some refinements of the generating functions $NR_\mu^{(k)}(p, q, t)$ depending on the prefix or suffix of the words. Given any non-empty words $u, v \in NR([k])$, we let

$$\begin{aligned} \mathcal{P}_{n,k,u} &= \{w \in NR([k]) : |w| = n, u \text{ is a prefix of } w\}, \\ \mathcal{S}_{n,k,v} &= \{w \in NR([k]) : |w| = n, v \text{ is a suffix of } w\}, \quad \text{and} \\ \mathcal{PS}_{n,k,u,v} &= \{w \in NR([k]) : |w| = n, u \text{ is a prefix of } w, \text{ and } v \text{ is a suffix of } w\}. \end{aligned}$$

Then we define the generating functions

$$NR_{\mu}^{u,(k)}(p,q,t) = \sum_{n\geq 1} NR_{n,\mu}^{u,(k)}(p,q)t^n,$$
$$NR_{\mu}^{(k),v}(p,q,t) = \sum_{n\geq 1} NR_{n,\mu}^{(k),v}(p,q)t^n, \quad \text{and}$$
$$NR_{\mu}^{u,(k),v}(p,q,t) = \sum_{n\geq 1} NR_{n,\mu}^{u,(k),v}(p,q)t^n,$$

where

$$NR_{n,\mu}^{u,(k)}(p,q) = \sum_{w\in\mathcal{P}_{n,k,u}} q^{\mathrm{triv}_\mu(w)}p^{\mathrm{ntriv}_\mu(w)},$$
$$NR_{n,\mu}^{(k),v}(p,q) = \sum_{w\in\mathcal{S}_{n,k,v}} q^{\mathrm{triv}_\mu(w)}p^{\mathrm{ntriv}_\mu(w)}, \text{ and}$$
$$NR_{n,\mu}^{u,(k),v}(p,q) = \sum_{w\in\mathcal{PS}_{n,k,u,v}} q^{\mathrm{triv}_\mu(w)}p^{\mathrm{ntriv}_\mu(w)}.$$

Note if $i,j\in[k]$, then $\langle i,j\rangle$ is a trivial (nontrivial) μ-match if and only if its reverse complement $\langle k+1-j, k+1-i\rangle$ is a trivial (nontrivial) μ-match. It follows that the map that sends any word $w=w_1\ldots w_n$ to its reverse complement $(k+1-w_n)\ldots(k+1-w_1)$ shows that for any $i,j\in[k]$,

$$NR_{\mu}^{i,(k)}(p,q,t) = NR_{\mu}^{(k),k+1-i}(p,q,t) \quad \text{and}$$
$$NR_{\mu}^{i,(k),j}(p,q,t) = NR_{\mu}^{k+1-j,(k),k+1-i}(p,q,t).$$

In [6], we studied the generating functions $NR_{\mu}^{i,(k)}(p,q,t)$ and $NR_{\mu}^{(k)}(p,q,t)$. The main focus of this paper is to study the generating functions $N^{i,j}(p,q,t) := NR_{\mu}^{i,(3),j}(p,q,t)$. That is, for $i,j\in\{1,2,3\}$,

$$N^{i,j}(p,q,t) = \sum_{n\geq 1} N_n^{i,j}(p,q)t^n,$$

where

$$N_n^{i,j}(p,q) = \sum_{w\in\mathcal{PS}_{n,3,i,j}} q^{\mathrm{triv}_\mu(w)}p^{\mathrm{ntriv}_\mu(w)}.$$

We shall show that the generating functions $N^{i,j}(p,q,t)$ are closely related to the combinatorics of convolutions of Fibonacci numbers. For example, we shall show that

$$N^{3,1}(p,1,t) = \sum_{r\geq 0} \frac{t^{2r+2}p^r}{(1-t-t^2)^{r+1}}$$

so that the coefficient of p^r in $N^{3,1}(p,1,t)$ is $\frac{t^{2r+2}}{(1-t-t^2)^{r+1}}$, which is just a shifted version of the generating function of the convolution of $r+1$ copies of the Fibonacci numbers.

Hence the coefficient of p^r in $N^{3,1}(p,q,t)$ is a q-analogue of the generating function of the convolution of $r+1$ copies of the Fibonacci numbers.

The outline of this paper is as follows. In Section 2, we show how to compute the generating functions $N^{i,j}(p,q,t)$ for all $i,j \in [3]$. In Section 3, we give methods to find exact formulas for the coefficients that appear in $N^{3,1}(p,q,t)$, since these coefficients come from q-analogues of convolutions of Fibonacci numbers. In Section 4, we briefly discuss the combinatorics of the coefficients that arise in the other $N^{i,j}(p,q,t)$ functions. In Section 5, we present some results on the parity of convolutions of Fibonacci numbers.

2 Computing the generating functions $N^{i,j}(p,q,t)$

In this section, we compute the generating functions $N^{i,j}(p,q,t)$ by showing that they satisfy simple recursions. This allows us to compute all the $N^{i,j}(p,q,t)$ functions at once by inverting a simple matrix. We note that this method does not scale well as k increases because the recursions become increasingly complicated and the number of equations quickly gets too large to solve even by computer. We developed an alternative method using finite automata to compute $NR^{(k)}(p,q,t)$ in [6], which can be extended to $k=4,5$. This method can be easily modified to compute the functions $N^{i,j}(p,q,t)$ as well.

There is one observation that we should make initially. If we start with an alphabet $[k]$ where $k \geq 3$, then the only factors that correspond to nontrivial μ-matches are of the form iuj where $i+1<j$ and u does not contain any letters s such that $i \leq s \leq j$. Of course, when $k=3$, then i must equal 1 and j must equal 3 in which case u must be empty. Therefore, the only way to get a nontrivial μ-match over the alphabet [3] is to have a consecutive occurrence of 13.

First we will show that the generating functions $N^{i,j}(p,q,t)$ satisfy some simple recursions. Let $\mathcal{D}_{i,j}$ denote the set of all nonempty words $w \in NR([3])$ that start with i and end with j. Then there are several cases.

Case 1. $N^{1,1}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start and end with 1 as follows. That is, either

(i) $w = 1$,

(ii) $w = 12u$ where $2u \in \mathcal{D}_{2,1}$,

(iii) $w = 131u$ where $1u \in \mathcal{D}_{1,1}$ (with u possibly empty), or

(iv) $w = 132u$ where $2u \in \mathcal{D}_{2,1}$.

Clearly, case (i) contributes t to $N^{1,1}(p,q,t)$. In case (ii), the fact that the second letter is 2 ensures that the initial 1 can only form a μ-match with the second letter in w so that the words in (ii) contribute $qtN^{2,1}(p,q,t)$ to $N^{1,1}(p,q,t)$. In case (iii), the fact that the third letter is 1 ensures that the initial 1 in w can only form a

μ-match with the second letter in w. Also, the 3 in position 2 in w cannot be part of any other μ-match in w. Thus it follows that the words in case (iii) contribute $pt^2N^{1,1}(p,q,t)$ to $N^{1,1}(p,q,t)$. In case (iv), the 2 in position 3 of w ensures that the initial 1 can only form a μ-match with second and third letters in w, and the 3 in position 2 in w cannot be part of any other μ-match in w, so that the words in case (iv) contribute $qpt^2N^{2,1}(p,q,t)$ to $N^{1,1}(p,q,t)$. Thus it follows that

$$N^{1,1}(p,q,t) = t + pt^2N^{1,1}(p,q,t) + (qt + qpt^2)N^{2,1}(p,q,t). \tag{1}$$

This type of reasoning can be used in all the other cases of $N^{i,j}(p,q,t)$. Rather than give a detailed explanation in each case, we shall just list the corresponding subcases and the contribution of each subcase to $N^{i,j}(p,q,t)$.

Case 2. $N^{1,2}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 1 and end with 2 as follows. That is, either

(i) $w = 12u$ where $2u \in \mathcal{D}_{2,2}$ which contributes $qtN^{2,2}(p,q,t)$ to $N^{1,2}(p,q,t)$,

(ii) $w = 131u$ where $1u \in \mathcal{D}_{1,2}$ which contributes $pt^2N^{1,2}(p,q,t)$ to $N^{1,2}(p,q,t)$, or

(iii) $w = 132u$ where $2u \in \mathcal{D}_{2,2}$ which contributes $qpt^2N^{2,2}(p,q,t)$ to $N^{1,2}(p,q,t)$.

Thus it follows that

$$N^{1,2}(p,q,t) = pt^2N^{1,2}(p,q,t) + (qt + qpt^2)N^{2,2}(p,q,t). \tag{2}$$

Case 3. $N^{1,3}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 1 and end with 3 as follows. That is, either

(i) $w = 12u$ where $2u \in \mathcal{D}_{2,3}$ which contributes $qtN^{2,3}(p,q,t)$ to $N^{1,3}(p,q,t)$,

(ii) $w = 13$ which contributes pt^2 to $N^{1,3}(p,q,t)$,

(iii) $w = 131u$ where $1u \in \mathcal{D}_{1,3}$ which contributes $pt^2N^{1,3}(p,q,t)$ to $N^{1,3}(p,q,t)$, or

(iv) $w = 132u$ where $2u \in \mathcal{D}_{2,3}$ which contributes $qpt^2N^{2,3}(p,q,t)$ to $N^{1,3}(p,q,t)$.

Thus it follows that

$$N^{1,3}(p,q,t) = pt^2 + pt^2N^{1,3}(p,q,t) + (qt + qpt^2)N^{2,3}(p,q,t). \tag{3}$$

Case 4. $N^{2,1}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 2 and end with 1 as follows. That is, either

(i) $w = 21$ which contributes t^2 to $N^{2,1}(p,q,t)$,

(ii) $w = 212u$ where $2u \in \mathcal{D}_{2,1}$ which contributes $qt^2N^{2,1}(p,q,t)$ to $N^{2,1}(p,q,t)$,

(iii) $w = 2131u$ where $1u \in \mathcal{D}_{1,1}$ which contributes $qpt^3N^{1,1}(p,q,t)$ to $N^{2,1}(p,q,t)$,

(iv) $w = 2132u$ where $2u \in \mathcal{D}_{2,1}$ which contributes $q^2pt^3N^{2,1}(p,q,t)$ to $N^{2,1}(p,q,t)$, or

(v) $w = 23u$ where $3u \in \mathcal{D}_{3,1}$ which contributes $qtN^{3,1}(p,q,t)$ to $N^{2,1}(p,q,t)$.

Notice that we must subdivide the words starting with 213 to determine whether the 1 in the second position is involved in a trivial $\langle 1,2\rangle$-match. Thus it follows that

$$N^{2,1}(p,q,t) = t^2 + qpt^3N^{1,1}(p,q,t) + (qt^2 + q^2pt^3)N^{2,1}(p,q,t) + qtN^{3,1}(p,q,t). \quad (4)$$

Case 5. $N^{2,2}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start and end with 2 as follows.

(i) $w = 2$ which contributes t to $N^{2,2}(p,q,t)$,

(ii) $w = 212u$ where $2u \in \mathcal{D}_{2,2}$ which contributes $qt^2N^{2,2}(p,q,t)$ to $N^{2,2}(p,q,t)$,

(iii) $w = 2131u$ where $1u \in \mathcal{D}_{1,2}$ which contributes $qpt^3N^{1,2}(p,q,t)$ to $N^{2,2}(p,q,t)$,

(iv) $w = 2132u$ where $2u \in \mathcal{D}_{2,2}$ which contributes $q^2pt^3N^{2,2}(p,q,t)$ to $N^{2,2}(p,q,t)$, or

(v) $w = 23u$ where $3u \in \mathcal{D}_{3,2}$ which contributes $qtN^{3,2}(p,q,t)$ to $N^{2,2}(p,q,t)$.

Thus it follows that

$$N^{2,2}(p,q,t) = t + qpt^3N^{1,2}(p,q,t) + (qt^2 + q^2pt^3)N^{2,2}(p,q,t) + qtN^{3,1}(p,q,t). \quad (5)$$

Case 6. $N^{2,3}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 2 and end with 3 as follows. That is, either

(i) $w = 212u$ where $2u \in \mathcal{D}_{2,3}$ which contributes $qt^2N^{2,3}(p,q,t)$ to $N^{2,3}(p,q,t)$,

(ii) $w = 213$ which contributes qpt^3 to $N^{2,3}(p,q,t)$,

(iii) $w = 2131u$ where $1u \in \mathcal{D}_{1,3}$ which contributes $qpt^3N^{1,3}(p,q,t)$ to $N^{2,3}(p,q,t)$,

(iv) $w = 2132u$ where $2u \in \mathcal{D}_{2,3}$ which contributes $q^2pt^3N^{2,3}(p,q,t)$ to $N^{2,3}(p,q,t)$, or

(v) $w = 23u$ where $3u \in \mathcal{D}_{3,3}$ which contributes $qtN^{3,3}(p,q,t)$ to $N^{2,3}(p,q,t)$.

Thus it follows that

$$N^{2,3}(p,q,t) = qpt^3 + qpt^3 N^{1,3}(p,q,t) + (qt^2 + q^2pt^3)N^{2,3}(p,q,t) + qtN^{3,3}(p,q,t). \quad (6)$$

Case 7. $N^{3,1}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 3 and end with 1 as follows. That is, either

(i) $w = 31u$ where $1u \in \mathcal{D}_{1,1}$ which contributes $tN^{1,1}(p,q,t)$ to $N^{3,1}(p,q,t)$, or

(ii) $w = 32u$ where $2u \in \mathcal{D}_{2,1}$ which contributes $tN^{2,1}(p,q,t)$ to $N^{3,1}(p,q,t)$.

Thus it follows that

$$N^{3,1}(p,q,t) = tN^{1,1}(p,q,t) + tN^{2,1}(p,q,t). \quad (7)$$

Case 8. $N^{3,2}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start with 3 and end with 2 as follows. That is, either

(i) $w = 31u$ where $1u \in \mathcal{D}_{1,2}$ which contributes $tN^{1,2}(p,q,t)$ to $N^{3,2}(p,q,t)$, or

(ii) $w = 32u$ where $2u \in \mathcal{D}_{2,2}$ which contributes $tN^{2,2}(p,q,t)$ to $N^{3,2}(p,q,t)$.

Thus it follows that

$$N^{3,1}(p,q,t) = tN^{1,2}(p,q,t) + tN^{2,2}(p,q,t). \quad (8)$$

Case 9. $N^{3,3}(p,q,t)$.
We can classify the words $w \in NR([3])$ that start and end with 3 as follows. That is, either

(i) $w = 3$ which contributes t to $N^{3,3}(p,q,t)$,

(ii) $w = 31u$ where $1u \in \mathcal{D}_{1,3}$ which contributes $tN^{1,3}(p,q,t)$ to $N^{3,3}(p,q,t)$, or

(iii) $w = 32u$ where $2u \in \mathcal{D}_{2,3}$ which contributes $tN^{2,3}(p,q,t)$ to $N^{3,1}(p,q,t)$.

Thus it follows that

$$N^{3,1}(p,q,t) = t + tN^{1,3}(p,q,t) + tN^{2,3}(p,q,t). \quad (9)$$

Putting (1), (2), (3), (4), (5), (6), (7), (8), and (9) together, we obtain the matrix equation

$$\begin{bmatrix} -t \\ 0 \\ -pt^2 \\ -t^2 \\ -t \\ -qpt^3 \\ 0 \\ 0 \\ -t \end{bmatrix} = M \begin{bmatrix} N^{1,1}(p,q,t) \\ N^{1,2}(p,q,t) \\ N^{1,3}(p,q,t) \\ N^{2,1}(p,q,t) \\ N^{2,2}(p,q,t) \\ N^{2,3}(p,q,t) \\ N^{3,1}(p,q,t) \\ N^{3,2}(p,q,t) \\ N^{3,3}(p,q,t) \end{bmatrix}, \quad (10)$$

where

$$M = \begin{bmatrix} pt^2-1 & 0 & 0 & qt+qpt^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & pt^2-1 & 0 & 0 & qt+qpt^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & pt^2-1 & 0 & 0 & qt+qpt^2 & 0 & 0 & 0 \\ qpt^3 & 0 & 0 & qt^2+q^2pt^3-1 & 0 & 0 & qt & 0 & 0 \\ 0 & qpt^3 & 0 & 0 & qt^2+q^2pt^3-1 & 0 & 0 & qt & 0 \\ 0 & 0 & qpt^3 & 0 & 0 & qt^2+q^2pt^3-1 & 0 & 0 & qt \\ t & 0 & 0 & t & 0 & 0 & -1 & 0 & 0 \\ 0 & t & 0 & 0 & t & 0 & 0 & -1 & 0 \\ 0 & 0 & t & 0 & 0 & t & 0 & 0 & -1 \end{bmatrix}.$$

Thus if we multiply both sides of (10) by M^{-1}, we can can solve for $N^{i,j}(p,q,t)$ for $i,j \in [3]$. We have carried out this computation in Mathematica and found that if

$$D(p,q,t) = 1 - 2qt^2 - q^2t^3 - pt^2(1 + q^2t + 2q(q-1)t^2), \tag{11}$$

then

$$\begin{aligned} N^{1,1}(p,q,t) = N^{3,3}(p,q,t) &= \frac{t - qt^3 + pq(1-q)t^4}{D(p,q,t)}, \\ N^{1,2}(p,q,t) = N^{2,3}(p,q,t) &= \frac{qt^2 + qpt^3}{D(p,q,t)}, \\ N^{1,3}(p,q,t) &= \frac{pt^2 + q^2t^3 + 2qp(q-1)t^4}{D(p,q,t)}, \\ N^{2,1}(p,q,t) = N^{3,2}(p,q,t) &= \frac{t^2 + qt^3 + p(q-1)t^4}{D(p,q,t)}, \\ N^{2,2}(p,q,t) &= \frac{t - pt^3}{D(p,q,t)}, \text{ and} \\ N^{3,1}(p,q,t) &= \frac{t^2 + t^3 - p(1-q)^2t^5}{D(p,q,t)}. \end{aligned}$$

Setting $p = 0$ and $q = 1$ in the equations above and simplifying where possible, we see that

$$\begin{aligned} N^{1,1}(0,1,t) = N^{3,3}(0,1,t) &= \frac{t(1-t)}{1-t-t^2}, \\ N^{1,2}(0,1,t) = N^{2,3}(0,1,t) &= \frac{t^2}{1-2t^2-t^3}, \\ N^{1,3}(0,1,t) &= \frac{t^3}{1-2t^2-t^3}, \\ N^{2,1}(0,1,t) = N^{3,2}(0,1,t) &= \frac{t^2}{1-t-t^2}, \\ N^{2,2}(0,1,t) &= \frac{t}{1-2t^2-t^3}, \text{ and} \\ N^{3,1}(0,1,t) &= \frac{t^2}{1-t-t^2}. \end{aligned}$$

One can obtain the generating function $N_\mu^{(3)}(p,q,t)$ by taking $1 + \sum_{i,j\in[3]} N^{i,j}(p,q,t)$, which gives

$$N_\mu^{(3)}(p,q,t) = \frac{(1+t)^2(1+t-p(q-1)^2t^3)}{1-2qt^2-q^2t^3-pt^2(1+q^2t+2q(q-1)t^2)}.$$

3 The combinatorics of the coefficients of $N^{3,1}(p,q,t)$

The generating function for the Fibonacci numbers F_n is $\sum_{n\geq 0} F_n t^n = \frac{1}{1-t-t^2}$. Our next theorem follows immediately from the generating functions for $N^{3,i}(0,1,t)$ for $i \in [3]$. However we can also give a simple combinatorial proof.

Theorem 1. *For all $n \geq 2$,*

$$N_n^{3,1}(0,1) = N_n^{3,2}(0,1) = F_{n-2}$$

and, for all $n \geq 3$,

$$N_n^{3,3}(0,1) = F_{n-3}.$$

Proof. Since we have set $p=0$, this means that we are only considering words in $NR([3])$ that have no nontrivial μ-matches. By our observation at the start of the last section, this means that we are only considering words in $NR([3])$ that have no consecutive occurrences of the pattern 13. Our proofs proceed by induction of the length of the words. Let $\mathcal{C}_{n,i,j}$ denote the set of all words of length n in $NR([3])$ that start with i, end with j, and have no consecutive occurrences of 13.

The only words in $\mathcal{C}_{2,3,1}$ and $\mathcal{C}_{2,3,2}$ are 31 and 32 so that $N_2^{3,1}(0,1) = N_2^{3,2}(0,1) = 1 = F_0$. The only words in $\mathcal{C}_{3,3,1}$ and $\mathcal{C}_{3,3,2}$ are 321 and 312 so that $N_3^{3,1}(0,1) = N_3^{3,2}(0,1) = 1 = F_1$. The only words in $\mathcal{C}_{4,3,1}$ and $\mathcal{C}_{4,3,2}$ are 3231, 3121, 3212, and 3232 so that $N_4^{3,1}(0,1) = N_4^{3,2}(0,1) = 2 = F_2$.

For $n \geq 5$, we classify the words $u \in \mathcal{C}_{n,3,1}$ by their last three letters. That is, u is either of the form (i) $u = 3v121$, (ii) $u = 3v321$, or (iii) $3v231$. In the first case, $3v1$ can be any word in $\mathcal{C}_{n-2,3,1}$ so that there are F_{n-4} such words by induction. In cases (ii) and (iii), we can just remove the second to last letter and we we get all words of the form $3v31$ and $3v21$ in $N([k])$ with no consecutive occurrence of 13. This clearly is all words in $\mathcal{C}_{n-1,3,1}$ so that the number of words in cases (ii) and (iii) is F_{n-3} by induction. Thus the number of words in $\mathcal{C}_{n,3,1}$ is $F_{n-4} + F_{n-3} = F_{n-2}$.

Similarly, for $n \geq 5$, all the words u in $\mathcal{C}_{n,3,2}$ are either of the form (i) $u = 3v12$ or (ii) $u = 3v232$. In case (i), $3v1$ can be any word in $\mathcal{C}_{n-1,3,1}$ so that the number of words in case (i) is F_{n-3} by our previous result. In case (ii), $3v2$ can be any word in $\mathcal{C}_{n-2,3,2}$ so that the number of words in case (ii) is F_{n-3} by induction. Thus the number of words in $\mathcal{C}_{n,3,2}$ is $F_{n-4} + F_{n-3} = F_{n-2}$.

For words in $u = \mathcal{C}_{n,3,3}$, for $n \geq 3$, u must be of the form $u = 3v23$. However, $3v2$ can be any word in $\mathcal{C}_{n-1,3,2}$ so that by our previous result, there are F_{n-3} such words. $\square$

For any function $G(x) = \sum_{n\geq 0} g_n x^n$, we shall let $G(x)|_{x^n} = g_n$ denote the coefficient of x^n in $G(x)$. We have the following corollary of Theorem 1.

Corollary 2. *For all $r \geq 0$,*

$$N^{3,1}(p,1,t)|_{p^r} = \frac{t^{2r+2}}{(1-t-t^2)^{r+1}},$$

so that for all $n, r \geq 0$, $N_n^{3,1}(p,1)|_{p^r} = \displaystyle\sum_{\substack{j_1,\ldots,j_{r+1}\geq 0 \\ n-2r-2=\sum_{i=1}^{r+1} j_i}} F_{j_1}\cdots F_{j_{r+1}}.$

Proof. Let $\mathcal{C}^{(r)}_{n,i,j}$ denote the set of words $w \in NR([3])$ that start with i, end with j, and have r consecutive occurrences of 13, and let $\mathcal{C}^{(r)}_{i,j} = \bigcup_{n\geq 0} \mathcal{C}^{(r)}_{n,i,j}$. Then

$$N^{3,1}(p,1,t)|_{p^r} = \sum_{w\in\mathcal{C}^{(r)}_{3,1}} t^{|w|}.$$

But every word $w \in \mathcal{C}^{(r)}_{3,1}$ can be factored uniquely as $v_1v_2\ldots v_{r+1}$, where each v_i is a word in $NR([3])$ which starts with 3, ends with 1, and has no consecutive occurrences of 13. It follows that

$$N^{3,1}(p,1,t)|_{p^r} = \prod_{i=1}^{r+1} N^{3,1}(0,1,t) = \frac{t^{2r+2}}{(1-t-t^2)^{r+1}}.$$

Since this is the generating function for F_{n-2} raised to the power $r+1$, or the generating function for the convolution of $r+1$ Fibonacci numbers, it holds that for all $n, r \geq 0$,

$$N^{3,1}_n(p,1)|_{p^r} = \sum_{\substack{j_1,\ldots,j_{r+1}\geq 0\\ n-2r-2=\sum_{i=1}^{r+1} j_i}} F_{j_1}\cdots F_{j_{r+1}}.$$

□

It follows from Theorem 1 and Corollary 2 that $N^{3,1}_n(0,q)$ is a q-analogue of F_{n-2} and $N^{3,1}_n(p,q)|_{p^r}$ is a q-analogue of $\sum_{\substack{j_1,\ldots,j_{r+1}\geq 0\\ n-2r-2=\sum_{i=1}^{r+1} j_i}} F_{j_1}\cdots F_{j_{r+1}}$. We compute the generating functions for these q-analogues in the next theorem.

Theorem 3.

$$N^{3,1}(0,q,t) = \frac{t^2(1+t)}{1-2qt^2-q^2t^3}, \tag{12}$$

and for $r \geq 1$,

$$N^{3,1}(p,q,t)|_{p^r} = \frac{t^{2r+2}(1+tq+q(q-1)t^2)^2(1+q^2t+2q(q-1)t^2)^{r-1}}{(1-2qt^2-q^2t^3)^{r+1}}. \tag{13}$$

n	$N_n^{3,1}(0,q)$	n	$N_n^{3,1}(0,q)$
0	0	11	$16q^4+38q^5+q^6$
1	0	12	$64q^5+25q^6$
2	1	13	$32q^5+104q^6+8q^7$
3	1	14	$144q^6+88q^7+q^8$
4	$2q$	15	$64q^6+272q^7+41q^8$
5	$2q+q^2$	16	$320q^7+280q^8+10q^9$
6	$5q^2$	17	$128q^7+688q^7+170q^9+q^{10}$
7	$4q^2+4q^3$	18	$704q^8+832q^9+61q^{10}$
8	$12q^3+q^4$	19	$256q^8+1696q^9+620q^{10}+12q^{11}$
9	$8q^3+13q^4$	20	$1536q^9+2352q^{10}+292q^{11}+q^{12}$
10	$28q^4+6q^5$	21	$512q^9+4096q^{10}+2072q^{11}+85q^{12}$

Table 1: The first few terms of $N^{3,1}(0,q,t)$.

Proof. Note that

$$\begin{aligned}
N^{3,1}(p,q,t) &= \frac{t^2+t^3-p(1-q)^2t^5}{1-2qt^2-q^2t^3-pt^2(1+q^2t+2q(q-1)t^2)} \\
&= \frac{t^2+t^3-p(1-q)^2t^5}{1-2qt^2-q^2t^3}\left(\frac{1}{1-p\left(\frac{t^2(1+q^2t+2q(q-1)t^2)}{1-2qt^2-q^2t^3}\right)}\right) \\
&= \frac{t^2+t^3-p(1-q)^2t^5}{1-2qt^2-q^2t^3}\left(1+\sum_{r\geq 1}p^r\left(\frac{t^2(1+q^2t+2q(q-1)t^2)}{1-2qt^2-q^2t^3}\right)^r\right) \\
&= \frac{t^2(1+t)}{1-2qt^2-q^2t^3}+ \\
&\quad \sum_{r\geq 1}p^r\frac{t^{2r+2}(1+tq+q(q-1)t^2)^2(1+q^2t+2q(q-1)t^2)^{r-1}}{(1-2qt^2-q^2t^3)^{r+1}}.
\end{aligned}$$

□

The first few terms of $N^{3,1}(0,q,t)$ are displayed in Table 1. Setting $q=1$ recovers the Fibonacci sequence, where the indices have been shifted by two.

Next, we shall explain all of the coefficients in this table. Given a polynomial $P(x)$, write $P(q)=aq^k+HOT$ if the lowest power of q that appears with a nonzero coefficient in $P(q)$ is q^k and its coefficient is a. Similarly, write $P(q)=bq^n+LOT$ if the highest power of q that appears with a nonzero coefficient in $P(q)$ is q^n and its coefficient is b. Thus LOT stands for "lower order terms" and HOT stands for "higher order terms."

Note that it follows from equation (11), or from Theorem 3, that for $n\geq 4$,

$$N_n^{3,1}(0,q)=2qN_{n-2}^{3,1}(0,q)+q^2N_{n-3}^{3,1}(0,q). \tag{14}$$

This is easily proved directly. That is, for $n \geq 4$, a word in $\mathcal{C}_{n,3,1}$ is either of the form (i) $3v121$, (ii) $3v1231$, (iii) $3v2321$ or (iv) $3v3231$. The words in (i) contribute $qN^{3,1}_{n-2}(0,q)$ to $N^{3,1}_{n}(0,q)$ and the words in (ii) contribute $q^2N^{3,1}_{n-3}(0,q)$ to $N^{3,1}_{n}(0,q)$. For the words in (iii) and (iv), we can remove the second to last and third to last letters to give words of the form (iii)* $3v21$ and (iv)* $3v31$ which together give us all words in $\mathcal{C}_{n-2,3,1}$. Thus the words in (iii) and (iv) contribute $qN^{3,1}_{n-2,\mu}(0,q)$ to $N^{3,1}_{n}(0,q)$.

Using the recursion (14) and initial values from Table 1, one can easily prove the following by induction.

Theorem 4. *1. For $n \geq 1$, $N^{3,1}_{2n}(0,q) = c_{2n}q^{n-1} + HOT$ where $c_2 = 1$, $c_4 = 2$, and $c_{2n} = 2c_{2n-2} + 2^{n-3}$ for $n \geq 3$.*

2. For $n \geq 1$, $N^{3,1}_{2n+1}(0,q) = 2^{n-1}q^{n-1} + HOT$.

3. For $n \geq 0$, $N^{3,1}_{3n+2}(0,q) = q^{2n} + LOT$.

4. For $n \geq 1$, $N^{3,1}_{3n+1}(0,q) = 2nq^{2n-1} + LOT$.

5. For $n \geq 1$, $N^{3,1}_{3n}(0,q) = (1 + 2n(n-1))q^{2n-2} + LOT$.

Proof. We shall only prove (1) and (2), as the proofs of (3), (4), and (5) are similar. We proceed by induction on n. Table 1 verifies that the results hold for $n \leq 3$.

We know from (14) that

$$N^{3,1}_{2n+1}(0,q) = 2qN^{3,1}_{2n-1}(0,q) + q^2N^{3,1}_{2n-2}(0,q).$$

By induction, the lowest power of q occurring in $N^{3,1}_{2n-1}(0,q)$ is q^{n-2} and the lowest power of q occurring in $N^{3,1}_{2n-2}(0,q)$ is q^{n-2}. Thus the lowest power of q occurring in $N^{3,1}_{2n+1}(0,q)$ is q^{n-1} and it must be the case that

$$N^{3,1}_{2n+1}(0,q)|_{q^{n-1}} = 2N^{3,1}_{2n-1}(0,q)|_{q^{n-2}} = 2(2^{n-2}) = 2^{n-1}.$$

Similarly, we know

$$N^{3,1}_{2n}(0,q) = 2qN^{3,1}_{2n-2}(0,q) + q^2N^{3,1}_{2n-3}(0,q).$$

By induction, the lowest power of q occurring in $N^{3,1}_{2n-2}(0,q)$ is q^{n-2} and the lowest power of q occurring in $N^{3,1}_{2n-3}(0,q)$ is q^{n-3}. Thus the lowest power of q occurring in $N^{3,1}_{2n+1}(0,q)$ is q^{n-1} and it must be the case that

$$\begin{aligned} N^{3,1}_{2n}(0,q)|_{q^{n-1}} &= 2N^{3,1}_{2n-2}(0,q)|_{q^{n-2}} + N^{3,1}_{2n-3}(0,q)|_{q^{n-3}} \\ &= 2c_{2n-2} + 2^{n-3}. \end{aligned}$$

□

One can also give direct combinatorial explanations of parts (2), (3) and (4) of the previous theorem. That is, the word in $\mathcal{C}_{3n+2,3,1}$ with the maximum number of trivial μ-matches is $3(123)^n1$, which has $2n$ trivial μ-matches, explaining (3). Then one can obtain a word $w \in \mathcal{C}_{3n+4,3,1}$ with $2n+1$ trivial μ-matches simply inserting either 23 or 12 immediately before any 1 in $3(123)^n1$. Thus the number of words in $\mathcal{C}_{3n+4,3,1}$ with $2n+1$ trivial μ-matches is $2(n+1)$, which justifies (4). For (2), to create a word in $\mathcal{C}_{2n+1,3,1}$ with $n-1$ nontrivial matches, start with the word 321. Then insert $a_1a_2\ldots a_{2n-3}a_{2n-2}$ after the 2 where for each $i=0,\ldots,n-1$, either $a_{2i+1}a_{2i+2}=12$ or $a_{2i+1}a_{2i+2}=32$. It is easy to see that each additional $a_{2i+1}a_{2i+2}$ creates exactly one new trivial μ-match so that there are 2^{n-1} such words.

Given the facts that $N_{2n}^{3,1}(0,q) = c_{2n}q^{n-1} + HOT$ and $N_{2n+1}^{3,1}(0,q) = 2^{n-1}q^{n-1} + HOT$, one can find formulas for all the coefficients in both $N_{2n}^{3,1}(0,q)$ and $N_{2n+1}^{3,1}(0,q)$.

To find formulas for the coefficients that appear in $N_{2n+1}^{3,1}(0,q)$, first let

$$F_1(q,t) := \frac{(1+t)}{1-2qt^2-q^2t^3} = \sum_{n\geq 0} N_{n+2}^{3,1}(0,q)t^n.$$

Thus

$$F_2(q,t) := \frac{1}{2t}(F_1(q,t)-F_1(q,-t)) = \sum_{n\geq 0} N_{2n+3}^{3,1}(0,q)t^{2n}.$$

Then one can compute that

$$\begin{aligned}
F_3(q,t) := F_2(q,(\frac{t}{q})^{1/2}) &= \sum_{n\geq 0} N_{2n+3}^{3,1}(0,q)\frac{t^n}{q^n} = \frac{1-2t+qt}{(1-2t)^2-qt^3} \\
&= \left(\frac{1}{1-2t}+\frac{qt}{(1-2t)^2}\right)\frac{1}{1-\frac{qt^3}{(1-2t)^2}} \\
&= \frac{1}{1-2t} + \sum_{k\geq 1} q^k \frac{(1-t)^2t^{3k-2}}{(1-2t)^{2k+1}}.
\end{aligned}$$

Therefore, for all $n \geq 0$,

$$N_{2n+3}^{3,1}(0,q)|_{q^n} = F_3(q,t)|_{q^0t^n} = \frac{1}{1-2t}|_{t^n} = 2^n,$$

as was proven in Theorem 4 part (2). For $k \geq 1$,

$$N_{2n+3}^{3,1}(0,q)|_{q^{n+k}} = F_3(q,t)|_{q^kt^n} = \frac{(1-t)^2t^{3k-2}}{(1-2t)^{2k+1}}|_{t^n}.$$

By Newton's binomial theorem

$$\frac{1}{(1-2t)^{2k+1}} = \sum_{m\geq 0}\binom{2k+m}{2k}2^mt^m.$$

Hence it follows that

1. $N^{3,1}_{2n+3}(0,q)|_{q^{n+k}} = 0$ for $n < 3k-2$,

2. $N^{3,1}_{2n+3}(0,q)|_{q^{n+k}} = \sum_{m\geq 0} \binom{2k+m}{2k} 2^m t^m|_{t^0} = 1$ for $n = 3k-2$,

3. $N^{3,1}_{2n+3}(0,q)|_{q^{n+k}} = (1-2t) \sum_{m\geq 0} \binom{2k+m}{2k} 2^m t^m|_{t^1} = 4k$ for $n = 3k-1$, and

4. for $n = 3k+j$ where $j \geq 0$,

$$\begin{aligned} N^{3,1}_{2n+3}(0,q)|_{q^{n+k}} &= (1-2t+t^2) \sum_{m\geq 0} \binom{2k+m}{2k} 2^m t^m|_{t^{j+2}} \\ &= \binom{2k+j+2}{2k} 2^{j+2} - 2\binom{2k+j+1}{2k} 2^{j+1} + \binom{2k+j}{2k} 2^j \\ &= 2^j \left(4\binom{2k+j+2}{2k} - 4\binom{2k+j+1}{2k} + \binom{2k+j}{2k} \right) \\ &= 2^j \left(4\binom{2k+j+1}{2k-1} + \binom{2k+j}{2k} \right). \end{aligned}$$

Thus we have proven the following theorem.

Theorem 5. *For $n \geq 0$,*

1. $N^{3,1}_{2n+3}(0,q)|_{q^m} = 0$ *for* $m < n$,

2. $N^{3,1}_{2n+3}(0,q)|_{q^n} = 2^n$, *and*

3. *for* $k \geq 1$,

$$N^{3,1}_{2n+3}(0,q)|_{q^{n+k}} = \begin{cases} 0 & \text{if } n < 3k-2, \\ 1 & \text{if } n = 3k-2, \\ 4k & \text{if } n = 3k-1, \text{ and} \\ 2^j \left(4\binom{2k+j+1}{2k-1} + \binom{2k+j}{2k}\right) & \text{if } n = 3k+j \text{ where } j \geq 0. \end{cases}$$

We can use the same methods to find formulas for the coefficients in $N^{3,1}_{2n}(0,q)$. That is, if we let

$$G_2(q,t) := \frac{1}{2}(F_1(q,t) + F_1(q,-t)) = \sum_{n\geq 0} N^{3,1}_{2n+2}(0,q) t^{2n},$$

then one can compute that

$$\begin{aligned} G_3(q,t) := G_2(q,(\frac{t}{q})^{1/2}) &= \sum_{n\geq 0} N^{3,1}_{2n+2}(0,q) \frac{t^n}{q^n} = \frac{(1-t)^2}{(1-2t)^2 - qt^3} \\ &= \frac{(1-t)^2}{(1-2t)^2} + \sum_{k\geq 1} q^k \frac{(1-t)^2 t^{3k}}{(1-2t)^{2k+2}}. \end{aligned}$$

Thus for all $n \geq 0$,

$$N_{2n+2}^{3,1}(0,q)|_{q^n} = \frac{(1-t)^2}{(1-2t)^2}|_{t^n}$$

and, for $k \geq 1$,

$$N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} = \frac{(1-t)^2 t^{3k}}{(1-2t)^{2k+2}}|_{t^n}.$$

In this case, Newton's binomial theorem says that

$$\frac{1}{(1-2t)^{2k+2}} = \sum_{m\geq 0}\binom{2k+1+m}{2k+1}2^m t^m.$$

We see that

$$\begin{aligned}
N_2^{3,1}(0,q)|_{q^0} &= \frac{(1-t)^2}{(1-2t)^2}|_{t^0} = 1,\\
N_4^{3,1}(0,q)|_{q^1} &= \frac{1}{(1-2t)^2}|_{t^1} - 2\frac{1}{(1-2t)^2}|_{t^0} = 4-2(1) = 2,
\end{aligned}$$

and, for $n \geq 2$,

$$\begin{aligned}
N_{2n}^{3,1}(0,q)|_{q^n} &= \frac{1}{(1-2t)^2}|_{t^n} - 2\frac{1}{(1-2t)^2}|_{t^{n-1}} + \frac{1}{(1-2t)^2}|_{t^{n-2}}\\
&= (n+1)2^n - 2n2^{n-1} + (n-1)2^{n-2}\\
&= (n+3)2^{n-2}.
\end{aligned}$$

Similarly for $k \geq 1$, we have

1. $N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} = 0$ for $n < 3k$,
2. $N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} = \sum_{m\geq 0}\binom{2k+1+m}{2k+1}2^m t^m|_{t^0} = 1$ for $n = 3k$,
3. $N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} = (1-2t)\sum_{m\geq 0}\binom{2k+1+m}{2k+1}2^m t^m|_{t^1} = 4k+2$ for $n = 3k+1$, and
4. for $n = 3k+2+j$ where $j \geq 0$,

$$\begin{aligned}
N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} &= (1-2t+t^2)\sum_{m\geq 0}\binom{2k+1+m}{2k+1}2^m t^m|_{t^{j+2}}\\
&= \binom{2k+j+3}{2k+1}2^{j+2} - 2\binom{2k+j+2}{2k+1}2^{j+1} + \binom{2k+j+1}{2k+1}2^j\\
&= 2^j\left(4\binom{2k+j+3}{2k+1} - 4\binom{2k+j+2}{2k+1} + \binom{2k+j+1}{2k+1}\right)\\
&= 2^j\left(4\binom{2k+j+2}{2k} + \binom{2k+j+1}{2k+1}\right).
\end{aligned}$$

n	$N_n^{3,1}(p,q)\|_p$	n	$N_n^{3,1}(p,q)\|_p$
0	0	11	$66q^4+62q^5+2q^6$
1	0	12	$28q^4+162q^5+45q^6$
2	0	13	$172q^5+230q^6+18q^7$
3	0	14	$64q^5+475q^6+202q^7+3q^8$
4	1	15	$432q^6+768q^7+108q^8$
5	$2q$	16	$144q^6+1320q^7+789q^8+32q^9$
6	$2q+3q^2$	17	$1056q^7+2388q^7+522q^9+4q^{10}$
7	$8q^2+2q^3$	18	$320q^7+3528q^8+2802q^9+215q^{10}$
8	$5q^2+14q^3+q^4$	19	$2528q^8+7048q^9+2196q^{10}+50q^{11}$
9	$24q^3+14q^4$	20	$704q^8+9152q^9+9281q^{10}+1142q^{11}+5q^{12}$
10	$12q^3+51q^4+8q^5$	21	$5952q^9+19984q^{10}+8376q^{11}+378q^{12}$

Table 2: The first few terms of $N^{3,1}(p,q,t)|_p$.

Thus we have proven the following theorem.

Theorem 6. *For $n \geq 0$,*

1. $N_{2n+2}^{3,1}(0,q)|_{q^m} = 0$ *for* $m < n$,

2. $$N_{2n+2}^{3,1}(0,q)|_{q^n} = \begin{cases} 1 & \text{if } n=0, \\ 2 & \text{if } n=1, \quad \text{and} \\ (n+3)2^{n-2} & \text{if } n \geq 2, \end{cases}$$

   and

3. *for all* $k \geq 1$,

$$N_{2n+2}^{3,1}(0,q)|_{q^{n+k}} = \begin{cases} 0 & \text{if } n < 3k, \\ 1 & \text{if } n = 3k, \\ 4k+2 & \text{if } n = 3k+1, \quad \text{and} \\ 2^j\left(4\binom{2k+j+2}{2k} + \binom{2k+j+1}{2k+1}\right) & \text{if } n = 3k+2+j \text{ where } j \geq 0. \end{cases}$$

One can carry out a similar analysis for

$$N^{3,1}(p,q,t)|_p = \frac{t^4(1+tq+q(q-1)t^2)^2}{(1-2qt^2-q^2t^3)^2}, \tag{15}$$

which comes from setting $r=1$ in (13) from Theorem 3. We have computed the first few terms of $N^{3,1}(p,q,t)|_p$, which are displayed in Table 2.

Note that since $(1-2q^2-q^2t^3)^2 = 1-4qt^2-2q^2t^3+4q^2t^4+4q^3t^5+q^4t^6$, it follows from the generating function of $N^{3,1}(0,q,t)$ that for $n \geq 7$,

$$\begin{aligned} N^{3,1}_n(p,q)|_p = 4qN^{3,1}_{n-2}(p,q)|_p + 2q^2N^{3,1}_{n-3}(p,q)|_p - 4q^2N^{3,1}_{n-4}(p,q)|_p \\ - 4q^3N^{3,1}_{n-5}(p,q)|_p - q^4N^{3,1}_{n-6}(p,q)|_p. \end{aligned} \tag{16}$$

Using the recursion (16) and the initial values from our table, one can easily prove the following theorem by induction.

Theorem 7. *1. For $n \geq 0$, $N^{3,1}_{2n+4}(p,q)|_p = d_{2n+4}q^n + HOT$ where $d_4 = 1$ and $d_{2n+4} = (n+3)2^{n-2}$ for $n \geq 1$.*

2. For $n \geq 0$, $N^{3,1}_{2n+5}(p,q)|_p = d_{2n+5}q^{n+1} + HOT$ where $d_5 = 2$ and $d_{2n+5} = (18+13n+n^2)2^{n-3}$ for $n \geq 1$.

3. For $n \geq 0$, $N^{3,1}_{3n+3}(p,q)|_p = n(2n^2-2n+3)q^{2n} + LOT$.

4. For $n \geq 1$, $N^{3,1}_{3n+4}(p,q)|_p = 2n^2q^{2n+1} + LOT$.

5. For $n \geq 1$, $N^{3,1}_{3n+5}(p,q)|_p = nq^{2n+2} + LOT$.

By using the same type of series manipulations that we did for $N^{3,1}(0,q,t)$, one can also show that

$$\begin{aligned} \sum_{n\geq 0} N^{3,1}_{2n+3}(p,q)|_p \frac{t^n}{q^n} &= \frac{O(p,q)}{((1-2t)^2-qt^3)^2} \\ &= \frac{O(p,q)}{(1-2t)^4} \frac{1}{\left(1-\frac{qt^3}{(1-2t)^2}\right)^2} \\ &= O(p,q)\sum_{m\geq 0}(m+1)\frac{q^mt^{3m}}{(1-2t)^{2m+4}}, \end{aligned} \tag{17}$$

where

$$O(p,q) = 2t(1-t)(1-2t)(1-t-t^2) + 2t^2(1-t)(1-3t^2)q + 2t^4(1-t)q^2.$$

Similarly, one can show that

$$\begin{aligned} \sum_{n\geq 0} N^{3,1}_{2n+4}(p,q)|_p \frac{t^n}{q^n} &= \frac{E(p,q)}{((1-2t^2)-qt^3)^2} \\ &= E(p,q)\sum_{m\geq 0}(m+1)\frac{q^mt^{3m}}{(1-2t)^{2m+4}}, \end{aligned} \tag{18}$$

where

$$E(p,q) = (t-1)^2(1-2t)^2 + t(3-10t+9t^2-2t^3+t^4)q + t^2(1-t^2-2t^3)q^2 + t^5q^3.$$

For any fixed $k \geq 0$, one can use (17) and (18) to extract closed forms for the series

$$\sum_{n\geq 0} \left(N^{3,1}_{2n+3}(p,q)|_{pq^{n+k}}\right) t^n \quad \text{and} \quad \sum_{n\geq 0} \left(N^{3,1}_{2n+4}(p,q)|_{pq^{n+k}}\right) t^n.$$

These closed forms will always be of the form $\frac{P(t)}{(1-2t)^j}$, which will in turn allow one to give explicit formulas for $N^{3,1}_{2n+3}(p,q)|_{pq^{n+k}}$ and $N^{3,1}_{2n+3}(p,q)|_{pq^{n+k}}$ as functions of n and k.

4 The combinatorics of the coefficients of $N^{i,j}(p,q,t)$ for $(i,j) \neq (3,1)$

In the last section, we focused on the combinatorics of the coefficients of $N^{3,1}(p,q,t)$ because they gave us q-analogues of the Fibonacci numbers and convolutions of Fibonacci numbers. There are similarly interesting phenomena for the other $N^{i,j}(p,q,t)$ functions.

In this section, we shall limit ourselves to understanding the coefficients of the highest and lowest powers of q in $N^{i,j}(0,q,t)$, although series manipulations and Newton's binomial theorem could be used to find formulas for the other coefficients, as in the previous section. Our calculations in Section 2 showed that there are only six different generating functions of the form $N^{i,j}(0,q,t)$ so that we need only consider the generating functions $N^{1,1}(p,q,t)$, $N^{1,2}(p,q,t)$, $N^{1,3}(p,q,t)$, $N^{2,1}(p,q,t)$, $N^{2,2}(p,q,t)$, and $N^{3,1}(p,q,t)$. When we evaluate these generating functions at $p=0$, we find that

$$\begin{aligned}
N^{1,1}(0,q,t) &= \frac{t(1-qt^2)}{1-2q^2-q^2t^3},\\
N^{1,2}(0,q,t) &= \frac{qt^2}{1-2q^2-q^2t^3},\\
N^{1,3}(0,q,t) &= \frac{q^2t^3}{1-2q^2-q^2t^3},\\
N^{2,1}(0,q,t) &= \frac{t^2(1+qt)}{1-2q^2-q^2t^3},\\
N^{2,2}(0,q,t) &= \frac{t}{1-2q^2-q^2t^3}, \quad \text{and}\\
N^{3,1}(p,q,t) &= \frac{t^2(1+t)}{1-2q^2-q^2t^3}.
\end{aligned}$$

Notice that $N^{1,2}(0,q,t) = qtN^{2,2}(0,q,t)$ and $N^{1,3}(0,q,t) = qtN^{1,2}(0,q,t) = q^2t^2N^{2,2}(0,q,t)$. Thus up to powers of t and q, there are only three other cases that we have to consider, namely, $N^{1,1}(0,q,t)$, $N^{2,1}(0,q,t)$, and $N^{2,2}(0,q,t)$.

First we consider what happens for $N^{i,j}_n(0,1)$ in such cases.

Theorem 8. *1.* $N^{2,2}_1(0,1) = 1$ *and for all* $n \geq 2$,

$$N^{2,2}_n(0,1) = F_{n-2} + (-1)^{n-1}.$$

2. $N_1^{1,1}(0,1) = 1$, $N_2^{1,1}(0,1) = 0$, *and for all* $n \geq 3$,

$$N_n^{1,1}(0,1) = F_{n-3}.$$

3. $N_1^{2,1}(0,1) = 0$ *and for all* $n \geq 2$,

$$N_n^{2,1}(0,1) = F_{n-2}.$$

Proof. Notice that $\mathcal{C}_{1,2,2} = \{2\}$ so $N_1^{2,2}(0,1) = 1$ as claimed. It is easy to see that $\mathcal{C}_{2,2,2} = \emptyset$ and $\mathcal{C}_{3,2,2} = \{212, 232\}$ so that $N_2^{2,2}(0,1) = 0 = F_0 + (-1)^{2-1}$ and $N_3^{2,2}(0,1) = 2 = F_1 + (-1)^{3-1}$. Additionally, $\mathcal{C}_{4,2,2} = \{2312\}$ so that $N_4^{2,2}(0,1) = 1 = F_2 + (-1)^{4-1}$.

For $n \geq 5$, the elements of $\mathcal{C}_{n,2,2}$ are either of the form (i) $2v212$, (ii) $2v2312$, or (iii) $2v232$. The words in cases (i) and (iii) are clearly each counted by $N_{n-2}^{2,2}(0,1)$ if we remove the last two letters. The words in case (ii) are counted by $N_{n-3}^{2,2}(0,1)$ if we remove the last three letters. Thus by induction,

$$\begin{aligned} N_n^{2,2}(0,1) &= 2(N_{n-2}^{2,2}(0,1)) + N_{n-3}^{2,2}(0,1) \\ &= 2(F_{n-4} + (-1)^{n-3}) + F_{n-5} + (-1)^{n-4} \\ &= F_{n-2} + (-1)^{n-1}, \end{aligned}$$

which proves part (1).

Now, observe that $\mathcal{C}_{1,1,1} = \{1\}$ so that $N_1^{1,1}(0,1) = 1$ and $\mathcal{C}_{2,1,1} = \emptyset$ so that $N_2^{1,1}(0,1) = 0$. Moreover, $\mathcal{C}_{3,1,1} = \{121\}$ so that $N_3^{1,1}(0,1) = 1 = F_0$ and $\mathcal{C}_{4,1,1} = \{1231\}$ so that $N_4^{1,1}(0,1) = 1 = F_1$.

For $n \geq 5$, the elements of $\mathcal{C}_{n,1,1}$ either take the form (i) $12v21$ or (ii) $12v231$. The words in case (i) are counted by $N_{n-2,\mu}^{2,2}(0,1)$ by removing the first and last letters. The words in case (ii) are counted by $N_{n-3,\mu}^{2,2}(0,1)$ by removing the first letter and the last two letters. The result of part (1) gives

$$\begin{aligned} N_n^{1,1}(0,1) &= N_{n-2,\mu}^{2,2}(0,1) + N_{n-3,\mu}^{2,2}(0,1) \\ &= F_{n-4} + (-1)^{n-3} + F_{n-5} + (-1)^{n-4} \\ &= F_{n-3}, \end{aligned}$$

which proves part (2).

To prove part (3), notice that $\mathcal{C}_{1,2,1} = \emptyset$ so that $N_1^{2,1}(0,1) = 0$. Also, $\mathcal{C}_{2,2,1} = \{21\}$ so that $N_2^{2,1}(0,1) = 1 = F_0$ and $\mathcal{C}_{3,2,1} = \{231\}$ so that $N_3^{2,1}(0,1) = 1 = F_1$.

For $n \geq 4$, the elements of $\mathcal{C}_{n,2,1}$ either take the form (i) $2v21$ or (ii) $2v231$. The words in case (i) are counted by $N_{n-1,\mu}^{2,2}(0,1)$ by removing the last letter, and the words in case (ii) are counted by $N_{n-2,\mu}^{2,2}(0,1)$ by removing the last two letters. The result of part (1) gives

$$\begin{aligned} N_n^{2,1}(0,1) &= N_{n-1,\mu}^{2,2}(0,1) + N_{n-2,\mu}^{2,2}(0,1) \\ &= F_{n-3} + (-1)^{n-2} + F_{n-4} + (-1)^{n-3}. \\ &= F_{n-2}. \end{aligned}$$

$\square$

n	$N_n^{1,1}(0,q)$	n	$N_n^{1,1}(0,q)$
0	0	11	$16q^5+18q^6$
1	0	12	$48q^6+7q^7$
2	0	13	$32q^6+56q^7+q^8$
3	q	14	$112q^7+32q^8$
4	q^2	15	$64q^7+160q^8+9q^9$
5	$2q^2$	16	$256q^8+120q^9+q^{10}$
6	$3q^3$	17	$128q^8+432q^9+50q^{10}$
7	$4q^3+q^4$	18	$576q^9+400q^{10}+11q^{11}$
8	$8q^4$	19	$256q^9+1120q^{10}+220q^{11}+q^{12}$
9	$8q^4+5q^4$	20	$1280q^{10}+1232q^{11}+72q^{12}$
10	$20q^5+q^6$	21	$512q^{10}+2816q^{11}+840q^{12}+13q^{13}$

Table 3: The first few terms of $N^{1,1}(0,q,t)$.

n	$N_n^{2,1}(0,q)$	n	$N_n^{2,1}(0,q)$
0	0	11	$48q^5+7q^6$
1	0	12	$32q^5+56q^6+q^7$
2	1	13	$112q^6+32q^7$
3	q	14	$64q^6+160q^7+9q^8$
4	$2q$	15	$256q^7+120q^8+q^9$
5	$3q^2$	16	$128q^7+432q^8+50q^9$
6	$4q^2+q^3$	17	$576q^8+400q^9+11q^{10}$
7	$8q^3$	18	$256q^8+1120q^9+220q^{10}+q^{11}$
8	$8q^3+5q^4$	19	$1280q^9+1232q^{10}+72q^{11}$
9	$20q^4+q^5$	20	$512q^9+2816q^{10}+840q^{11}+13q^{12}$
10	$16q^4+18q^5$	21	$2816q^{10}+3584q^{11}+364q^{12}+q^{13}$

Table 4: The first few terms of $N^{2,1}(0,q,t)$.

Recall our earlier observation that $N^{1,2}(0,q,t) = qtN^{2,2}(0,q,t)$ and $N^{1,3}(0,q,t) = qtN^{1,2}(0,q,t) = q^2t^2N^{2,2}(0,q,t)$. It is easy to see that every $u \in \mathcal{C}_{n,1,2}$ is of the form $12v2$ where $2v2 \in \mathcal{C}_{n-1,2,2}$ so that $N_n^{1,2}(0,1,t) = N_{n-1}^{2,2}(0,q,t)$ for $n \geq 3$. Similarly, every $u \in \mathcal{C}_{n,1,3}$ is of the form $1v23$ where $1v2 \in \mathcal{C}_{n-1,1,2}$ so that $N_n^{1,3}(0,1,t) = N_{n-1}^{1,2}(0,q,t)$ for $n \geq 3$. Thus the three cases in the previous theorem are sufficient to determine $N^{i,j}(0,q,t)$ for any pair $(i,j) \neq (3,1)$.

We have computed the first few terms of $N^{1,1}(0,q,t)$, $N^{2,1}(0,q,t)$, and $N^{2,2}(0,q,t)$ in Tables 3, 4, and 5, respectively.

For all $(i,j) \in \{(1,1),(2,1),(2,2)\}$, we have that

$$N_n^{i,j}(0,q) = 2qN_{n-2}^{i,j}(0,q) + q^2N_{n-3}^{i,j}(0,q)$$

n	$N_n^{2,2}(0,q)$	n	$N_n^{2,2}(0,q)$
0	0	11	$32q^5+24q^6$
1	0	12	$80q^6+8q^7$
2	0	13	$64q^6+80q^7+q^8$
3	$2q$	14	$192q^7+40q^8$
4	q^2	15	$128q^7+240q^8+10q^9$
5	$4q^2$	16	$488q^8+160q^9+q^{10}$
6	$4q^3$	17	$256q^8+672q^9+60q^{10}$
7	$8q^3+q^4$	18	$1024q^9+560q^{10}+12q^{11}$
8	$12q^4$	19	$512q^9+1792q^{10}+280q^{11}+q^{12}$
9	$16q^4+6q^5$	20	$2304q^{10}+1792q^{11}+84q^{12}$
10	$32q^5+q^6$	21	$1024q^{10}+4608q^{11}+1120q^{12}+14q^{13}$

Table 5: The first few terms of $N^{2,2}(0,q,t)$.

for $n \geq 4$. We can use this recursion to find the coefficients of the lowest and highest powers of q that occur in each $N_n^{i,j}(0,q)$, just as we did in Theorem 4. Thus we will simply state the next results without proof.

Theorem 9. *1. For $n \geq 2$, $N_{2n}^{1,1}(0,q) = n2^{n-3}q^n + HOT$.*

2. For $n \geq 1$, $N_{2n+1}^{1,1}(0,q) = 2^{n-1}q^n + HOT$.

3. For $n \geq 1$, $N_{3n}^{1,1}(0,q) = (2n-1)q^{2n-1} + LOT$.

4. For $n \geq 0$, $N_{3n+1}^{1,1}(0,q) = q^{2n} + LOT$.

5. For $n \geq 0$, $N_{3n+2}^{1,1}(0,q) = 2n^2q^{2n} + LOT$.

Theorem 10. *1. For $n \geq 1$, $N_{2n}^{2,1}(0,q) = 2^{n-1}q^{n-1} + HOT$.*

2. For $n \geq 1$, $N_{2n+1}^{2,1}(0,q) = (n+1)2^{n-2}q^n + HOT$.

3. For $n \geq 1$, $N_{3n}^{2,1}(0,q) = q^{2n-1} + LOT$.

4. For $n \geq 1$, $N_{3n+1}^{2,1}(0,q) = 2n^2q^{2n-1} + LOT$.

5. For $n \geq 0$, $N_{3n+2}^{2,1}(0,q) = (2n+1)q^{2n} + LOT$.

Theorem 11. *1. For $n \geq 1$, $N_{2n}^{2,2}(0,q) = (n-1)2^{n-2}q^n + HOT$.*

2. For $n \geq 0$, $N_{2n+1}^{2,2}(0,q) = 2^nq^n + HOT$.

3. For $n \geq 1$, $N_{3n}^{2,2}(0,q) = 2nq^{2n-1} + LOT$.

4. For $n \geq 0$, $N_{3n+1}^{2,2}(0,q) = q^{2n} + LOT$.

5. For $n \geq 0$, $N_{3n+2}^{2,2}(0,q) = 2n(n+1)q^{2n} + LOT$.

5 Parity Patterns

In this section, we shall briefly discuss some results on how our work relates to the parity of convolutions of Fibonacci numbers. First we consider the parity of the Fibonacci numbers themselves. It is well known that if we define F_n by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, then F_{3n} and F_{3n+1} are odd for all $n \geq 0$ and F_{3n+2} are even for all $n \geq 0$. This is straightforward to prove by induction. The work of this section is inspired by a question of Bruce Sagan, who, when looking at the table of $N_n^{3,1}(0,q)$ on page 177, asked about the parity of the coefficients appearing in $N_n^{3,1}(0,q)$. It turns out we can prove the following theorem, which gives an alternative way to prove the parity conditions satisfied by the Fibonacci numbers.

Theorem 12. *1. For all $n \geq 1$, all of coefficients that appear in $N_{3n+1}^{3,1}(0,q)$ are even.*

2. For all $n \geq 0$, $N_{3n+2}^{3,1}(0,q)$ is a degree $2n$ polynomial in q, $N_{3n+2}^{3,1}(0,q)|_{q^{2n}}$ is odd, and for all $i < 2n$, $N_{3n+2}^{3,1}(0,q)|_{q^i}$ is even.

3. For all $n \geq 1$, $N_{3n}^{3,1}(0,q)$ is a degree $2n-2$ polynomial in q, $N_{3n}^{3,1}(0,q)|_{q^{2n-2}}$ is odd, and for all $i < 2n-2$, $N_{3n}^{3,1}(0,q)|_{q^i}$ is even.

Proof. For part (1), we note from (12) that

$$N^{3,1}(0,q,t) = \sum_{n\geq 2} N_n^{3,1}(0,q)t^n = \frac{t^2(1+t)}{1-2qt^2-q^2t^3}.$$

Thus if we let

$$M^{3,1}(q,t) := \frac{t(1+t)}{1-2qt^2-q^2t^3},$$

then one can compute that

$$\begin{aligned}\sum_{n\geq 1} N_{3n+1}^{3,1}(0,q)t^{3n} &= \frac{1}{3}\left(M^{3,1}(q,t) + M^{3,1}(q,e^{\frac{2\pi i}{3}}t) + M^{3,1}(q,e^{\frac{4\pi i}{3}}t)\right)\\ &= \frac{2qt^3(1+2qt^3-q^2t^3)}{1-3q^2t^3-8q^3t^6+3q^4t^6-q^6t^9}.\end{aligned}$$

The factor of 2 in the numerator shows that all of the coefficients that appear in $N_{3n+1}^{3,1}(0,q)$ are even.

For part (2), we have already shown in Theorem 4 part (3) that for $n \geq 0$, the highest power of q which appears in $N_{3n+2}^{3,1}(0,q)$ is q^{2n} and $N_{3n+2}^{3,1}(0,q)|_{q^{2n}} = 1$. Now let

$$R^{3,1}(q,t) := \frac{(1+t)}{1-2qt^2-q^2t^3}$$

Then

$$\sum_{n\geq 1} N_{3n+2}^{3,1}(0,q)t^{3n} = \frac{1}{3}\left(R^{3,1}(q,t) + R^{3,1}(q,e^{\frac{2\pi i}{3}}t) + R^{3,1}(q,e^{\frac{4\pi i}{3}}t)\right).$$

But then one can compute that

$$\left(\sum_{n\geq 1} N^{3,1}_{3n+2}(0,q)t^{3n}\right) - \frac{1}{1-q^2t^3} = \frac{2qt^3(1+q^2t^3)^2}{1-4q^2t^3-8q^3t^6+6q^4t^6+8q^5t^9-4q^6t^9+q^8t^{12}},$$

which shows that for all $i<2n$, the coefficient of q^i in $N^{3,1}_{3n+2}(0,q)$ is even.

For part (3), note that we have already shown in Theorem 4 part (5) that for $n\geq 1$, the highest power of q which appears in $N^{3,1}_{3n}(0,q)$ is q^{2n-2} and $N^{3,1}_{3n}(0,q)|_{q^{2n-2}} = 1+(2n(n-1))$, which is clearly odd. To show that the coefficients of the remaining powers of q are even, we can proceed as follows. First let

$$\begin{aligned} P^{3,1}(q,t) &:= \sum_{n\geq 1} N^{3,1}_{3n}(0,q)t^{3n} \\ &= \frac{1}{3}\left(N^{3,1}(0,q,t)+N^{3,1}(0,q,e^{\frac{2\pi i}{3}}t)+N^{3,1}(0,q,e^{\frac{4\pi i}{3}}t)\right). \end{aligned}$$

Then let

$$\begin{aligned} P^{3,1}_1(q,t) &:= \frac{1}{t^2}P^{3,1}(q,t^{2/3}) \\ &= \sum_{n\geq 0} N^{3,1}_{3n+3}(0,q)t^{2n}. \end{aligned}$$

Since the highest power of q that appears in $N^{3,1}_{3n+3}(0,q)$ is q^{2n}, it follows that coefficient of the highest power of q in $N^{3,1}_{3n+3}(0,q)$ is the coefficient of the constant term in $q^{2n}N^{3,1}_{3n+3}(0,1/q)$ and for all $i<2n$,

$$N^{3,1}_{3n+3}(0,q)|_{q^i} = \left(q^{2n}N^{3,1}_{3n+3}(0,1/q)\right)|_{q^{2n-i}}.$$

By replacing q with $1/q$ and t with qt in $P^{3,1}_1(q,t)$, one can compute that the resulting function is

$$\begin{aligned} P^{3,1}_2(q,t) &:= \sum_{n\geq 0} q^{2n}N^{3,1}_{3n+3}(0,1/q)t^{2n} \\ &= \frac{(1+t^2)^2}{1-3t^2+(8q-3)t^4+t^6}. \end{aligned}$$

Thus, plugging $q=0$ into this rational function gives

$$P^{3,1}_2(0,t) = \frac{(1+t^2)^2}{(1-t^2)^3},$$

which is the generating function for the coefficients appearing in Theorem 4 part (5). We have that

$$P^{3,1}_2(q,t) - P^{3,1}_2(0,t) = \frac{8qt^4(1-t^2)^2}{(1-t^2)^3(1-3t^3+(3-8q)t^4-t^6)},$$

which shows that for all $i<2n$, $N^{3,1}_{3n+3}(0,q)|_{q^i}$ is divisible by 8 and, hence, is certainly even. □

We can apply similar techniques to determine the parity of convolutions of the Fibonacci numbers. That is, let

$$L^{3,1}(q,t) := N^{3,1}(p,q,t)|_p = \frac{t^4(1+tq+q(q-1)t^2)^2}{(1-2qt^2-q^2t^3)^2}$$

from (15). Then we know that

$$L^{3,1}(q,t) = \sum_{n\geq 0} L_n^{3,1}(q)t^n,$$

where

$$L_n^{3,1}(1) = \begin{cases} 0 & \text{if } n \leq 3 \text{ and} \\ \sum_{i+j=n-4} F_iF_j & \text{if } n \geq 4. \end{cases}$$

Hence $L^{3,1}(1,t)$ is the generating function of the convolution of two Fibonacci numbers. Thus we can study the parity of convolutions of two Fibonacci numbers by studying the generating function $L^{3,1}(q,t)$. First observe that

$$L^{3,1}(1,t) = \frac{t^4(1+t)}{(1-2t^2-t^3)^2} \tag{19}$$

and that

$$(1-2t^2-t^3)^2 = 1-4t^2-2t^3+4t^4+4t^5+t^6.$$

This means that for $n \geq 6$,

$$L_n^{3,1}(1) = 4L_{n-2}^{3,1}(1) + 2L_{n-3}^{3,1}(1) - 4L_{n-4}^{3,1}(1) - 4L_{n-5}^{3,1}(1) - L_{n-6}^{3,1}(1). \tag{20}$$

Thus for all $n \geq 6$, $L_n^{3,1}(1)$ and $L_{n-6}^{3,1}(1)$ have the same parity. One can compute that the initial terms of the sequence $\{L_n^{3,1}(1)\}_{n\geq 4}$ are

$$1, 2, 5, 10, 20, 38, 71, 130, 235, 420, 744, 1308, 2285, 3970, 6865, \ldots,$$

which appear in "The on-line encyclopedia of integer sequences" as sequence A001629 [7]. Notice that these are the same numbers that appear if we set $q = 1$ in the table for $N_n^{(3,1)}(p,q)|_p$ on page 182. Using these initial values and the recursion (20), one can show that the following theorem holds.

Theorem 13. *For all $n \geq 1$,*

(0) $L_{6n}^{3,1}$ is odd,	(1) $L_{6n+1}^{3,1}$ is even,	(2) $L_{6n+2}^{3,1}$ is even,
(3) $L_{6n+3}^{3,1}$ is even,	(4) $L_{6n+4}^{3,1}$ is odd, and	(5) $L_{6n+5}^{3,1}$ is even.

We also have an analogue of Theorem 12 in this case. That is, by using the same type of techniques that we used to prove Theorem 12, we can prove the following.

Theorem 14. *1. For $n \geq 1$, the highest power of q that appears in $L_{6n}^{3,1}(q)$ is $q^{2+4(n-1)}$, $L_{6n}^{3,1}(q)|_{q^{2+4(n-1)}}$ is odd, and for $i < 2+4(n-1)$, $L_{6n}^{3,1}(q)|_{q^i}$ is even.*

2. *For $n \geq 1$, the highest power of q that appears in $L_{6n+1}^{3,1}(q)$ is $q^{3+4(n-1)}$ and all the coefficients that appear in $L_{6n+1}^{3,1}(q)$ are even.*

3. *For $n \geq 1$, the highest power of q that appears in $L_{6n+2}^{3,1}(q)$ is q^{4n}, $L_{6n+2}^{3,1}(q)|_{q^{4n}}$ is odd, $L_{6n+2}^{3,1}(q)|_{q^{4n-1}}$ is even, $L_{6n+2}^{3,1}(q)|_{q^{4n-2}}$ is odd, and for $i < 4n-2$, $L_{6n+2}^{3,1}(q)|_{q^i}$ is even.*

4. *For $n \geq 1$, the highest power of q that appears in $L_{6n+3}^{3,1}(q)$ is q^{4n} and all the coefficients that appear in $L_{6n+3}^{3,1}(q)$ are even.*

5. *For $n \geq 1$, the highest power of q that appears in $L_{6n+4}^{3,1}(q)$ is q^{4n+1}, $L_{6n+4}^{3,1}(q)|_{q^{4n+1}}$ is even, $L_{6n+4}^{3,1}(q)|_{q^{4n}}$ is odd, and for $i < 4n$, $L_{6n+4}^{3,1}(q)|_{q^i}$ is even.*

6. *For $n \geq 1$, the highest power of q that appears in $L_{6n+5}^{3,1}(q)$ is q^{4n+2} and all the coefficients that appear in $L_{6n+5}^{3,1}(q)$ are even.*

One might be led to conjecture that the period for the parity of convolutions of r Fibonacci numbers is $3r$, but this is not the case. For example, let

$$J^{3,1}(q,t) := N^{3,1}(p,q,t)|_{p^2} = \frac{t^6(1+tq+q(q-1)t^2)^2(1+q^2t+2q(q-1)t^2)}{(1-2qt^2-q^2t^3)^3},$$

which comes from setting $r = 2$ in (13) from Theorem 3. Then we know that

$$J^{3,1}(q,t) = \sum_{n\geq 0} J_n^{3,1}(q)t^n,$$

where

$$J_n^{3,1}(1) = \begin{cases} 0 & \text{if } n \leq 5 \text{ and} \\ \sum_{i+j+k=n-6} F_iF_jF_k & \text{if } n \geq 6. \end{cases}$$

Hence $J^{3,1}(1,t)$ is the generating function of the convolution of three Fibonacci numbers. Thus we can study the parity of convolutions of three Fibonacci numbers by studying the generating function $J^{3,1}(q,t)$. First observe that

$$\begin{aligned} J^{3,1}(1,t) &= \frac{t^6(1+t)^3}{(1-2t^2-t^3)^3} \\ &= \frac{t^6(1+t)^3(1-2t^2-t^3)}{(1-2t^2-t^3)^4}. \end{aligned}$$

Since

$$(1-2t^2-t^3)^4 = 1-8t^2-4t^3+24t^4+24t^5-26t^6-48t^8-8t^8+28t^9+24t^{10}+8t^{11}+t^{12},$$

this implies that for $n \geq 12$,

$$\begin{aligned} J_n^{3,1}(1) = {}& 8J_{n-2}^{3,1}(1) + 4J_{n-3}^{3,1}(1) - 24J_{n-4}^{3,1}(1) - 24J_{n-5}^{3,1}(1) + 26J_{n-6}^{3,1}(1) + \\ & 48J_{n-7}^{3,1}(1) + 8J_{n-8}^{3,1}(1) - 28J_{n-9}^{3,1}(1) - 24J_{n-10}^{3,1}(1) - 8J_{n-11}^{3,1}(1) - J_{n-12}^{3,1}(1). \end{aligned}$$

Thus for all $n \geq 6$, $J_n^{3,1}(1)$ and $J_{n-12}^{3,1}(1)$ have the same parity. One can compute that the initial terms of the sequence $\{J_n^{3,1}(1)\}_{n\geq 6}$ are

$$1, 3, 9, 22, 51, 111, 233, 474, 942, 1836, 3522, 6666, 12473, 23109, 42447, 77378, 140109,$$
$$252177, 451441, \ldots,$$

which form sequence A001628 in "The on-line encyclopedia of integer sequences" [7]. With these initial terms, we can prove the following theorem.

Theorem 15. *For all $n \geq 1$,*

(0) $J_{12n}^{3,1}$ is odd,	(1) $J_{12n+1}^{3,1}$ is even,	(2) $J_{12n+2}^{3,1}$ is even,
(3) $J_{12n+3}^{3,1}$ is even,	(4) $J_{12n+4}^{3,1}$ is even,	(5) $J_{12n+5}^{3,1}$ is even,
(6) $J_{12n+6}^{3,1}$ is odd,	(7) $J_{12n+7}^{3,1}$ is odd,	(8) $J_{12n+8}^{3,1}$ is odd,
(9) $J_{12n+9}^{3,1}$ is even,	(10) $J_{12n+10}^{3,1}$ is odd, and	(11) $J_{12n+11}^{3,1}$ is odd.

Similarly, the parity patterns for convolutions of four Fibonacci numbers will also exhibit a period of size 12. That is, let

$$H^{3,1}(q,t) := N^{3,1}(p,q,t)|_{p^3} = \frac{t^8(1+qt(1-t)+q^2t^2)^2(1-2qt^2+q^2t(1+2t)^2}{(1-2qt^2+q^2t^3)^4}.$$

Then we know that

$$H^{3,1}(q,t) = \sum_{n\geq 0} H_n^{3,1}(q)t^n,$$

where

$$H_n^{3,1}(1) = \begin{cases} 0 & \text{if } n \leq 7 \text{ and} \\ \sum_{i+j+k+\ell=n-8} F_iF_jF_kF_\ell & \text{if } n \geq 8. \end{cases}$$

Hence $H^{3,1}(1,t)$ is the generating function of the convolution of four Fibonacci numbers. Thus we can study the parity of convolutions of four Fibonacci numbers by studying the generating function $H^{3,1}(q,t)$. First observe that

$$H^{3,1}(1,t) = \frac{t^8(1+t)^4}{(1-2t^2-t^3)^4}. \tag{21}$$

So we can argue as in the case of $J^{3,1}(1,t)$ that for $n > 12$, the parity of $H_n^{3,1}(1)$ and $H_{n-12}^{3,1}(1)$ are the same.

By examining the first few terms of the generating function for $H^{3,1}(1,t)$, one can show that the following theorem holds.

Theorem 16. *For all $n \geq 1$,*

(0) $H_{12n}^{3,1}$ is odd,	(1) $H_{12n+1}^{3,1}$ is even,	(2) $H_{12n+2}^{3,1}$ is even,
(3) $H_{12n+3}^{3,1}$ is even,	(4) $H_{12n+4}^{3,1}$ is even,	(5) $H_{12n+5}^{3,1}$ is even,
(6) $J_{12n+6}^{3,1}$ is even,	(7) $H_{12n+7}^{3,1}$ is even,	(8) $H_{12n+8}^{3,1}$ is odd,
(9) $H_{12n+9}^{3,1}$ is even	(10) $H_{12n+10}^{3,1}$ is even, and	(11) $H_{12n+11}^{3,1}$ is even.

It is not difficult to prove by induction that $(1-2t^2-t^3)^{2n}$ is a polynomial of degree $6n$ with constant term 1, leading coefficient 1, and all other coefficients even. This will allow one to show that patterns in the parities of convolutions of $2k-1$ Fibonacci numbers, and also $2k$ Fibonacci numbers, exhibit periods of size $6k$.

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(Received 15 Sep 2014; revised 17 Sep 2015)