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Closed-form expansions for the universal edge elimination polynomial

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Abstract

We establish closed-form expansions for the universal edge elimination polynomial of paths and cycles and their generating functions. This includes closed-form expansions for the bivariate matching polynomial, the bivariate chromatic polynomial, and the covered components polynomial.

1 Introduction

As a generalization of several well-known graph polynomials, Averbouch, Godlin and Makowsky [1] introduced the so-called *universal edge elimination polynomial* $\xi(G,x,y,z)$, whose recursive definition involves three kinds of edge elimination:

G_{-e}**:** The graph obtained from G by removing the edge e.

$G_{/e}$**:** The graph obtained from G by removing e and identifying its endpoints,

$G_{\dagger e}$**:** The graph obtained from G by removing e and all incident vertices.

All graphs are considered as finite and undirected, and may have loops and multiple edges. We use P_n to denote the simple path with n vertices ($n = 0, 1, \dots$), and $\oplus$ to denote the disjoint union of graphs. According to [1], $\xi(G,x,y,z)$ is defined by

$$\xi(P_0,x,y,z) = 1, \quad \xi(P_1,x,y,z) = x, \tag{1}$$

$$\xi(G,x,y,z) = \xi(G_{-e},x,y,z) + y\xi(G_{/e},x,y,z) + z\xi(G_{\dagger e},x,y,z), \tag{2}$$

$$\xi(G_1 \oplus G_2,x,y,z) = \xi(G_1,x,y,z)\xi(G_2,x,y,z). \tag{3}$$

The universal edge elimination polynomial $\xi(G,x,y,z)$ generalizes, among others, the bivariate matching polynomial $M(G,x,y) = \xi(G,x,0,y)$ (provided G is loop-free), the bivariate chromatic polynomial $P(G,x,y) = \xi(G,x,-1,x-y)$, and the covered components polynomial $C(G,x,y,z) = \xi(G,x,y,xyz-xy)$. The implications of our results on $\xi(G,x,y,z)$ for these polynomials are new as well. We refer to [1–4] for the definitions of the various graph polynomials and the relationships among them.

2 Closed-form expansions for paths and cycles

We use $\mathbb{N}$ to denote the set of positive integers. The following theorem provides a closed-form expansion for the universal edge elimination polynomial of a path.

Theorem 2.1. *Let $n \in \mathbb{N}$, and $x, y, z \in \mathbb{R}$. If $z > -\left(\frac{x+y}{2}\right)^2$, then*

$$\xi(P_n, x, y, z) = \frac{\sqrt{D} - x + y}{2\sqrt{D}} \left(\frac{x + y - \sqrt{D}}{2}\right)^n + \frac{\sqrt{D} + x - y}{2\sqrt{D}} \left(\frac{x + y + \sqrt{D}}{2}\right)^n \tag{4}$$

where

$$D := x^2 + 2xy + y^2 + 4z. \tag{5}$$

If $z < -\left(\frac{x+y}{2}\right)^2$, then

$$\xi(P_n, x, y, z) = (-z)^{n/2} \left(\cos(n\varphi) + \frac{x - y}{\sqrt{-D}} \sin(n\varphi)\right) \tag{6}$$

where

$$\varphi = \begin{cases} \arctan \frac{\sqrt{-D}}{x+y} & \text{if } x + y > 0, \\ \pi/2 & \text{if } x + y = 0, \\ \pi + \arctan \frac{\sqrt{-D}}{x+y} & \text{if } x + y < 0. \end{cases} \tag{7}$$

If $z = -\left(\frac{x+y}{2}\right)^2$, then

$$\xi(P_n, x, y, z) = \frac{(n+1)x - (n-1)y}{2} \left(\frac{x + y}{2}\right)^{n-1}. \tag{8}$$

Proof. By choosing e as an end edge of P_n, Eqs. (2) and (3) yield the recurrence

$$\xi(P_n, x, y, z) = (x + y)\xi(P_{n-1}, x, y, z) + z\xi(P_{n-2}, x, y, z) \quad (n \geq 2), \tag{9}$$

where the initial conditions are given by Eq. (1). This is a homogeneous linear recurrence of degree 2 with constant coefficients. We solve this recurrence by applying the method of characteristic roots. The characteristic equation of the recurrence is

$$r^2 - (x + y)r - z = 0, \tag{10}$$

with discriminant D, given by Eq. (5). In our three cases, we have $D > 0$, $D < 0$, and $D = 0$, respectively. In the first two cases, the solution to Eq. (9) is of the form

$$\xi(P_n, x, y, z) = c_1 r_1^n + c_2 r_2^n \tag{11}$$

where r_1, r_2 are the distinct roots of Eq. (10) and c_1, c_2 are chosen to satisfy Eq. (1). In the first case we have

$$\begin{aligned} r_1 &= \frac{x+y-\sqrt{D}}{2}, & c_1 &= \frac{\sqrt{D}-x+y}{2\sqrt{D}}, \\ r_2 &= \frac{x+y+\sqrt{D}}{2}, & c_2 &= \frac{\sqrt{D}+x-y}{2\sqrt{D}}, \end{aligned} \tag{12}$$

and in the second case,

$$\begin{aligned} r_1 &= \frac{x+y}{2} - \frac{\sqrt{-D}}{2}i, & c_1 &= \frac{1}{2} + \frac{x-y}{2\sqrt{-D}}i, \\ r_2 &= \frac{x+y}{2} + \frac{\sqrt{-D}}{2}i, & c_2 &= \frac{1}{2} - \frac{x-y}{2\sqrt{-D}}i. \end{aligned} \tag{13}$$

A little bit of extra work is needed in the second case in order to get rid of the imaginary parts: Representing r_1 and r_2 in polar form and applying Euler's formula we obtain

$$\begin{aligned} r_1^n &= \left(\sqrt{-z}\, e^{-i\varphi}\right)^n = (-z)^{n/2} \left(\cos(n\varphi) - \sin(n\varphi)i\right), \\ r_2^n &= \left(\sqrt{-z}\, e^{i\varphi}\right)^n \ = (-z)^{n/2} \left(\cos(n\varphi) + \sin(n\varphi)i\right), \end{aligned} \tag{14}$$

with φ as in Eq. (7). Thus, Eq. (11) becomes $\xi(P_n, x, y, z) =$

$$(-z)^{n/2} \left(\frac{1}{2}\cos(n\varphi) + \frac{x-y}{2\sqrt{-D}}\cos(n\varphi)i - \frac{1}{2}\sin(n\varphi)i + \frac{x-y}{2\sqrt{-D}}\sin(n\varphi) \right.$$
$$\left. + \frac{1}{2}\cos(n\varphi) - \frac{x-y}{2\sqrt{-D}}\cos(n\varphi)i + \frac{1}{2}\sin(n\varphi)i + \frac{x-y}{2\sqrt{-D}}\sin(n\varphi) \right).$$

This shows that the imaginary parts cancel out. This proves Eq. (6).

In the third case, the solution to Eq. (9) is $\xi(P_n, x, y, z) = (c_1 + c_2 n)r^n$ where $r = \frac{x+y}{2}$ is the unique root of Eq. (10) and $c_1, c_2 \in \mathbb{R}$ are determined by Eq. (1). If $x + y = 0$, then $\xi(P_n, x, y, z) = 0$. Thus, in this case, Eq. (8) holds. If $x + y \neq 0$, then by Eq. (1), $c_1 = 1$ and $c_2 = \frac{x-y}{x+y}$; hence,

$$\xi(P_n, x, y, z) = \left(1 + \frac{x-y}{x+y}n\right)\left(\frac{x+y}{2}\right)^n,$$

which coincides with Eq. (8). This completes the proof. □

For any $n \in \mathbb{N}$, we use C_n to denote the connected 2-regular graph with n vertices. We adopt the convention that C_0 is the empty graph. By Eq. (2) we have

$$\xi(C_1, x, y, z) = x + xy + z, \tag{15}$$
$$\xi(C_2, x, y, z) = x^2 + 2xy + 2z + xy^2 + yz. \tag{16}$$

The following theorem generalizes Eqs. (15) and (16) to cycles of any finite length.

Theorem 2.2. *Let $n \in \mathbb{N}$ and $x, y, z \in \mathbb{R}$. Let D and φ be defined as in Eq.* (5) *resp.* (7). *If $z \geq -\left(\frac{x+y}{2}\right)^2$, then*

$$\xi(C_n, x, y, z) = \left(\frac{x + y - \sqrt{D}}{2}\right)^n + \left(\frac{x + y + \sqrt{D}}{2}\right)^n + y^{n-1}(xy - y + z). \quad (17)$$

If $z \leq -\left(\frac{x+y}{2}\right)^2$, then

$$\xi(C_n, x, y, z) = 2(-z)^{n/2} \cos(n\varphi) + y^{n-1}(xy - y + z). \quad (18)$$

Proof. For $n = 1, 2$ the theorem agrees under both conditions on z with Eqs. (15) and (16). This is easy to see for $z \geq -\left(\frac{x+y}{2}\right)^2$, while for $z \leq -\left(\frac{x+y}{2}\right)^2$ the identities $\cos(\arctan(t)) = 1/\sqrt{1+t^2}$ and $\cos(\alpha) = 2\cos^2(\alpha) - 1$ reveal the coincidence.

For the rest of this proof, we assume $n \geq 3$. We may further assume that $z \neq -\left(\frac{x+y}{2}\right)^2$ as the remaining case follows for reasons of continuity by taking limits on both sides of Eqs. (17) and (18) as $z \downarrow -\left(\frac{x+y}{2}\right)^2$ resp. $z \uparrow -\left(\frac{x+y}{2}\right)^2$. By Eq. (2) we have the non-homogeneous recurrence

$$\xi(C_n, x, y, z) = \xi(P_n, x, y, z) + y\xi(C_{n-1}, x, y, z) + z\xi(P_{n-2}, x, y, z) \quad (n \geq 3)$$

with initial condition as in Eq. (16). Iterating this recurrence gives

$$\begin{aligned} \xi(C_n, x, y, z) &= \sum_{j=0}^{n-2} y^j \Big(\xi(P_{n-j}, x, y, z) + z\xi(P_{n-j-2}, x, y, z)\Big) + y^{n-1}(x + xy + z) \\ &= \xi(P_n, x, y, z) + y\xi(P_{n-1}, x, y, z) + (y^2 + z) \sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) \\ &\qquad + y^{n-3}\left(xz + yz + xy^2 + xy^3 + y^2 z\right). \quad (19) \end{aligned}$$

Using Eq. (11) with c_1, r_1, c_2, r_2 from Eqs. (12) and (13) in the preceding proof, the sum on the right-hand side of Eq. (19) can be written as

$$\begin{aligned} \sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) &= \sum_{j=0}^{n-4} y^j \left(c_1 r_1^{n-j-2} + c_2 r_2^{n-j-2}\right) \\ &= c_1 r_1^{n-2} \sum_{j=0}^{n-4} \left(\frac{y}{r_1}\right)^j + c_2 r_2^{n-2} \sum_{j=0}^{n-4} \left(\frac{y}{r_2}\right)^j. \end{aligned}$$

If $z \neq -xy$, then $y \neq r_1$ and $y \neq r_2$. In this case, by applying the formula for finite geometric series the preceding equation simplifies to

$$\sum_{j=0}^{n-4} y^j \xi(P_{n-j-2}, x, y, z) = c_1 r_1^2 \frac{r_1^{n-3} - y^{n-3}}{r_1 - y} + c_2 r_2^2 \frac{r_2^{n-3} - y^{n-3}}{r_2 - y}.$$

Substituting this latter expression into Eq. (19) and taking into account that r_1 and r_2 are given as in Eqs. (12) and (13) leads to

$$\begin{aligned}\xi(C_n, x, y, z) &= c_1 r_1^n + c_2 r_2^n + y(c_1 r_1^{n-1} + c_2 r_2^{n-1}) \\ &\quad + (y^2+z)\left(c_1 r_1^2 \frac{r_1^{n-3} - y^{n-3}}{r_1 - y} + c_2 r_2^2 \frac{r_2^{n-3} - y^{n-3}}{r_2 - y}\right) \\ &\quad + y^{n-3}\left(xz + yz + xy^2 + xy^3 + y^2 z\right) \\ &= r_1^n + r_2^n + y^{n-1}(xy - y + z)\,, \qquad (20)\end{aligned}$$

where the last equality follows by substituting $c_1 = -\frac{r_1 - y}{\sqrt{D}}$, $c_2 = \frac{r_2 - y}{\sqrt{D}}$, $\sqrt{D} = -\frac{r_1^2 - r_2^2}{x+y}$, and rearranging and cancelling terms (note that $\sqrt{D} = i\sqrt{-D}$ if $D < 0$). Now, for $z > -\left(\frac{x+y}{2}\right)^2$ Eq. (17) follows from Eqs. (20) and (12), whereas for $z < -\left(\frac{x+y}{2}\right)^2$ Eq. (18) follows from Eqs. (20) and (14) after cancelling out the imaginary parts, in analogy to the proof of Theorem 2.1.

If $z = -xy$, then $z > -\left(\frac{x+y}{2}\right)^2$. In this remaining case, the result follows for reasons of continuity by taking limits on both sides of Eq. (17) as $z \downarrow -xy$. □

Remark 2.3. For $z = -\left(\frac{x+y}{2}\right)^2$, Eqs. (17) and (18) coincide. In this case,

$$\xi(C_n, x, y, z) = 2\left(\frac{x+y}{2}\right)^n - \frac{x^2 - 2xy + y^2 + 4y}{4}\, y^{n-1}.$$

Alternatively, this can be shown by combining Eqs. (8) and (19) and applying the formula for finite geometric series.

Remark 2.4. The preceding closed-form expansions can also be proved by induction. A computer algebra system might be helpful. In Sage [5], for instance, the following lines of code prove Eqs. (4) and (17) by induction on the number of vertices.

```
var("n x y z")
D = x^2+2*x*y+y^2+4*z
path = (sqrt(D)-x+y)/(2*sqrt(D))*((x+y-sqrt(D))/2)^n \
      +(sqrt(D)+x-y)/(2*sqrt(D))*((x+y+sqrt(D))/2)^n
cycle = ((x+y-sqrt(D))/2)^n+((x+y+sqrt(D))/2)^n+y^(n-1)*(x*y-y+z)
bool(path(n=0)==1 and path(n=1)==x \
      and (x+y)*path(n=n-1)+z*path(n=n-2)==path)
bool(cycle(n=1)==x+x*y+z and path+y*cycle(n=n-1)+z*path(n=n-2)==cycle)
```

We proceed with a corollary on the generating function of $\xi(G, x, y, z)$.

Corollary 2.5.

$$\sum_{n=0}^{\infty} \xi(P_n, x, y, z) t^n = \frac{1 - yt}{1 - (x+y)t - zt^2}\,,$$

$$\sum_{n=0}^{\infty} \xi(C_n, x, y, z) t^n = \frac{1 + zt^2}{1 - (x+y)t - zt^2} + \frac{(xy - y + z)t}{1 - yt}\,.$$

Proof. Corollary 2.5 is an immediate consequence of Theorem 2.1, Theorem 2.2 and the geometric series formula. □

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