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A generalized class of modified R-polynomials*

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Abstract

In this paper we define a family of polynomials closely related to the modified R-polynomials of the symmetric group and begin work toward a classification of the polynomials by using a combinatorial interpretation involving subwords of the maximal element in the Bruhat order. The problem of determining the precise conditions which make one of these polynomials zero motivates our work. We state several properties of these polynomials and symmetries which they satisfy that were discovered while pursuing a resolution to this problem.

1 Introduction

The Iwahori-Hecke algebra, $H_n(q)$, is a single parameter deformation of the group algebra $\mathbb{C}[\mathfrak{S}_n]$, where $\mathfrak{S}_n$ is the *symmetric group*. In particular, $H_n(q)$ specializes to $\mathbb{C}[\mathfrak{S}_n]$ when $q = 1$. While working on representations of $H_n(q)$, Kazhdan and Lusztig [12] introduced a family of polynomials which are now known as *Kazhdan-Lusztig polynomials*. These polynomials were used to construct a basis and irreducible modules for $H_n(q)$. In order to define their polynomials, Kazhdan and Lusztig introduced a set of polynomials in $\mathbb{Z}[q]$ known as the *R-polynomials*. Due to the recursive way in which all of these polynomials are defined, they are difficult to work with. An alternative way of working with them is to make use of *modified R-polynomials* in $\mathbb{N}[q]$. The coefficients of the modified R-polynomials and their combinatorial interpretations have been studied by Brenti [2–6], Deodhar [7], and Dyer [9].

The construction of representations of $H_n(q)$ was the underlying reason for the introduction of the Kazhdan-Lusztig and R-polynomials. In the same manner study of the representation theory of quantum groups by Lusztig [14] and Kashiwara [11] eventually lead to interest in the quantum matrix bialgebra $\mathcal{A}(n;q)$, a single parameter deformation of the commutative polynomial ring $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$. In order

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to construct transition matrices between special bases in $\mathcal{A}(n;q)$, a set of polynomials appear which satisfy recursive relations very similar to those of the modified R-polynomials. In Lambright's dissertation [13], a family of polynomials was defined combinatorially and shown not only to be the polynomials appearing in the transition matrices, but also containing the modified R-polynomials as a subfamily.

In [13], several features of these polynomials were identified, including several symmetries they satisfy. In this paper we will define these polynomials recursively in an analogous manner to Kazhdan and Lusztig's definition. Following this we interpret the polynomials combinatorially and take advantage of facts from [13] to identify more features and properties which they satisfy. Next we will state results that we have found in pursuing a classification of these polynomials.

2 Relevant Background Information and Notation

The symmetric group $\mathfrak{S}_n$ has a standard presentation given by the generators $s_1, \ldots, s_{n-1}$ and the relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{2.1}$$

Let $[n]$ denote the set $\{1, \ldots, n\}$. Let $\mathfrak{S}_n$ act on rearrangements of the letters $[n]$ by

$$s_i \circ v_1 \cdots v_n = v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n. \tag{2.2}$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n = s_{i_1} \circ (\cdots (s_{i_\ell} \circ 1 \cdots n)) \cdots). \tag{2.3}$$

The one-line notation does not depend on the expression $s_{i_1} \cdots s_{i_\ell}$ for w. When an expression for w is as short as possible, we say that expression is *reduced*. Furthermore, we call the number of generators in a reduced expression for w the *length* of w and denote it by $\ell(w)$.

The *Bruhat order* on $\mathfrak{S}_n$ is defined by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword (See [1] for more information). A generator is called a *left ascent* for v if $sv > v$, and a *left descent* otherwise. Right ascents and descents are defined analogously. The unique maximal element in the Bruhat order will be denoted by w_0. This permutation has one line notation $n(n-1) \cdots 2\,1$.

It is known that left or right multiplication by w_0 induces an antiautomorphism of the Bruhat order. Thus if $u < v$, then we have

$$w_0 v < w_0 u, \qquad \text{and} \qquad v w_0 < u w_0. \tag{2.4}$$

We also have that

$$\ell(w_0 v) = \ell(v w_0) = \ell(w_0) - \ell(v), \tag{2.5}$$

for all $v \in \mathfrak{S}_n$.

We also consider the *reflections* of $\mathfrak{S}_n$, a set defined by

$$T = \{wsw^{-1} | w \in \mathfrak{S}_n, s \text{ a generator of } \mathfrak{S}_n\}. \tag{2.6}$$

The *Bruhat graph* of S_n is the directed graph with vertex set S_n and edge connecting u to v if and only if $v = ut$ with $t \in T$ and $\ell(u) < \ell(v)$.

The *absolute length* of u, denoted by $a\ell(u)$ is the minimum number of reflections used to express u. Following [10], the absolute length of the pair (u, v), denoted $a\ell(u, v)$, is the (oriented) distance between u and v in the Bruhat graph of $\mathfrak{S}_n$.

We wish to have a way to indicate particular subwords that are related to u, v, and $w \in \mathfrak{S}_n$. We fix a reduced expression for the word $w = w_1 w_2 \cdots w_k$ of length k. Following the notation and terminology of [8] a *mask* σ associated to w is a binary word $\sigma_1\sigma_2\cdots\sigma_k$. Every mask corresponds to a subexpression of w, denoted $w^\sigma = w_1^{\sigma_1} w_2^{\sigma_2} \cdots w_k^{\sigma_k}$ where

$$w_i^{\sigma_i} = \begin{cases} w_i & \text{if } \sigma_i = 1, \\ e & \text{if } \sigma_i = 0. \end{cases} \tag{2.7}$$

We consider a sequence of masks related to σ where $\sigma[j] = \sigma_1\sigma_2\cdots\sigma_j$. These masks in turn define a sequence of subwords of w where $w^{\sigma[j]} = w_1^{\sigma_1} w_2^{\sigma_2} \cdots w_j^{\sigma_j}$. Thus, for example, $w^{\sigma[1]} = w_1^{\sigma_1}$, $w^{\sigma[2]} = w_1^{\sigma_1} w_2^{\sigma_2}$, and $w^{\sigma[k]} = w^\sigma$.

We will say that the mask σ is *appropriate* for the elements $u, v, w \in \mathfrak{S}_n$ if and only if

1. $u^{-1}v = w^\sigma$.
2. If $uw^{\sigma[j-1]}w_j > uw^{\sigma[j-1]}$ then $\sigma_j = 1$.

For further notation, we will let $S_{u,v,w}$ be the set of all masks that are appropriate for the triple u, v, w and σ^{-1} denote the reverse ordering of σ. For example, $10011^{-1} = 11001$. We also define the function $z : S_{u,v,w} \to \mathbb{N}$ where $z(\sigma)$ is the number of zeros in the mask σ.

3 The Polynomials of Interest

In [12] Kazhdan and Lusztig defined a family of polynomials with integer coefficients associated to a Coxeter group, of which $\mathfrak{S}_n$ is one. They also defined a second family of polynomials which came to be known as the R-polynomials. These led to the definition of the modified R-polynomials, which are polynomials with non-negative integer coefficients. For a nice introduction to these polynomials see Chapter 5 of [1]. The modified R-polynomials are the unique family $\{\widetilde{R}_{v,w}(q) \mid v, w \in \mathfrak{S}_n\}$ of polynomials in $\mathbb{N}[q]$ satisfying

1. $\widetilde{R}_{v,w}(q) = 0$ if $v \not\leq w$.
2. $\widetilde{R}_{w,w}(q) = 1$ for all w.

3. For each left ascent s of v

$$\widetilde{R}_{v,w}(q) = \begin{cases} \widetilde{R}_{sv,sw}(q) & \text{if } sw > w, \\ \widetilde{R}_{sv,sw}(q) + q\widetilde{R}_{sv,w}(q) & \text{otherwise.} \end{cases} \tag{3.1}$$

The polynomials which are the focus of this paper were initially defined using conditions that resemble those satisfied by the modified R-polynomials.

Definiton 3.1. *For $u, v, w \in \mathfrak{S}_n$ define the polynomial $p_{u,v,w}(q) \in \mathbb{N}[q]$ by*

1. $p_{u,v,w}(q) = 0$ *if* $u^{-1}v \not\leq w$.
2. $p_{u,v,u^{-1}v}(q) = 1$.
3. *For each right descent s of u,*

$$p_{u,v,w}(q) = \begin{cases} p_{us,v,sw}(q) & \text{if } sw > w \\ p_{us,v,sw}(q) + qp_{us,v,w}(q) & \text{otherwise.} \end{cases} \tag{3.2}$$

In what follows we will refer to these polynomials as the p-polynomials. These polynomials appear in [13] in a different guise, as they were defined combinatorially by counting certain paths in the diagram of the Bruhat order associated with $\mathfrak{S}_n$. We will use the following alternative combinatorial definition for what we will initally refer to as the $\tilde{p}$-polynomials.

Definiton 3.2. *For $u, v, w \in \mathfrak{S}_n$ define the polynomial $\tilde{p}_{u,v,w}(q) \in \mathbb{N}[q]$:*

$$\tilde{p}_{u,v,w}(q) = \sum_{\sigma \in S_{u,v,w}} q^{z(\sigma)}. \tag{3.3}$$

This definition of the masks $S_{u,v,w}$ depends upon the reduced expression chosen for w, however, as we will see that $\tilde{p}_{u,v,w}(q) = p_{u,v,w}(q)$, the $\tilde{p}$-polynomials are well defined. In order to justify the identification between the p-polynomials and the $\tilde{p}$-polynomials there are a few straightfoward implications that follow from Definition 3.2.

Proposition 3.3. *If $u^{-1}v \not\leq w$ then $\tilde{p}_{u,v,w}(q) = 0$.*

Proof. If $u^{-1}v \not\leq w$ then there is no appropriate mask for the triple u, v, w, hence $\tilde{p}_{u,v,w}(q) = 0$. □

Proposition 3.4. *$\tilde{p}_{u,v,w}(q) = 1$ if and only if $u^{-1}v = w$.*

Proof. This follows from the fact that $u^{-1}v = w$ if and only if the only appropriate mask for u, v, w is that containing all ones and no zeros. □

Proposition 3.5. *For all $u, v, w \in \mathfrak{S}_n$ with s a right descent of u*

$$\tilde{p}_{u,v,w}(q) = \begin{cases} \tilde{p}_{us,v,sw}(q) & \text{if } sw > w, \\ \tilde{p}_{us,v,sw}(q) + q\tilde{p}_{us,v,w}(q) & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. Let s be a right descent of u. If $\sigma \in S_{u,v,w}$ then $u^{-1}v = w^\sigma$ and $su^{-1}v = sw^\sigma$.

First suppose that $sw > w$. We will show that $\tau = 1\sigma \in S_{us,v,sw}$. It is clear that $su^{-1}v = (sw)^\tau$. Because $(us)s > us$ it follows that all appropriate masks for us, v, sw must begin with 1. Since $z(\sigma) = z(\tau)$, $\tilde{p}_{u,v,w}(q) = \tilde{p}_{us,v,sw}(q)$.

Now suppose that $sw < w$. We may express $w = sw'$. There are two cases to consider.

If $\sigma_1 = 1$ then a modification of the above argument shows that $\sigma \in S_{us,v,sw}$.

If $\sigma_1 = 0$ then $su^{-1}v = s(sw')^\sigma = (sw')^{1\sigma_2\cdots\sigma_k} = w^\kappa$. Here $\kappa \in S_{us,v,w}$ and we note that $z(\kappa) = z(\sigma) - 1$. Hence $\tilde{p}_{u,v,w}(q) = \tilde{p}_{us,v,sw}(q) + q\tilde{p}_{us,v,w}(q)$.

□

Now we provide a connection between these two families of polynomials. Due to the identification in Theorem 3.6, in all that follows we refer to the $\tilde{p}$-polynomials as p-polynomials.

Theorem 3.6. *For all $u, v, w \in \mathfrak{S}_n$, $\tilde{p}_{u,v,w}(q) = p_{u,v,w}(q)$.*

Proof. The identity follows by induction on the length of u by 3.2, Proposition 3.3, Proposition 3.4, and Proposition 3.5.

□

In [13] the p-polynomials are shown to satisfy the following symmetry relations,

$$p_{u,v,w}(q) = p_{u^{-1},w,v}(q) = p_{v,u,w^{-1}}(q) = p_{v^{-1},w^{-1},u}(q) = p_{w,u^{-1},v^{-1}}(q) = p_{w^{-1},v^{-1},u^{-1}}(q). \tag{3.5}$$

This string of symmetries is a combination of two symmetries: $p_{u,v,w}(q) = p_{u^{-1},w,v}(q)$ and $p_{u,v,w}(q) = p_{v,u,w^{-1}}$. In [13] the first is established through a combinatorial proof, however the proof of the second is rooted in the algebraic properties of the Hecke algebra. Here we will show that both of these symmetries can be proved using the combinatorial setting of Definition 3.2.

Proposition 3.7. *For all $u, v, w \in \mathfrak{S}_n$, $p_{u,v,w}(q) = p_{v,u,w^{-1}}(q)$.*

Proof. Let σ be an appropriate mask of $u, v, w \in \mathfrak{S}_n$. Since $u^{-1}v = w^\sigma$ if and only if $v^{-1}u = (w^{-1})^{\sigma^{-1}}$ and $z(\sigma) = z(\sigma^{-1})$, we need only show that $\sigma^{-1} \in S_{v,u,w^{-1}}$.

For simplicity of notation, let $y = w^{-1}$ and $\tau = \sigma^{-1}$. For w of length k we have that $y_j = w_{k+1-j}$.

Suppose that $vy^{\tau[j-1]}y_j > vy^{\tau[j-1]}$. Since $uw^\sigma = v$ we have

$$(uw^\sigma)(y^{\tau[j-1]}y_j) > (uw^\sigma)(y^{\tau[j-1]}). \tag{3.6}$$

By the fact that $w^\sigma(w^{-1})^{\sigma^{-1}[j-1]} = w^{\sigma[k+1-j]}$ we see that

$$uw^{\sigma[k-j]}w_{k+1-j}^{\sigma_{k+1-j}}w_{k+1-j} > uw^{\sigma[k-j]}w_{k+1-j}^{\sigma_{k+1-j}}. \tag{3.7}$$

Suppose by way of contradiction that $\tau_j = 0$. Thus $\sigma_{k+1-j} = 0$ and (3.7) states

$$uw^{\sigma[k-j]}w_{k+1-j} > uw^{\sigma[k-j]}. \tag{3.8}$$

Since $u, v, w \in S_{u,v,w}$ it follows that $\sigma_{k+1-j} = 1$, which is a contradiction. Hence $\sigma^{-1} \in S_{v,u,w^{-1}}$. □

The other equality used to generate the symmetries in (3.5) is given by the following.

Proposition 3.8. *For all* $u, v, w \in \mathfrak{S}_n$, $p_{u,v,w}(q) = p_{v^{-1},w^{-1},u}(q)$.

Proof. The strategy of proof is similar to the above identification of $\tilde{p}$-polynomials with the p-polynomials. We demonstrate that $\tilde{p}_{v^{-1},w^{-1},u}(q)$ satisfies the three conditions of the p-polynomials in Defintion 3.1.

1. If $u^{-1}v \not\leq w$ and we suppose that there is a mask $\sigma \in S_{v^{-1},w^{-1},u}$ then $vw^{-1} = u^\sigma$. This relation implies that $u^{-1}v$ can be compared to w in the Bruhat order, and thus $u^{-1}v > w$. However, by condition 2 of appropriate masks, if $v^{-1}u^{\sigma[j-1]}u_j > v^{-1}u^{\sigma[j-1]}$ then $\sigma_j = 1$, there can be no such mask σ. This implies that $\tilde{p}_{v^{-1},w^{-1},u}(q) = 0$.

2. If $u^{-1}v = w$ then it is straightforward to show that $\tilde{p}_{v^{-1},w^{-1},u}(q) = 1$.

3. An argument similar to Proposition 3.5 shows that $\tilde{p}_{v^{-1},w^{-1},u}(q)$ satisfies the recursive condition. Let s be a right descent of v^{-1}. If $\sigma \in S_{v^{-1},w^{-1},u}$ then $vw^{-1} = u^\sigma$ and $svw^{-1} = su^\sigma$. First suppose that $su > u$. We will show that $\tau = 1\sigma \in S_{v^{-1}s,w^{-1},su}$. It is clear that $svw^{-1} = (su)^\tau$. Because $(v^{-1}s)s > v^{-1}s$ it follows that all appropriate masks for $v^{-1}s, w, su$ must begin with 1. Since $z(\sigma) = z(\tau)$, $\tilde{p}_{v^{-1},w^{-1},u}(q) = \tilde{p}_{v^{-1}s,w^{-1},su}(q)$.

   Now suppose that $su < u$. We may express $u = su'$. There are two cases to consider.

   If $\sigma_1 = 1$ then a modification of the above argument shows that $\sigma \in S_{v^{-1}s,w^{-1},su}$.

   If $\sigma_1 = 0$ then $svw^{-1} = s(su')^\sigma = (su')^{1\sigma_2\cdots\sigma_k} = u^{1\sigma_2\cdots\sigma_k}$. Here $1\sigma_2\cdots\sigma_k \in S_{v^{-1}s,w^{-1},u}$ and we note that $z(1\sigma_2\cdots\sigma_k) = z(\sigma) - 1$. Hence $\tilde{p}_{v^{-1},w^{-1},u}(q) = \tilde{p}_{v^{-1}s,w^{-1},su}(q) + q\tilde{p}_{v^{-1}s,w^{-1},u}(q)$.

The identification of $\tilde{p}_{v^{-1},w^{-1},u}(q) = p_{u,v,w}(q)$ follows by the above and induction on the length of u. □

The modified R-polynomials are in fact special cases of the p-polynomials. For $v, w \in \mathfrak{S}_n$,

$$\widetilde{R}_{v,w}(q) = p_{w_0,w_0v,w}(q). \tag{3.9}$$

Using the symmetries in (3.5) we can see that for the polynomials in (3.9) the recursive definitions for the p-polynomials and modified R-polynomials are in fact equivalent.

Many facts about the modified R-polynomials are known due to work of Brenti, Deodhar, and others. We are interested in seeing what of these facts extend to the p-polynomials, or whether we can state any analogous properties which specialize to those known about the modified R-polynomials. For example, it is known that

$$\widetilde{R}_{v,w}(q) = \widetilde{R}_{w_0w,w_0v}(q) = \widetilde{R}_{ww_0,vw_0}(q). \tag{3.10}$$

The first equality in (3.10) can be explained using (3.9) and (3.5). The following proposition, will allow us to explain the second equality in (3.10).

Proposition 3.9. *For any* $u, v, w \in \mathfrak{S}_n$,

$$p_{u,v,w}(q) = p_{w_0uw_0,w_0vw_0,w_0ww_0}(q). \tag{3.11}$$

Proof. This theorem is a consequence of the fact that conjugation of a word by w_0 is an automorphism of the Bruhat order. Let $\sigma \in S_{u,v,w}$. Since in $\mathfrak{S}_n$ we have $w_0s_iw_0 = s_{n-1}$ we calculate all conjugations by w_0 by mapping the word $w_1w_2 \cdots w_k$ to $(w_0w_1w_0)(w_0w_2w_0) \cdots (w_0w_kw_0)$. If $u^{-1}v = w^\sigma$ then $w_0u^{-1}vw_0 = (w_0ww_0)^\sigma$

Suppose that $(w_0uw_0)(w_0ww_0)^{\sigma[j-1]}(w_0w_jw_0) > (w_0uw_0)(w_0ww_0)^{\sigma[j-1]}$. For all generators, we switch s_i to s_{n-i} and see that since $uw^{\sigma[j-1]}w_j > uw^{\sigma[j-1]}$ we have $\sigma_j = 1$. Therefore $\sigma \in S_{w_0uw_0,w_0vw_0,w_0ww_0}$.

□

Now, we can justify the second equality in (3.10) using the first equality in (3.10), the identity (3.9), and Proposition 3.9 to say,

$$\widetilde{R}_{v,w}(q) = \widetilde{R}_{w_0w,w_0v}(q) = p_{w_0,w,w_0v}(q) = p_{w_0,w_0ww_0,vw_0}(q) = \widetilde{R}_{ww_0,vw_0}(q). \tag{3.12}$$

4 Work toward a classification of p-polynomials

The p-polynomials are a generalization of the modified R-polynomials. Some of the features of the modified R-polynomials generalize to the p-polynomials, while other facts seems to be specific to the modified R-polynomials. A classification of the p-polynomials would help us to understand this phenomenon.

Computation of examples of p-polynomials quickly becomes tedious and tiresome. Although Proposition 3.5 provides a means to calculate any p-polynomial recursively, this strategy leaves a lot to be desired. With only a few sample calculations patterns begin to emerge. There appears to be a specific structure to the p-polynomials, and they are quite rigid. For a specific instance of this, despite there being $(3!)^3 = 216$ ordered triples of permutations in $\mathfrak{S}_3$ there are only five distinct p-polynomials that arise from these elements.

Several questions surface. How many unique polynomials exist for $\mathfrak{S}_n$? Which polynomials are p-polynomials for some $u, v, w \in \mathfrak{S}_n$ for some n? The next two sections contain results working toward answering questions of this nature.

Proposition 3.4 provides a necessary and sufficient condition for when a p-polynomial is 1. So we turn our attention to classifying exactly when $p_{u,v,w}(q) = 0$. In this case, the converse of the statement in the definition is not true, since the polynomial $p_{s_2,s_2s_1,s_1s_2}(q) = 0$, and $u^{-1}v < w$. However, if we use the symmetries in (3.5) along with Proposition 3.3 we can explain why this specific example is zero.

Corollary 4.1. *If one of the following conditions holds:* $u^{-1}v \not\leq w$, $uw \not\leq v$, *or* $vw^{-1} \not\leq u$, *then* $p_{u,v,w}(q) = 0$.

While Corollary 4.1 allows us to conclude $p_{s_2,s_2s_1,s_1s_2}(q) = 0$, since $uw \not\leq v$, it turns out that once $n > 3$ for $\mathfrak{S}_n$ we begin to see p-polynomials that are zero, but where none of the conditions in this corollary hold. For example, $p_{s_1s_2s_3s_2s_1,s_1s_2s_3s_2s_1,s_1s_2s_3}(q)$

$= 0$ even though $u^{-1}v < w$, $uw < v$, and $vw^{-1} < u$ all are satified. These *mystery zeros* show that Corollary 4.1 is not the whole story. A necessary and sufficient condition for when the p-polynomial $p_{u,v,w}(q)$ vanishes is that the triple u, v, w has no appropriate mask. We see from Definition 3.2 that the triple u, v, w fails to have an appropriate mask if there is no σ such that $u^{-1}v = w^\sigma$, or if for every σ such that $u^{-1}v = w^\sigma$, there is a position j such that $uw^{\sigma[j-1]}w_j > uw^{\sigma[j-1]}$ and $\sigma_j = 0$.

Also of interest is the degree of a p-polynomial.

Proposition 4.2. *The degree of the p-polynomial $p_{u,v,w}(q)$ is bounded above by min$\{\ell(w) - \ell(u^{-1}v), \ell(v) - \ell(uw), \ell(u) - \ell(vw^{-1})\}$.*

Proof. The degree of the p-polynomial $p_{u,v,w}(q)$ is the maximal number of zeros in a mask σ such that $u^{-1}v = w^\sigma$. Thus deg $p_{u,v,w}(q) \leq \ell(w) - \ell(u^{-1}v)$. By (3.5), the degree of $p_{u,v,w}(q)$ is also bounded above by $\ell(v) - \ell(uw)$ and $\ell(u) - \ell(vw^{-1})$, from which the statement of the theorem follows.

□

While not as sharp an upper bound, the following corollary is helpful in determining degree as it does not require any composition of symmetric group elements.

Corollary 4.3. *The degree of the p-polynomial $p_{u,v,w}(q)$ is bounded above by* $\min\{\ell(w) - |\ell(u) - \ell(v)|, \ell(v) - |\ell(u) - \ell(w)|, \ell(u) - |\ell(v) - \ell(w)|\}$.

Proof. This follows from the fact that $|\ell(u) - \ell(v)| \leq \ell(u^{-1}v) \leq \ell(u) + \ell(v)$. □

Proposition 4.4. *For the p-polynomial $p_{u,v,w}(q) \neq 0$ the minimal power of q with a nonzero coefficient is bounded below by max $\{a\ell(u^{-1}v, w), a\ell(uw, v), a\ell(vw^{-1}, u)\}$.*

Proof. Denote the minimal power of q with a nonzero coefficient by a. The proof is by induction on the length of u. If $\ell(u) = 0$ then in order for $p_{u,v,w}(q) \neq 0$ we must have $v = w$. Thus $p_{e,v,v}(q) = 1$ and $a\ell(ev, v) = 0$.

Now suppose by induction that the statement of the proposition holds for $\ell(u) < m$. Let u be such that $\ell(u) = m$. By Proposition 3.5 if there is a generator s with $us < u$ and $sw > w$, $p_{u,v,w}(q) = p_{us,v,sw}(q)$. Thus $a \geq a\ell((us)(sw), v) = a\ell(uw, v)$.

Otherwise, for all s with $us < u$ we have $sw < w$. Thus $p_{u,v,w}(q) = p_{us,v,sw}(q) + qp_{us,v,w}(q)$. Thus $a \geq$ min $\{a\ell((us)(sw), v), a\ell(usw, v) + 1\}$. Since $a\ell((us)(sw), v) \leq a\ell(usw, v)$ we have $a \geq a\ell((us)(sw), v) = a\ell(uw, v)$.

By (3.5) $a \geq a\ell(u^{-1}v, w)$ and $a \geq a\ell(vw^{-1}, u)$, hence the statement of the proposition follows.

□

On many occasions the degree of a p-polynomials equals the upper bound from Proposition 4.2, however there are examples where this is not true. In $\mathfrak{S}_4$, the polynomial $p_{s_2w_0,s_2w_0,s_2w_0}(q) = 2q^3 + q$. Here the upper bound from Proposition 4.2 is 5 while the degree is only 3. Although we may not be able to determine the degree of an arbitrary p-polynomial, we can determine its parity.

Proposition 4.5. *If $p_{u,v,w}(q) \neq 0$, then all terms with nonzero coefficents have exponents with the same parity as $\ell(u) + \ell(v) + \ell(w)$.*

Proof. Suppose by way of contradiction that there exists a p-polynomial $p_{u,v,w}(q)$ with terms q^{2m+1} and q^{2n}, with $m, n \in \mathbb{N}$, that have nonzero coefficients. This implies that there are masks $\sigma, \tau \in S_{u,v,w}$ with $z(\sigma) = 2m+1$ and $z(\tau) = 2n$. This means that $u^{-1}v = w^\sigma = w^\tau$. Thus $e = w^\sigma (w^{-1})^{\tau^{-1}}$.

We have obtained an unreduced expression for e that has length $\ell(w) - (2m + 1) + \ell(w) - 2n = 2\ell(w) - 2m - 2n - 1$. However, since this is an odd number, we have a contradiction.

If $p_{u,v,w}(q) \neq 0$ then there is at least one appropriate mask $\sigma \in S_{u,v,w}$.

$$\begin{aligned} z(\sigma) &\equiv \ell(w) - \ell(u^{-1}v) \bmod 2 \\ &\equiv \ell(w) - |\ell(u^{-1}) - \ell(v)| \bmod 2 \\ &\equiv \ell(w) + \ell(u) + \ell(v) \bmod 2. \end{aligned} \tag{4.1}$$

□

Corollary 4.6. *If $p_{u,v,w}(q) \neq 0$, then the degree of $p_{u,v,w}(q)$ has the same parity as $\ell(u) + \ell(v) + \ell(w)$.*

5 Classification for Low Degrees

Further classification is possible for p-polynomials of low degree. In this section we make use of the notation $[q^i](p(q))$, indicating the coefficient of the term q^i in the polynomial $p(q)$.

Proposition 5.1. *For any p-polynomial with $p_{u,v,w}(q) \neq 1$, it has constant term of zero.*

Proof. The constant term of the polynomial $p_{u,v,w}(q)$ comes from appropriate masks $\sigma \in S_{u,v,w}$ with no zeros. The only mask of this type is that containing all ones. If $\sigma = 111 \cdots 1 \in S_{u,v,w}$ then $u^{-1}v = w^\sigma = w$, and $p_{u,v,w}(q) = 1$. Thus if $u^{-1}v \neq w$ then the mask $\sigma = 111 \cdots 1 \notin S_{u,v,w}$ and the constant term of $p_{u,v,w}(q)$ is zero. □

Corollary 5.2. *If deg $p_{u,v,w}(q) = 0$, then $p_{u,v,w}(q) = 1$.*

Proposition 5.3. *For any p-polynomial $p_{u,v,w}(q)$, $[q](p_{u,v,w}(q)) \in \{0, 1\}$.*

Proof. Suppose by way of contradiction that the coefficient of q is greater than 1. Thus there are two distinct masks $\sigma, \tau \in S_{u,v,w}$ such that $z(\sigma) = z(\tau) = 1$. Let $\sigma_i = 0$ and $\tau_j = 0$. By supposition $i \neq j$. Write a reduced word for w as $w = w' w_i w'' w_j w'''$ where w', w'', w''' are reduced words. Since $w^\sigma = w^\tau$ we see

$$w^\sigma = w' w'' w_j w''' = w' w_i w'' w''' = w^\tau.$$

It follows that $w'' w_j = w_i w''$, hence $w = w' w_i (w'' w_j) w''' = w' w_i (w_i w'') w''' = w' w'' w'''$ a word with length shorter than an already reduced word. This contradiction implies that there can be at most one mask $\sigma \in S_{u,v,w}$ such that $z(\sigma) = 1$. □

Corollary 5.4. *If deg $p_{u,v,w}(q) = 1$, then $p_{u,v,w}(q) = q$.*

Statements similar to Proposition 5.3 and Corollary 5.4 are true for p-polynomials of degree 2.

Proposition 5.5. *For any p-polynomial $p_{u,v,w}(q)$, $[q^2](p_{u,v,w}(q)) \in \{0,1\}$.*

Proof. The proof is by induction on the length of u. If $\ell(u) < 2$ then $[q^2](p_{u,v,w}(q)) = 0$. We consider $\ell(u) = 2$ and suppose $[q^2](p_{u,v,w}(q)) \neq 0$.

Let $u = s_i s_j$. Since there is an acceptable mask σ such that $vw^{-1} = u^\sigma$ with $z(\sigma) = 2$ we have $vw^{-1} = e$ and thus $v = w$. Furthermore since the p-polynomial is nonzero, $u^{-1}v < w$ and so $s_j s_i v < v$, implying that $s_i v < v$. Similarly because $uw < v$ we know that $s_i s_j v < v$ and thus $s_j v < v$. We use Proposition 3.5 to calculate $p_{s_i s_j,v,v}(q)$:

$$\begin{aligned} p_{s_i s_j,v,v}(q) &= p_{s_i,v,s_j v}(q) \quad + q p_{s_i,v,v}(q) \\ &= \quad q p_{e,v,s_i v}(q) \quad + q^2 p_{e,v,v}(q) \\ &= \quad q^2. \end{aligned} \tag{5.1}$$

Now suppose by induction that the statement is true for all p-polynomials with $\ell(u) < m$. Let $\ell(u) = m$. If there is a s such that $us < u$ and $sw > w$ then $p_{u,v,w}(q) = p_{us,v,sw}(q)$, and by the inductive hypothesis we are done.

If for all s such that $us < u$, $sw < w$ then by Proposition 3.5 we have $p_{u,v,w}(q) = p_{us,v,sw}(q) + q p_{us,v,w}(q)$. By the inductive hypothesis, we need only show that the possibility that $[q^2](p_{us,v,sw}(q)) = 1$ and $[q](p_{us,v,w}(q)) = 1$ does not occur. If this is the case then $a\ell(su^{-1}v, sw) = 2$ and $a\ell(su^{-1}v, w) = 1$. Since $a\ell(su^{-1}v, sw) \leq a\ell(su^{-1}v$, this is a contradiction and the statement of the proposition follows. ☐

Corollary 5.6. *If deg $p_{u,v,w}(q) = 2$, then $p_{u,v,w}(q) = q^2$.*

Thus far these propositions concerning classification reveal nothing new from what occurs for modified R-polynomials as q and q^2 are the only modified R-polynomials of degree 1 and 2, respectively. The p-polynomials of degree 3 are more interesting, and diverge from the modified R-polynomials in a significant way. The polynomials q^3 and $q^3 + q$ are the only modified R-polynomials of degree 3. These are also p-polynomials, as are $aq^3 + q$ in $\mathfrak{S}_n$ where $a < n$.

Proposition 5.7. *Let $u_m = s_1 s_2 \cdots s_{m-1} s_m s_{m-1} \cdots s_2 s_1 \in \mathfrak{S}_n$, where $m < n$. The p-polynomial $p_{u_m,u_m,u_m}(q) = (m-1)q^3 + q$.*

Proof. The proof is by induction on m. We note that

- $p_{s_1,s_1,s_1}(q) = 0q^3 + q$.
- $p_{s_1 s_2 s_1, s_1 s_2 s_1, s_1 s_2 s_1}(q) = q^3 + q$.
- For $u_3 = s_1 s_2 s_3 s_2 s_1$, the polynomial $p_{u_3,u_3,u_3}(q) = 2q^3 + q$.

Suppose by induction that for all $k < m$ we have $p_{u_k,u_k,u_k}(q) = (k-1)q^3 + q$. Let $k = m$

We note that s_m is both a left and right descent of u_m. Furthermore $u_m = s_m u_{m-1} s_m = u_{m-1} s_m u_{m-1}$. For simplicity of notation, in the following proof we will let $s = s_m$ and $u = u_{m-1}$.

$$\begin{aligned}
p_{u_m,u_m,u_m}(q) &= p_{sus,sus,sus}(q) \\
&= p_{su,sus,us}(q) + qp_{su,sus,sus}(q) \\
&= p_{us,us,sus}(q) + qp_{us,sus,sus}(q) \\
&= p_{u,us,us}(q) + qp_{u,us,sus}(q) + qp_{u,sus,us}(q) + q^2 p_{u,sus,sus}(q) \\
&= p_{us,u,su}(q) + 2qp_{u,us,sus}(q) + q^2 p_{sus,u,sus}(q) \qquad (5.2) \\
&= p_{us,u,su}(q) + q^2 p_{sus,u,sus}(q) \\
&= p_{u,u,u}(q) + p_{u,u,su}(q) + q^2 p_{su,u,us}(q) + q^3 p_{su,u,sus} \\
&= p_{u,u,u}(q) + q^3 p_{su,u,sus} \\
&= p_{u,u,u}(q) + q^3.
\end{aligned}$$

By (5.2) and the inductive hypothesis,

$$p_{u_m,u_m,u_m}(q) = p_{u_{m-1},u_{m-1},u_{m-1}}(q) + q^3 = (m-2)q^3 + q + q^3 = (m-1)q^3 + q.$$

□

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