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Sharp bounds on the size of pairable graphs and pairable bipartite graphs

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Abstract

The k-pairable graphs, introduced by Chen in 2004, constitute a wide class of graphs with a new type of symmetry, which includes many graphs of theoretical and practical importance, such as hypercubes, Hamming graphs of even order, antipodal graphs (also called diametrical graphs, or symmetrically even graphs), S-graphs, etc. Let k be a positive integer. A connected graph G is said to be k-pairable if the automorphism group of G contains an involution ϕ with the property that the distance between x and $\phi(x)$ is at least k for any vertex x of G. The pair length of G is k if G is k-pairable but not $(k+1)$-pairable. It is known that any graph of pair length $k > 0$ has even order at least $2k$. In this paper, we give sharp bounds for the size of a graph G of order n and pair length k for any integer $k > 0$ and any even integer $n \geq 2k$, when G is bipartite and when G is not restricted to be bipartite, respectively.

1 Introduction

All graphs considered in this paper are finite simple connected graphs unless otherwise specified. Motivated by an elegant result of Graham, Entringer and Székely [11] on spanning trees of a graph with an antipodal isomorphism, Chen [5] introduced the following concept of k-pairable graphs in 2004.

Definition 1.1 [5] Let k be a positive integer. A graph G is said to be *k-pairable* if its automorphism group contains an involution ϕ such that each vertex v has distance at least k from its image $\phi(v)$.

An *involution* on a set is a bijective map that is the inverse of itself. In Definition 1.1, the involution ϕ is also called a *k-pair partition* of G, and $\phi(v)$ is called the *mate of vertex v* under ϕ (and vice versa).

A k-pairable graph is briefly called a *pairable graph* if there is no need to specify the positive integer k, and a graph is said to be *non-pairable* if it is not k-pairable for any positive integer k. The k-pairable graphs constitute a wide class of graphs with a new type of symmetry, which includes many graphs of theoretical and practical importance, such as hypercubes, Hamming graphs of even order, antipodal graphs (also called diametrical graphs, or symmetrically even graphs), S-graphs, etc. (cf. [1, 3, 10, 12, 13]).

A new graph parameter called the *pair length* of a graph G was also introduced by Chen [5] to further study the k-pairable graphs.

Definition 1.2 [5] The *pair length* of a graph G, denoted as $p(G)$, is the maximum integer k such that G is k-pairable; $p(G) = 0$ if G is not k-pairable for any positive integer k.

This parameter measures the maximum distance, in some sense, between a subgraph induced by half the vertices of G and its isomorphic image induced by the other half of $V(G)$. Thanks to a referee, we got informed that if we drop the requirement that the automorphism be an involution, then the above definition for the pair length of a graph becomes the same as the absolute mobility of a graph, a concept defined in [14] by Potočnik, Šajna, and Verret.

Chen [5] obtained a result involving spanning trees and cycles, which seems to have the potential to be applied to the study of networks. That is, if G is a k-pairable graph ($k > 1$), then for every spanning tree T of G, there exists an edge e of G outside T whose addition to T forms a cycle of length at least $2k$. This extends a result by Graham et al. [11] on graphs with antipodal isomorphisms to this larger class of k-pairable graphs. Chen [5] also showed that the pair length of a Cartesian product graph is at least the sum of pair lengths of its factors and posted the question that if the equality is always true in general. Christofides [9] answered the question affirmatively.

Many interesting results [5, 6, 7, 8] have been obtained on k-pairable graphs since Definitions 1.1 and 1.2 were introduced. A necessary and sufficient condition for a graph to have pair length equal to a positive integer k was given by Che and Chen in [8], But it remains open on the characterizations of graphs with pair length

k, which was raised by Chen [5] in 2004. Special types of pairable graphs have been studied by many researchers. A characterization of uniquely k-pairable graphs in terms of the prime factor decomposition with respect to Cartesian product was given by Che [7]. Another special type of pairable graphs which are called *S-graphs* has also been studied, see [1, 3, 12] and the references therein. Since prime graphs are building blocks of all graph structures, Che and Chen [8] gave the minimum order of an r-regular prime graph with pair length k, and constructed such a graph with the minimum order for any given integers $r, k \geq 2$ (excluding $r = k = 2$). With this approach, the minimum order of an r-regular graph with pair length k for any integers $r, k \geq 2$ was also obtained.

When we study a class of connected simple graphs of given order n, one of the basic questions is to find bounds for the size of those graphs. In this paper, we give the sharp bounds for the size of a pairable graph G of order n and pair length k for any positive integer k and any even integer $n \geq 2k$, when G is bipartite and when G is not restricted to be bipartite, respectively.

2 Preliminaries

We follow [4] for basic terminologies and notations. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The cardinality of a set S is denoted by $|S|$. The degree of a vertex x of G is the number of vertices adjacent to x in G and denoted as $\deg_G(x)$. The distance $d_G(u, v)$ between two vertices u and v of G is the length of a shortest path between u and v in G.

If G is a pairable graph, then the automorphism group of G contains an involution which is fixed-point free, and so the order of G must be a positive even integer. Pair lengths of simple graphs such as cycles, complete graphs can be obtained by definition immediately: $p(C_n) = \frac{n}{2}$ and $p(K_n) = 1$ if n is even, $p(C_n) = p(K_n) = 0$ if n is odd, see [5]. Pair lengths of some special bipartite graphs were also given in [5]. For example, hypercubes Q_n have pair lengths $p(Q_n) = n$, complete bipartite graphs $K_{n,n}$ have pair lengths 2 or 1 depending on whether n is even or odd, and the pair length of a tree is at most 1. Moreover, a tree T has $p(T) = 1$ if and only if there is an edge $e = xy$ in T such that there exists an isomorphism f between the two connected components of $T - e$ satisfying $f(x) = y$, see [6]. The pair length of a complete bipartite graph $K_{m,n}$ is at most 2, and a characterization for complete bipartite graphs with pair length 1 or 2 is provided later in this section.

An induced cycle C of a graph G is called a *strongly induced cycle* if $d_C(x, y) = d_G(x, y)$ for any two vertices x, y of C. It is clear that if $n = 3, 4, 5$, then an induced n-cycle of G is just a strongly induced n-cycle of G, but this is not necessarily true when $n > 5$. The concept of a strongly induced cycle was introduced in [8]. It should be noted that this concept is very useful when studying structures of k-pairable graphs.

Below, we present some basic results on pairable graphs.

Theorem 2.1 [8] *Let G be a connected graph with pair length $p(G) > 0$. Then* *(i)* $p(G) = 1$ *if and only if* G *is* 1*-pairable and for any* 1*-pair partition* ϕ *of* G *there is an edge* e *of* G *such that* ϕ *maps* e *onto itself.*

(ii) $p(G) = k(> 1)$ *if and only if* G *is* k*-pairable and for any* k*-pair partition* ϕ *of* G *there is a strongly induced* $2k$*-cycle* C *of* G *such that* ϕ *maps* C *onto itself.*

Note: In general, the above edge e and the strongly induced $2k$-cycle C are not fixed, they depend on the pair partition ϕ considered.

If G is a graph of pair length $k > 0$, then by Theorem 2.1, the order of G must be an even integer at least $2k$. In particular, if $n = 2k$, then the graph G of order n and pair length k is unique and bipartite, i.e., $G = K_2$ when $k = 1$ and $G = C_n$ when $k > 1$.

If the k-pairable graphs are also bipartite, then they have the following property.

Theorem 2.2 *Let* G *be a connected bipartite graph of pair length* $p(G) = k > 0$*. Assume that* ϕ *is a* k*-pair partition of* G*. Then for any vertex* x *of* G*,*

(i) if k *is even, then* x *and* $\phi(x)$ *are in the same color class of* G*;*

(ii) if k *is odd, then* x *and* $\phi(x)$ *are from distinct color classes of* G*.*

Proof. In a connected bipartite graph, the unique color classes are clearly preserved setwise by any automorphism and so the theorem immediately follows. $\square$

Recall that any pairable graph has even order. It follows that the cardinalities of two color classes of a pairable bipartite graph must have the same parity. By Theorem 2.2, the following property of pairable bipartite graphs follows immediately.

Corollary 2.3 *Let* G *be a connected bipartite graph with color classes* B *and* W*. Assume that* G *has order* n *and pair length* $p(G) = k > 0$*. Then* n *is an even integer at least* $2k$*. Moreover,*

(i) if k *is even, then both* $|B|$ *and* $|W|$ *are even and* G *is a subgraph of* $K_{\frac{n}{2}-t,\frac{n}{2}+t}$ *where* $t \geq 0$ *has the same parity as* $\frac{n}{2}$*;*

(ii) if k *is odd, then* $|B| = |W| = \frac{n}{2}$*, and* G *is a subgraph of* $K_{\frac{n}{2},\frac{n}{2}}$*.*

The pair length of any complete bipartite graph $K_{m,n}$ can be determined by the above corollary with the trivial observation that $p(K_{m,n}) \leq 2$.

Corollary 2.4 *A complete bipartite graph* $K_{m,n}$ *has its pair length*

$$p(K_{m,n}) = \begin{cases} 2, & \text{if both } m \text{ and } n \text{ are even;} \\ 1, & \text{if } m = n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

We conclude this section with two well known trivial results on the bounds of graph sizes (see [4] and [2], for example):

1. Let G be a connected graph of order n. Then $n-1 \leq |E(G)| \leq n(n-1)/2$. The bounds are attained by a tree of order n and the complete graph K_n, respectively.

2. Let G be a bipartite graph of order n. Then $n-1 \leq |E(G)| \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. The bounds are attained by a tree of order n and the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, respectively.

Although these two results are easy to establish, the situation often becomes quite different when the graphs in concern have some specific restrictions. In next section, we shall give the sharp bounds for the size of a pairable graph G of order n and pair length k for any positive integer k and any even integer $n \geq 2k$, when G is bipartite and when G is not restricted to be bipartite, respectively.

3 Main Results

3.1 Pairable Graphs

Let k be a positive integer and n be an even integer at least $2k$. In this section, we give sharp bounds for the size of a pairable graph of order n and pair length k.

Theorem 3.1 *Let G be a connected pairable graph of order n. Then*

$$|E(G)| \geq \begin{cases} n-1, & \text{if } p(G) = 1; \\ n, & \text{if } p(G) > 1. \end{cases}$$

And

$$|E(G)| \leq \begin{cases} \frac{1}{2}n(n-1), & \text{if } p(G) = 1; \\ \frac{1}{2}n(n-2), & \text{if } p(G) = 2; \\ \frac{1}{2}(n-2k)(\frac{n}{2}-k+5)+2k, & \text{if } p(G) = k \geq 3. \end{cases}$$

Both bounds are sharp.

Proof. Let G be a pairable graph of order n. Then n is a positive even integer.

The sharp lower bounds on $|E(G)|$ can be easily seen as follows. If $p(G) = 1$, then it is trivial that $|E(G)| \geq n-1$ and a tree of order n has size $n-1$. If $p(G) = k > 1$, then G cannot be a tree by the fact [5] that the pair length of a tree is at most 1 as the center of a tree is either a vertex or an edge and it is fixed by every automorphism. Hence, $|E(G)| \geq n$ and a cycle of order n has size n.

We then show the sharp upper bounds on $|E(G)|$ based on the pair length $p(G)$.

Case 1. $p(G) = 1$. It is well known [4] that $|E(G)| \leq \frac{1}{2}n(n-1)$. This upper bound is sharp, since a complete graph of even order n has pair length $p(K_n) = 1$.

Case 2. $p(G) = 2$. Since there is no edge between a vertex x and its mate $x' = \phi(x)$ for any 2-pair partition ϕ of G, the complement of G contains a perfect matching M. Hence, $|E(G)| \leq |E(K_n - M)| = \frac{1}{2}n(n-2)$. This bound is attained by the size of $K_n - M$, which has order n and pair length 2.

Case 3. $p(G) = k \geq 3$. Let ϕ be a k-pair partition of G. By Theorem 2.1, G contains a strongly induced $2k$-cycle C such that the mate of any vertex of C under ϕ belongs to C. Let $H = G - C$ be the induced subgraph of G obtained by removing all vertices of C together with their incident edges. (H might be disconnected.) Then, the mate of any vertex of H under ϕ belongs to H, and so $V(H) = \cup_{i=1}^{\frac{n}{2}-k}\{h_i, h_i'\}$ where $h_i' = \phi(h_i)$. Note that $E(G) = E(H,C) \cup E(H) \cup E(C)$, where $E(H,C)$ denotes the set of edges between H and C. Since $|E(C)| = 2k$, an upper bound for $|E(G)|$ can be obtained by considering upper bounds on $|E(H,C)|$ and $|E(H)|$.

Since C is a strongly induced cycle, any vertex of H can be adjacent to at most three vertices of C, and those three vertices must form a path of length two along C. So $|E(H,C)| \leq 3(n-2k)$. We claim that each vertex h_i of H is adjacent to at most one vertex of h_j, h_j' where $j \neq i$ and $1 \leq j \leq \frac{n}{2} - k$. Otherwise, if h_i is adjacent to both h_j and h_j' for some $j \neq i$ and $1 \leq j \leq \frac{n}{2} - k$, then h_i' is adjacent to both h_j and h_j' and so $d_H(h_i, h_i') = 2$. This is impossible since $d_H(h_i, h_i') \geq d_G(h_i, h_i') \geq 3$. Hence, each vertex of H can be adjacent to at most $\frac{n}{2} - k - 1$ vertices of H and so $|E(H)| \leq \frac{1}{2}(n-2k)(\frac{n}{2} - k - 1)$. It follows that

$$\begin{aligned} |E(G)| &\leq 3(n-2k) + \frac{1}{2}(n-2k)(\frac{n}{2} - k - 1) + 2k \\ &= \frac{1}{2}(n-2k)(\frac{n}{2} - k + 5) + 2k. \end{aligned}$$

To show that the above upper bound is sharp, we can construct a desired graph G as follows. Let $C = u_1u_2u_3 \cdots u_ku_1'u_2'u_3' \cdots u_k'$ be an induced $2k$-cycle where $k \geq 3$. Replace each of u_1 and u_1' by a clique with $\frac{1}{2}(n-2k+2)$ vertices, denoted as V_1 and V_1' respectively. Then join all vertices of V_1 with two vertices u_2, u_k', and join all vertices of V_1' with two vertices u_k, u_2'. It is easy to verify that the resulted graph G is a pairable graph of order n, pair length $k \geq 3$, and size $\frac{1}{2}(n-2k)(\frac{n}{2} - k + 5) + 2k$. □

3.2 Pairable Bipartite Graphs

Let k be a positive integer and n be an even integer at least $2k$. In this section, we give sharp bounds for the size of a bipartite graph of order n and pair length k.

Theorem 3.2 *Let G be a connected pairable bipartite graph of order n. Then we have the following sharp bounds for $|E(G)|$:*

$$|E(G)| \geq \begin{cases} n-1, & \text{if } p(G) = 1; \\ n, & \text{if } p(G) > 1. \end{cases}$$

And

$$|E(G)| \leq \begin{cases} \dfrac{n^2}{4}, & \textit{if } p(G)=1 \textit{ and } n \equiv 2 \pmod 4, \\ & \textit{or } p(G)=2 \textit{ and } n \equiv 0 \pmod 4; \\ \dfrac{n^2}{4}-1, & \textit{if } p(G)=1 \textit{ and } n \equiv 0 \pmod 4, \\ & \textit{or } p(G)=2 \textit{ and } n \equiv 2 \pmod 4; \\ \dfrac{n^2}{4}-\dfrac{n}{2}, & \textit{if } p(G)=3; \\ 2\left\lfloor \dfrac{\frac{n}{2}-k}{2} \right\rfloor \left\lceil \dfrac{\frac{n}{2}-k}{2} \right\rceil + 2n-2k, & \textit{if } p(G)=k>3. \end{cases}$$

Proof. Let G be a pairable bipartite graph of order n. Then n is a positive even integer. The proof for the sharp lower bounds for $|E(G)|$ is trivial and so omitted. To obtain the upper bound for the size of a pairable bipartite graph G of order n and pair length k is much more complicated. We distinguish four cases based on $p(G) = 1, 2, 3$ or larger than 3.

Case 1. $p(G) = 1$. By Corollary 2.3 (ii), G is a subgraph of $K_{\frac{n}{2},\frac{n}{2}}$.

If $\frac{n}{2}$ is odd, that is, $n \equiv 2 \pmod 4$, then $|E(G)| \leq \frac{n^2}{4}$. This bound is sharp, since the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ has pair length 1 and size $\frac{n^2}{4}$.

If $\frac{n}{2}$ is even, that is, $n \equiv 0 \pmod 4$, then by Corollary 2.4, G cannot be a complete bipartite graph. Hence, $|E(G)| \leq \frac{n^2}{4} - 1$. This bound is sharp, since the graph $K_{\frac{n}{2},\frac{n}{2}} - e$ obtained by removing an edge e from $K_{\frac{n}{2},\frac{n}{2}}$ has pair length 1 and size $\frac{n^2}{4} - 1$.

Case 2. $p(G) = 2$. By Corollary 2.3 (i), G is a subgraph of $K_{\frac{n}{2}-t,\frac{n}{2}+t}$ where $t \geq 0$ has the same parity as $\frac{n}{2}$. So $|E(G)| \leq (\frac{n}{2} - t)(\frac{n}{2} + t) \leq \frac{n^2}{4} - t^2$.

If $\frac{n}{2}$ is even, that is, $n \equiv 0 \pmod 4$, then t is even and so $t \geq 0$, we have $|E(G)| \leq \frac{n^2}{4}$. This bound is sharp, since $K_{\frac{n}{2},\frac{n}{2}}$ has pair length 2 and size $\frac{n^2}{4}$.

If $\frac{n}{2}$ is odd, that is, $n \equiv 2 \pmod 4$, then t is odd and so $t \geq 1$, we have $|E(G)| \leq \frac{n^2}{4} - 1$. This bound is sharp, since $K_{\frac{n}{2}-1,\frac{n}{2}+1}$ has pair length 2 and size $\frac{n^2}{4} - 1$.

Case 3. $p(G) = 3$. By Theorem 2.2, we may write the two color classes of G as $B = \{x_1, x_2, \ldots, x_{\frac{n}{2}}\}$ and $W = \{x'_1, x'_2, \ldots, x'_{\frac{n}{2}}\}$, where $x'_i = \phi(x_i)$ is the mate of x_i under a 3-pair partition ϕ of G. Then x_i is not adjacent with x'_i since $d_G(x_i, x'_i) \geq 3$. Thus the degree of each vertex of G is at most $\frac{n}{2} - 1$ and so $|E(G)| \leq \frac{1}{2}n(\frac{n}{2} - 1) = \frac{n^2}{4} - \frac{n}{2}$. This bound is sharp, since it can be reached by the graph $K_{\frac{n}{2},\frac{n}{2}} - M$ obtained by deleting a perfect matching M from $K_{\frac{n}{2},\frac{n}{2}}$.

Case 4. $p(G) = k > 3$. Recall that n is an even integer at least $2k$. If $n = 2k$, then it is trivial since G is a $2k$-cycle by Theorem 2.1. Assume that $n \geq 2k+2$. Let ϕ be a k-pair partition of G. Then by Theorem 2.1, G contains a strongly induced $2k$-cycle C such that the mate of any vertex of C under ϕ belongs to C. Let $H = G - C$ be the induced subgraph obtained from G by removing all vertices of C together with their incident edges. (H might be disconnected.) Then the mate of any vertex of H under ϕ belongs to H. Note that $E(G) = E(H) \cup E(C) \cup E(H, C)$, where $E(H, C)$ denotes the set of edges between H and C. It is clear that $|E(C)| = 2k$. Since C is a strongly induced $2k$-cycle of G where $k > 3$ and since G is bipartite, any vertex of H can be adjacent to at most two vertices on C, and those two vertices must be in distance 2 on C. So $|E(H, C)| \leq 2(n - 2k)$. Then it remains to show the following upper bound for $|E(H)|$:

$$|E(H)| \leq 2 \left\lfloor \frac{\frac{n}{2} - k}{2} \right\rfloor \left\lceil \frac{\frac{n}{2} - k}{2} \right\rceil .$$

Assume that B and W are two color classes of G. Let $H_B = H \cap B$ and $H_W = H \cap W$. We distinguish two subcases based on the parity of $p(G) = k$.

Subcase 4.1. $p(G) = k > 3$ is even. We first show that each vertex of H_B can be adjacent to at most half of the vertices in H_W. By Corollary 2.3(i), $G = G(B, W)$ is a subgraph of $K_{\frac{n}{2}-t,\frac{n}{2}+t}$ where $t \geq 0$ has the same parity as $\frac{n}{2}$. Without loss generality, we may assume that $|B| = \frac{n}{2} - t$ and $|W| = \frac{n}{2} + t$, both of them are even. Then $H = G - C$ is a subgraph of $K_{\frac{n}{2}-k-t,\frac{n}{2}-k+t}$, where both $\frac{n}{2} - k - t$ and $\frac{n}{2} - k + t$ are even since k is even. By Theorem 2.2, any pair of mates under ϕ are contained in the same color class of G. Since the mate of any vertex of H under ϕ is contained in H, it follows that each of H_B and H_W is a union of pairs of mates under ϕ. Let x be an arbitrary vertex in H_B. Suppose that x is adjacent to both vertices of a pair of mates $y, y' = \phi(y) \in H_W$. Then its mate $x' = \phi(x)$ in H_B is adjacent to both $y, y' \in H_W$. It follows that xyx' is a path of length 2 in $H (\subseteq G)$. This is impossible since $d_H(x, x') \geq d_G(x, x') \geq k > 3$. Therefore, $x \in H_B$ can be adjacent to at most one vertex of each pair of mates in H_W, and so $x \in H_B$ can be adjacent to at most half of the vertices in H_W. Then

$$|E(H)| \leq (\frac{n}{2} - k - t) \cdot \frac{\frac{n}{2} - k + t}{2} = \frac{1}{2}[(\frac{n}{2} - k)^2 - t^2].$$

Recall that k is even and t has the same parity as $\frac{n}{2}$. If $\frac{n}{2} - k$ is even, then both $\frac{n}{2}$ and t are even, so $t \geq 0$; if $\frac{n}{2} - k$ is odd, then both $\frac{n}{2}$ and t are odd, so $t \geq 1$. Therefore, $|E(H)| \leq \frac{1}{2}[(\frac{n}{2} - k)^2 - \delta]$, where $\delta = 0$ when $\frac{n}{2} - k$ is even; and $\delta = 1$ when $\frac{n}{2} - k$ is odd. That is, $|E(H)| \leq 2 \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil$.

Subcase 4.2. $p(G) = k > 3$ is odd. By Corollary 2.3 (ii), G is a subgraph of $K_{\frac{n}{2},\frac{n}{2}}$. Then $H = G - C$ is a subgraph of $K_{\frac{n}{2}-k,\frac{n}{2}-k}$. By Theorem 2.2, any pair of mates under ϕ are from distinct color classes of G. Since the mate of any vertex of H under ϕ is contained in H, it follows that the mate of any vertex in H_B under ϕ

is in H_W. So we can write $H_B = \{x_1, x_2, \cdots, x_{\frac{n}{2}-k}\}$ and $H_W = \{x'_1, x'_2, \cdots, x'_{\frac{n}{2}-k}\}$ where $x'_i = \phi(x_i)$ is the mate of x_i under ϕ for $1 \leq i \leq \frac{n}{2} - k$. Hence, H has an involution ϕ switching its two color classes and $d_H(x_i, x'_i) \geq d_G(x_i, x'_i) = k > 3$ for each $1 \leq i \leq \frac{n}{2} - k$.

Below, we give the following claim which will enable us to get the desired upper bound for $|E(H)|$.

Claim. Let Q be a bipartite graph (connected or disconnected) of order $2p$ with $p \geq 1$. If the automorphism group of Q contains an involution ϕ such that ϕ switches its two color classes Q_B and Q_W and $d_Q(u, \phi(u)) > 3$ for each vertex u of Q, then

(i) there is a vertex of Q_B adjacent to at most half of the vertices in Q_W, and

(ii) $|E(Q)| \leq 2\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil$.

The claim can be proved as follows.

Proof of (i): It is trivial when $p = 1$ or 2. Let $p \geq 3$. Suppose for contradiction that each vertex of Q_B is adjacent to more than half of the vertices in Q_W. Then any two vertices in Q_B have a common neighbor in Q_W. Without loss of generality, we can write that $Q_B = \{u_i | 1 \leq i \leq p\}$ and $Q_W = \{u'_i | 1 \leq i \leq p\}$ where $u'_i = \phi(u_i)$ for $1 \leq i \leq p$. Assume that $u_1 \in Q_B$ is adjacent to a vertex $u'_j \in Q_W$. Then their mates $u'_1 \in Q_W$ and $u_j \in Q_B$ are adjacent. Note that since $d_Q(u_1, u'_1) > 3$, we have $j \neq 1$ and $u_j \neq u_1$. Moreover, $u_1, u_j \in Q_B$ have a common neighbor $u'_i \in Q_W$ by the assumption. Clearly, $u'_i \neq u'_1$ and $u'_i \neq u'_j$. It then follows that $u_1 u'_i u_j u'_1$ is a path of length 3 in Q. This contradicts the condition that $d_Q(u_1, u'_1) > 3$. Thus, (i) is proved.

Proof of (ii): To show that $|E(Q)| \leq 2\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil$, we apply induction on $p = |Q_B| = |Q_W|$. If $p = 1$, then $Q_B = \{u_1\}$ and $Q_W = \{u'_1\}$. So $|E(Q)| = 0$ since u_1, u'_1 are not adjacent. If $p = 2$, then $Q_B = \{u_1, u_2\}$ and $Q_W = \{u'_1, u'_2\}$. The only possible edges among vertices in Q are $u_1u'_2$ and $u_2u'_1$. So $|E(Q)| \leq 2$. Thus, the desired upper bound holds for $p = 1, 2$. For the case when $p \geq 3$, by (i) we may assume that vertex $u_1 \in Q_B$ is adjacent to at most $\left\lfloor\frac{p}{2}\right\rfloor$ vertices in Q_W. Clearly, similar conclusion holds for the vertex $u'_1 \in Q_W$. Now we consider the subgraph $Q - \{u_1, u'_1\}$ of order $2(p-1)$. Note that the restriction of ϕ on $Q - \{u_1, u'_1\}$ is an automorphism and involution of $Q - \{u_1, u'_1\}$ switching its two color classes $Q_B \setminus \{u_1\}$ and $Q_W \setminus \{u'_1\}$. Moreover, $d_{Q-\{u_1,u'_1\}}(u, \phi(u)) \geq d_Q(u, \phi(u)) > 3$ for each vertex u of $Q - \{u_1, u'_1\}$. By induction hypothesis, $|E(Q - \{u_1, u'_1\})| \leq 2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lceil\frac{p-1}{2}\right\rceil$. Recall that $\deg_Q(u_1) = \deg_Q(u'_1) \leq \left\lfloor\frac{p}{2}\right\rfloor$. Then

$$\begin{aligned}
|E(Q)| &= |E(Q \setminus \{u_1, u'_1\})| + \deg_Q(u_1) + \deg_Q(u'_1) \\
&\leq 2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lceil\frac{p-1}{2}\right\rceil + 2\left\lfloor\frac{p}{2}\right\rfloor = 2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p}{2}\right\rfloor + 2\left\lfloor\frac{p}{2}\right\rfloor \\
&= 2\left\lfloor\frac{p}{2}\right\rfloor\left(\left\lfloor\frac{p-1}{2}\right\rfloor + 1\right) = 2\left\lfloor\frac{p}{2}\right\rfloor\left\lceil\frac{p}{2}\right\rceil.
\end{aligned}$$

This ends the proof of the claim.

Now, applying the claim to the graph H where H has order $2p$ ($p = \frac{n}{2} - k$), we obtain the same upper bound for $|E(H)|$ as in Subcase 4.1.

So, we get $|E(H)| \le 2 \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil$ when $p(G) = k > 3$. Thus, we see that when $p(G) = k > 3$, $|E(G)| = |E(H)| + |E(C)| + |E(H,C)| \le 2 \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil + 2n - 2k$.

To show that the above upper bound is sharp, we construct a desired graph G as follows. (See Figures 1 and 2).

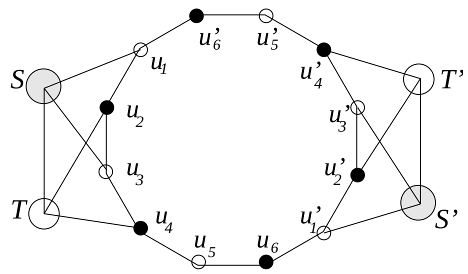

Figure 1: Example of a bipartite graph G with $p(G) = 6$ constructed in Theorem 3.2 for the case when $p(G) > 3$ is even, where any pair of mates under a 6-pair partition are in the same color class of G, $|S| = |S'| = \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil$ and $|T| = |T'| = \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor$.

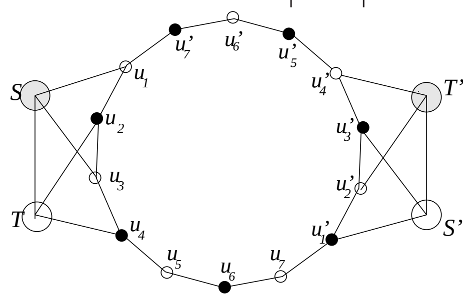

Figure 2: Example of a bipartite graph G with $p(G) = 7$ constructed in Theorem 3.2 for the case when $p(G) > 3$ is odd, where any pair of mates under a 7-pair partition are from distinct color classes of G, $|S| = |S'| = \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil$ and $|T| = |T'| = \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor$.

Let $C = u_1u_2u_3 \cdots u_k u'_1 u'_2 u'_3 \cdots u'_k$ be a $2k$-cycle and $H = (S,T)$, $H' = (S', T')$ be two complete bipartite graphs with $|S| = |S'| = \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil$ and $|T| = |T'| = \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor$. By adding edges to join all the vertices of S (respectively, S') with the two vertices u_1, u_3 (respectively, u'_1, u'_3) on C, and adding edges to join all the vertices of T (respectively, T') with the two vertices u_2, u_4 (respectively, u'_2, u'_4) on C, we get the desired graph G. It is easy to verify that G is a bipartite graph of order n and pair length k, and its size reaches the upper bound $2 \left\lfloor \frac{\frac{n}{2}-k}{2} \right\rfloor \left\lceil \frac{\frac{n}{2}-k}{2} \right\rceil + 2n - 2k$. □

Finally, we make a remark that for the case $n = 2k$ where $k = p(G)$, the lower bound and the upper bound given in Theorem 3.1 (respectively, Theorem 3.2) for the size of a bipartite graph G of order n and pair length k are equal. This is consistent with the fact that if $n = 2k$, then the graph G of order n and pair length k is unique and bipartite, i.e., $G = K_2$ when $k = 1$ and $G = C_n$ when $k > 1$.

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