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Partitioning a cyclic group into well-dispersed subsets with a common gap sequence

Tomoki Nakamigawa

Department of Information Science
Shonan Institute of Technology
1-1-25 Tsujido-Nishikaigan, Fujisawa
Kanagawa 251-8511
Japan
nakami@info.shonan-it.ac.jp

Abstract

Let k, ℓ and n be positive integers with $n = k\ell$. Let $\boldsymbol{a} = \{a_1, a_2, \ldots, a_\ell\}$ be a multiset of positive integers with $a_1 + a_2 + \cdots + a_\ell = n$. Let $\mathbb{Z}_n = [0, n-1]$ be the cyclic group of order n.

A partition of $\mathbb{Z}_n$ into k parts $T_1, T_2, \ldots, T_k$ is called a well-dispersed partition with a gap sequence $\boldsymbol{a}$, if (1) the cardinality of $T_i \cap [(j-1)k, jk-1]$ is 1 for $1 \le i \le k$, $1 \le j \le \ell$ and (2) a set of ℓ distances of consecutive elements of T_i coincides with $\boldsymbol{a}$ as a multiset for $1 \le i \le k$.

Amongst other results, it is proved that if k is a power of 2, then there exists a well-dispersed partition with a gap sequence $\{1, 2, \ldots, 2k-1\}$.

1 Introduction

Let k, ℓ and n be positive integers with $n = k\ell$. Let $\mathbb{Z}$ be the set of integers, and let $\mathbb{Z}_n$ be the cyclic group of order n. We regard $\mathbb{Z}_n$ as $[0, n-1] = \{0, 1, \ldots, n-1\}$.

For a subset $T = \{x_1, x_2, \ldots, x_k\} \subset [0, n-1]$, where $x_1 < x_2 < \cdots < x_k$, we define a *gap sequence* of T as a multiset of positive integers $x_2 - x_1, \ldots, x_k - x_{k-1}, x_1 - x_k + n$. Note that if $k = 1$, then the gap sequence of $T = \{x_1\}$ is just $\{n\}$.

Our aim is to find a partition of $\mathbb{Z}_n = T_1 \cup T_2 \cup \cdots \cup T_k$ such that a gap sequence of T_i is common for $1 \le i \le k$.

A related problem is finding a partition of integers $\mathbb{Z}$ into subsets of common gaps. Let a and b be positive integers with $a > b$. It is known that there exists a finite interval $I \subset \mathbb{Z}$ such that I is partitioned into $I = T_1 \cup T_2 \cup \cdots \cup T_k$, where each T_i is a translation of $\{0, a, a+b\}$ or a translation of $\{0, b, a+b\}$ [2]. Let $a = \lambda b + r$ with $\lambda \ge 1$, $0 \le r < b$. In this partition, it is known that the cardinality of the shortest interval I is at most $5a + c_\lambda b + r$, where $c_\lambda \equiv \lambda \pmod 6$ with $2 \le c_\lambda \le 7$

[1, 3]. Related problems are discussed in [4]. Let $T = \{x_1, \dots, x_n\} \subset \mathbb{Z}$. We call a multiset $x_2 - x_1,\ x_3 - x_2,\ \dots,\ x_n - x_{n-1}$ a gap sequence of T. One interesting unsolved problem is as follows.

Problem 1 *[4] For any sequence of positive integers $a_1, a_2, \dots, a_{n-1}$, can $\mathbb{Z}$ be partitioned into a family of n-sets with a common gap sequence $a_1, \dots, a_{n-1}$?*

Problem 1 is open, even for $n = 4$.

In this paper, we consider $\mathbb{Z}_n$ instead of $\mathbb{Z}$ as an underlying space. The aim of the paper is to study a problem of partitioning $\mathbb{Z}_n$ into a family of subsets with a common gap sequence.

We note a connection between a partition of $\mathbb{Z}$ and a partition of $\mathbb{Z}_n$. Let $\mathbb{Z}_n = [0, n-1] = T_1 \cup \cdots \cup T_k$ be a partition with a common gap sequence $\boldsymbol{a} = (a_1, \dots, a_\ell)$. Then we have a partition of $\mathbb{Z}$ with a common gap sequence $\boldsymbol{a}' = \boldsymbol{a} \setminus \{a_u\}$ for any u with $1 \le u \le \ell$. Indeed, we can lift a partition of $\mathbb{Z}_n$ to a partition $\mathbb{Z} = \cup_{1 \le i \le k}(\mathbb{Z}_n + T_i)$. Each $\mathbb{Z}_n + T_i$ has a partition in $\mathbb{Z}$ into copies of S with a gap sequence $\boldsymbol{a}'$, where the distance between consecutive copies is a_u.

Now we introduce an additional condition for a partition of $\mathbb{Z}_n$. For $1 \le j \le \ell$, we put $B_j = \{x \in [0, n-1] : (j-1)k \le x \le jk - 1\}$, which is the j-th run of size k. We call a partition $\mathbb{Z}_n = [0, n-1] = T_1 \cup \cdots \cup T_k$ a *well-dispersed partition with a gap sequence* $\boldsymbol{a} = \{a_1, \dots, a_\ell\}$, if (1) $|T_i \cap B_j| = 1$ for $1 \le i \le k$, $1 \le j \le \ell$ and (2) a gap sequence of T_i coincides with $\boldsymbol{a}$ for $1 \le i \le k$.

For positive integers k, ℓ and $n = k\ell$, $\mathcal{G}_{k,\ell}$ denotes a family of multisets $\boldsymbol{a} = \{a_1, \dots, a_\ell\}$ such that there exists a well-dispersed partition of $\mathbb{Z}_n = T_1 \cup \cdots \cup T_k$, where a gap sequence of T_i is $\boldsymbol{a}$ for $1 \le i \le k$. Furthermore, we define $\mathcal{G}_k = \cup_{\ell=1}^{\infty} \mathcal{G}_{k,\ell}$.

It is convenient for us to denote a well-dispersed partition $\mathbb{Z}_n = T_1 \cup \cdots \cup T_k$ by a matrix $M = (m_{i,j})_{1 \le i \le k, 0 \le j \le n-1}$ having k rows and n columns. For $1 \le i \le k$ and $0 \le j \le n-1$, if $j \in T_i$, then set $m_{i,j} = 1$, and otherwise set $m_{i,j} = 0$. Then the i-th row of M corresponds to T_i for $1 \le i \le k$, and M is a concatenation of permutation matrices $M_0, M_1, \dots, M_{\ell-1}$ of size k. We write M as $M = M_0|M_1|\cdots|M_{\ell-1}$.

$$
\begin{array}{c}
 \\ T_1 \\ T_2 \\ T_3
\end{array}
\begin{array}{ccc|ccc|ccc}
 & M_0 & & & M_1 & & & M_2 & \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
$$

Figure 1. A well-dispersed partition of $\mathbb{Z}_9$ into a family of subsets with a common gap sequence $\{1, 4, 4\}$.

Example 1 *Let $k = 3$ and $\ell = 3$. $\mathbb{Z}_9 = [0, 8]$ can be partitioned into $\mathbb{Z}_9 = T_1 \cup T_2 \cup T_3$, where $T_1 = \{0, 4, 8\}$, $T_2 = \{1, 5, 6\}$, $T_3 = \{2, 3, 7\}$. Then the partition is a well-dispersed partition with a gap sequence $\boldsymbol{a} = \{1, 4, 4\}$. Hence we have $\{1, 4, 4\} \in \mathcal{G}_{3,3}$ (see Fig. 1).*

Before closing this section, we will fix some notation on multisets. Let $\boldsymbol{a} = \{a_1, \ldots, a_\ell\}$ and $\boldsymbol{b} = \{b_1, \ldots, b_{\ell'}\}$ be multisets. For two scalars s and t, $s\boldsymbol{a} + t$ denotes a multiset $\{sa_i + t : 1 \leq i \leq \ell\}$. The *disjoint union* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} \uplus \boldsymbol{b} = \{a_1, \ldots, a_\ell, b_1, \ldots, b_{\ell'}\}$. The *sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} + \boldsymbol{b} = \{a_i + b_j : 1 \leq i \leq \ell, 1 \leq j \leq \ell'\}$.

2 Main Results

We will first show some observations for $\mathcal{G}_k$ and constructions of well-dispersed partitions.

Observation 1 *Let $\boldsymbol{a} = \{a_1, \ldots, a_\ell\} \in \mathcal{G}_{k,\ell}$. Then we have*
(1) $a_1 + \cdots + a_\ell = k\ell$, and
(2) $1 \leq a_i \leq 2k - 1$ for $1 \leq i \leq \ell$.

Suppose that $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}_k$. Let $P = P_0|P_1|\cdots|P_{\ell_1-1}$ and $Q = Q_0|Q_1|\cdots|Q_{\ell_2-1}$ be matrices corresponding to well-dispersed partitions with gap sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. We may assume $P_0 = Q_0$, since we can interchange two rows of Q, if necessary. Then $M = P|Q$ corresponds to a well-dispersed partition with a gap sequence $\boldsymbol{a} \uplus \boldsymbol{b}$.

Observation 2 *If $\boldsymbol{a}_i \in \mathcal{G}_{k,\ell_i}$ for $i = 1, 2$, then we have $\boldsymbol{a}_1 \uplus \boldsymbol{a}_2 \in \mathcal{G}_{k,\ell_1+\ell_2}$.*

For $k = 2$, we have $\{1, 3\} \in \mathcal{G}_{2,2}$, since $\mathbb{Z}_4 = [0, 3] = \{0, 3\} \cup \{1, 2\}$. We also have $\{2\} \in \mathcal{G}_{2,1}$. Hence, by Observation 2, it follows that for $k = 2$, the conditions (1) and (2) of Observation 1 are sufficient for a gap sequence of $\mathcal{G}_2$. However, for $k \geq 3$, the conditions (1) and (2) of Observation 1 are not sufficient for a gap sequence of $\mathcal{G}_k$. For example, we have $\{1, 2k - 1\} \notin \mathcal{G}_{k,2}$ for $k \geq 3$.

Let σ be a permutation of $\{1, 2, \ldots, k\}$ such that σ consists of a single cycle, and let P_σ be a permutation matrix of size k whose (i, j) entry is $\delta_{\sigma(i),j}$ for $1 \leq i, j \leq k$. We define $M = I|P_\sigma|P_\sigma^2|\cdots|P_\sigma^{k-1}$, where I is the identity matrix of size k. Then M corresponds to a well-dispersed partition.

Observation 3 *Let σ be a permutation of $\{1, 2, \ldots, k\}$. If σ consists of a single cycle, then $\boldsymbol{a} = \{\sigma(i) - i + k : 1 \leq i \leq k\} \in \mathcal{G}_{k,k}$.*

Example 2 *Let $k = 7$ and $\sigma = (1\,7\,2\,6\,3\,5\,4)$, denoted by cycle notation. Then we have a well-dispersed partition $\mathbb{Z}_{49} = [0, 48] = \{0, 13, 15, 26, 30, 39, 45\} \cup \{1, 12, 16, 25, 31, 35, 48\} \cup \{2, 11, 17, 21, 34, 36, 47\} \cup \{3, 7, 20, 22, 33, 37, 46\} \cup \{4, 10, 14, 27, 29, 40, 44\} \cup \{5, 9, 18, 24, 28, 41, 43\} \cup \{6, 8, 19, 23, 32, 38, 42\}$.*

Hence, we have $\{2, 4, 4, 6, 9, 11, 13\} \in \mathcal{G}_{7,7}$.

Next, let us consider a reflection of each run. Let us define a permutation matrix R of size k such that (i, j) entry of R is $\delta_{i,k+1-j}$ for $1 \leq i, j \leq k$. For a matrix $P = P_0|P_1|\cdots|P_{\ell-1}$ corresponding to a well-dispersed partition with a gap sequence $\boldsymbol{a}$, let us define $Q_i = P_iR$ for $1 \leq i \leq \ell - 1$. Then $Q = Q_0|Q_1|\cdots|Q_{\ell-1}$ corresponds to another well-dispersed partition.

Observation 4 *If $\boldsymbol{a} \in \mathcal{G}_{k,\ell}$, then we have $2k - \boldsymbol{a} \in \mathcal{G}_{k,\ell}$.*

Example 3 *We have $\{2, 7, 9\} \in \mathcal{G}_{6,3}$, because $\mathbb{Z}_{18} = [0, 17] = \{0, 9, 16\} \cup \{1, 8, 17\} \cup \{2, 11, 13\} \cup \{3, 10, 12\} \cup \{4, 6, 15\} \cup \{5, 7, 14\}$. By considering reflections of three runs $[0, 5]$, $[6, 11]$, $[12, 17]$, we have $\mathbb{Z}_{18} = \{5, 8, 13\} \cup \{4, 9, 12\} \cup \{3, 6, 16\} \cup \{2, 7, 17\} \cup \{1, 11, 14\} \cup \{0, 10, 15\}$. Hence, we have $\{3, 5, 10\} \in \mathcal{G}_{6,3}$.*

In a well-dispersed partition of run size k, every gap size is at most $2k-1$. Hence, it is a natural question whether $[1, 2k - 1]$ is contained in $\mathcal{G}_k$ or not. For $k = 3$, by using a computer, we have $[1, 5] \notin \mathcal{G}_3$. The main result of the paper is that there exist infinitely many k such that $[1, 2k - 1] \in \mathcal{G}_k$.

Theorem 5 *Let b be a non-negative integer. Let $k = a2^b$. If $a = 2, 5, 6, 7$, then $[1, 2k - 1] \in \mathcal{G}_k$.*

This is an open problem for other cases.

Problem 2 *For $k \neq 3$, is $[1, 2k - 1]$ contained in $\mathcal{G}_k$?*

3 Proof of Theorem 5

The key ingredient for the proof is the following lemma.

Lemma 6 *Let $\boldsymbol{a} = \{a_1, \ldots, a_\ell\} \in \mathcal{G}_{k,\ell}$ and $\boldsymbol{b} = \{b_1, \ldots, b_{\ell'}\} \in \mathcal{G}_{k',\ell'}$. If ℓ and ℓ' are relatively prime, then $\boldsymbol{c} = k'(\boldsymbol{a} - 1) + \boldsymbol{b} \in \mathcal{G}_{kk',\ell\ell'}$.*

Proof of Lemma 6. Let $n = k\ell$ and $n' = k'\ell'$. Let $P = P_0|P_1|\cdots|P_{\ell-1}$ be a $k \times n$ matrix corresponding to a well-dispersed partition of $\mathbb{Z}_n$ with a gap sequence $\boldsymbol{a}$. Similarly, let $Q = Q_0|Q_1|\cdots|Q_{\ell'-1}$ be a $k' \times n'$ matrix corresponding to a well-dispersed partition of $\mathbb{Z}_{n'}$ with a gap sequence $\boldsymbol{b}$. For $0 \leq t \leq \ell\ell' - 1$, we define a permutation matrix M_t of size kk' as $M_t = P_i \otimes Q_j$, the Kronecker product of P_i and Q_j, where $i \equiv t \pmod{\ell}$ with $0 \leq i \leq \ell - 1$ and $j \equiv t \pmod{\ell'}$ with $0 \leq j \leq \ell' - 1$. Furthermore, we define a $(kk') \times (nn')$ matrix M as $M = M_0|M_1|\cdots|M_{\ell\ell'-1}$.

We claim that M corresponds to a well-dispersed partition of $\mathbb{Z}_{nn'}$ with a gap sequence $\boldsymbol{c}$. Indeed, firstly, M is a concatenation of $\ell\ell'$ permutation matrices of size kk'. Secondly, let a row of M, the i-th row with $0 \leq i \leq kk' - 1$, be fixed. Let $i = i_1k' + i_2$, with $0 \leq i_1 \leq k - 1$, $0 \leq i_2 \leq k' - 1$. For every pair of gaps $a \in \boldsymbol{a}$ and

$b \in \boldsymbol{b}$, there exists a pair of indices α and β such that a is the α-th gap of the i_1-th row of P (the gap between two 1's contained in $P_{\alpha-1}$ and P_α) and b is the β-th gap of the i_2-th row of Q (the gap between two 1's contained in $Q_{\beta-1}$ and Q_β). Put t with $0 \le t \le \ell\ell' - 1$ as $t \equiv \alpha \pmod{\ell}$ and $t \equiv \beta \pmod{\ell'}$. Then, the t-th gap of the i-th row of M (the gap between two 1's contained in M_{t-1} and M_t) is $k'(a-1)+b$.

Hence, in each row of M, we have gaps $k'(a_i - 1) + b_j$ for $0 \le i \le \ell - 1$, $0 \le j \le \ell' - 1$, as required. □

Example 4 *Let* $\boldsymbol{a} = \{1, 3\}$. *We have* $\boldsymbol{a} \in \mathcal{G}_{2,2}$, *since there exists a matrix* $P = P_0|P_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$, *which corresponds to a well-dispersed partition with a gap sequence* $\boldsymbol{a}$. *Let* $\boldsymbol{b} = \{1, 2, 3\}$. *We have* $\boldsymbol{b} \in \mathcal{G}_{2,3}$, *since there exists a matrix* $Q = Q_0|Q_1|Q_2 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right)$, *which corresponds to a well-dispersed partition with a gap sequence* $\boldsymbol{b}$.

By Lemma 6, we have $\boldsymbol{c} = 2(\boldsymbol{a}-1)+\boldsymbol{b} = \{1, 2, 3, 5, 6, 7\} \in \mathcal{G}_{4,6}$. *By the construction in the proof of Lemma 6, the corresponding matrix is*
$M = P_0 \otimes Q_0 | P_1 \otimes Q_1 | P_0 \otimes Q_2 | P_1 \otimes Q_0 | P_0 \otimes Q_1 | P_1 \otimes Q_2$

$$= \left(\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc} 1&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0 \\ 0&0&1&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&1&0&0 \\ 0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&0&0&0&0&1&1&0&0&0 \end{array}\right).$$

Proposition 7 *Let* $m \ge 2$. *If* $[1, 2m-1] \in \mathcal{G}_m$, *then* $[1, 4m-1] \in \mathcal{G}_{2m}$.

Proof of Proposition 7. Note that $\{1, 3\} \in \mathcal{G}_{2,2}$. We apply Lemma 6 with $k = 2$, $\ell = 2$, $\boldsymbol{a} = \{1, 3\}$, $k' = m$, $\ell' = 2m-1$ and $\boldsymbol{b} = [1, 2m-1]$. Then we have $\boldsymbol{c} = k'(\boldsymbol{a}-1)+\boldsymbol{b} = [1, 2m-1] \cup [2m+1, 4m-1] \in \mathcal{G}_{2m}$. Since $\{2m\} \in \mathcal{G}_{2m}$, by Observation 2, we have $\boldsymbol{c} \uplus \{2m\} = [1, 4m-1] \in \mathcal{G}_{2m}$. □

By Proposition 7, in order to complete the proof of Theorem 5, it suffices to show that $[1, 2k-1] \in \mathcal{G}_k$ for $k = 2, 5, 6, 7$. Note that for $k = 5, 6, 7$, the following constructions were found by using computer.

Let $k = 2$. We have a partition $\mathbb{Z}_6 = \{0, 2, 5\} \cup \{1, 3, 4\}$. Then the partition is a well-dispersed partition with a gap sequence $[1, 3]$.

Let $k = 5$. We have a partition

$$\begin{array}{rcl} \mathbb{Z}_{45} & = & \{0, 5, 13, 17, 23, 26, 33, 35, 44\} \\ & \cup & \{1, 7, 12, 16, 24, 25, 34, 36, 43\} \\ & \cup & \{2, 8, 11, 18, 20, 29, 30, 38, 42\} \\ & \cup & \{3, 9, 10, 19, 22, 27, 31, 39, 41\} \\ & \cup & \{4, 6, 14, 15, 21, 28, 32, 37, 40\}. \end{array}$$

Then the partition is a well-dispersed partition with a gap sequence $[1, 9]$.

Let $k = 6$. We have a partition

$$\begin{aligned}
\mathbb{Z}_{54} &= \{0, 10, 16, 20, 27, 32, 40, 42, 53\} \\
&\cup \{1, 11, 12, 23, 25, 31, 39, 44, 51\} \\
&\cup \{2, 9, 14, 22, 24, 35, 36, 46, 52\} \\
&\cup \{3, 8, 15, 19, 29, 30, 41, 43, 49\} \\
&\cup \{4, 6, 17, 18, 28, 34, 38, 45, 50\} \\
&\cup \{5, 7, 13, 21, 26, 33, 37, 47, 48\}.
\end{aligned}$$

Then the partition is a well-dispersed partition with a gap sequence $\{1, 2, 4, 5, 6, 7, 8, 10, 11\}$. On the other hand, we have a partition $\mathbb{Z}_{12} = \{0, 9\} \cup \{1, 10\} \cup \{2, 11\} \cup \{3, 6\} \cup \{4, 7\} \cup \{5, 8\}$. Then the partition is a well-dispersed partition with a gap sequence $\{3, 9\}$. By Observation 2, we have $\{1, 2, 4, 5, 6, 7, 8, 10, 11\} \uplus \{3, 9\} = [1, 11] \in \mathcal{G}_6$.

Let $k = 7$. We have a partition

$$\begin{aligned}
\mathbb{Z}_{91} &= \{0, 10, 16, 21, 33, 36, 47, 51, 60, 67, 75, 77, 90\} \\
&\cup \{1, 11, 19, 25, 30, 39, 46, 50, 62, 63, 76, 78, 89\} \\
&\cup \{2, 12, 14, 27, 28, 40, 43, 54, 58, 66, 73, 79, 88\} \\
&\cup \{3, 13, 15, 26, 31, 35, 48, 49, 61, 64, 72, 81, 87\} \\
&\cup \{4, 9, 20, 22, 34, 37, 44, 52, 56, 69, 70, 80, 86\} \\
&\cup \{5, 7, 17, 24, 29, 41, 42, 55, 59, 65, 74, 82, 85\} \\
&\cup \{6, 8, 18, 23, 32, 38, 45, 53, 57, 68, 71, 83, 84\}.
\end{aligned}$$

Then the partition is a well-dispersed partition with a gap sequence $[1, 13]$.

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