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Edge decompositions of hypercubes by paths

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Abstract

Many authors have investigated edge decompositions of graphs by the edge sets of isomorphic copies of special subgraphs. For q-dimensional hypercubes Q_q various researchers have done this for certain trees, paths and cycles. In this paper we shall say that "H divides G" if $E(G)$ is the disjoint union of $\{E(H_i) \mid H_i \simeq H\}$. Our main result is that for q odd, the path of length m, P_m, divides Q_q if and only if $m \leq q$ and $m \mid q \cdot 2^{q-1}$.

1 Introduction

Edge decompositions of graphs by subgraphs have a long history. For example, there is a Steiner triple system of order n if and only if the complete graph K_n has an edge-decomposition by K_3. In 1847 Kirkman [9] proved that for a Steiner triple system to exist it is necessary that $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. In 1850 [10] he proved the converse holds also.

Theorem 1 *A Steiner system of order* $n \geq 3$ *exists if and only if* $n \equiv 1 \pmod 6$ *or* $n \equiv 3 \pmod 6$.

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In more modern times (1964) Ringel [17] stated the following conjecture, which is still open.

Ringel's Conjecture
If T is a fixed tree with m edges then K_{2m+1} is edge-decomposable into $2m+1$ copies of T.

Still more recently, the n-dimensional hypercube graph Q_n has been studied extensively, largely because of its usefulness as the architecture for distributed parallel processing supercomputers [12]. Communication problems such as "broadcasting" in these networks (see [8], [3]) have led to research on constructions of maximum size families of edge-disjoint spanning trees (maximum is $\lfloor n/2 \rfloor$ for Q_n [2]; see [11] for results on more general product networks.) Fink [6] and independently Ramras [14] proved that Q_n could be decomposed into 2^{n-1} isomorphic copies of any tree on n edges. Wagner and Wild [19] proved that Q_n is edge-decomposable into n copies of a *specific* tree on 2^{n-1} edges. Horak, Siran, and Wallis [7] showed that Q_n has an edge decomposition by isomorphic copies of any graph G with n edges each of whose blocks is either an even cycle or an edge. Ramras [15] proved that for a certain class of trees on $2n$ edges, isomorphic copies of these trees edge-decompose Q_n. Other researchers have demonstrated edge decompositions by Hamiltonian cycles for Cartesian products of cycles [16], [1], [4]. Song [18] applies a different construction of this to even-dimensional hypercubes.

We concentrate in this work on the important question of edge decompositions of hypercubes into paths of equal length. Literature on this specific question is not extensive. The cases of n odd and n even are very different, with the theory of edge decompositions of Q_n for n even being dominated by Hamiltonian cycle considerations as noted above. Mollard and Ramras [13] found edge decompositions of Q_n into copies of P_4, the path on 4 edges, for all $n \geq 5$. Our principal result goes far beyond that: we answer the general question of when Q_n for n odd can be edge decomposed into length-m paths. The method of proof involves construction of two new graph-theoretic concepts, the "m-stretch of a graph" (see Definition 2), and an analog of the topological cylinder (see Definition 4) and a variant thereof (Definition 5) that may have wide applicability to edge decomposition studies.

In an earlier version of this article (submitted to the arXiv in August 2013) we conjectured this principal result, and proved it for all m and all odd $n < 2^{2^5}$. We did not initially see how to complete the theorem within our framework but the ## construction in Definition 5 came to us later as the way to bridge the gap, making a satisfying whole of our approach. Meanwhile the truth of the theorem was independently established by J. Erde [5].

2 Notation and Preliminaries

Definition 1 For graphs H and G we say that H *divides* G if there is a collection of subgraphs $\{H_i\}$ each isomorphic to H ($H_i \simeq H$ for all i) for which $E(G)$ is the disjoint union of $\{E(H_i)\}$.

Notation We shall denote "H divides G" by $H <_D G$, since the relation $<_D$ is clearly reflexive and transitive and thus a partial order.

For the q-dimensional hypercube Q_q the vertices are the 2^q q-tuples of 0's and 1's. $V(Q_q)$ has an additive structure of $\mathbb{Z}_2^q$. The edge set $E(Q_q)$ consists of those (unordered) pairs of vertices that differ in exactly one coordinate. The group $\mathbb{Z}_2^q$ acts on the set of edges in the obvious way: for $\gamma \in \mathbb{Z}_2^q$ and $e = \{\alpha, \alpha'\}$ an unordered pair representing an edge of $E(Q_q)$, $\gamma + e$ will denote the edge $\{\gamma + \alpha, \gamma + \alpha'\}$.

The parity of a q-tuple $\alpha = (a_1, \ldots, a_q) \in \mathbb{Z}_2^q$ is $\rho(\alpha) = a_1 + \cdots + a_q$, defined (mod 2). Let B_q be the subgroup of $V(Q_q)$ consisting of those q-tuples with parity 0. For $q \geq 1$, clearly $|\, B_q \,| = |\, V(Q_q) \,|/2 = 2^{q-1}$.

Given an integer $j, 1 \leq j \leq q$, and a vertex $\alpha = (a_1, \ldots, a_q) \in V(Q_q)$, some helpful notation is as follows. Let

$$\overline{j} \cdot \alpha = (a_1, \ldots, 1 + a_j, \ldots, a_q)$$

i.e. alter a_j only. Let

$$j^0 \cdot \alpha = (a_1, \ldots, c, \ldots, a_q),$$

where $c = \rho(\alpha) + a_j$. The idea of j^0 is "alter the j^{th} coordinate if necessary so that the parity is 0". It should be obvious that $j^0 \cdot \alpha = j^0 \cdot (\overline{j} \cdot \alpha) \in B_q$. Likewise, put

$$j^1 \cdot \alpha = \overline{j} \cdot (j^0 \cdot \alpha),$$

i.e. alter the j^{th} coordinate if necessary so that the parity is 1. Notice that $\{\alpha, \overline{j} \cdot \alpha\}$ is an edge of Q_q and that $\{j^0 \cdot \alpha, j^1 \cdot \alpha\}$ is the same edge. Our notation for this edge is $\hat{j} \cdot \alpha$. Then $\hat{j}$ is compatible with the $\mathbb{Z}_2^q$-action, i.e. $\hat{j} \cdot (\gamma + \alpha) = \gamma + \hat{j} \cdot \alpha$. Clearly $\hat{j} \cdot \alpha = \hat{j} \cdot (j^0 \cdot \alpha) = \hat{j} \cdot (j^1 \cdot \alpha) = \hat{j} \cdot (\overline{j} \cdot \alpha)$.

The path P_q of length q is a graph with a vertex set $\{0, 1, \ldots, q\}$ and an edge set $\{\hat{1}, \ldots, \hat{q}\}$, $\hat{k}$ denoting the edge joining $k - 1$ and k. We define graph embeddings $f_\gamma : P_q \longrightarrow Q_q$, for $\gamma \in B_q$, as follows. For $0 \leq k \leq q$ let

$$1^k 0^{q-k} = (\underbrace{1, \ldots, 1}_{k\, 1'\text{s}}, \underbrace{0, \ldots, 0}_{q-k\, 0'\text{s}}) \in V(Q_q)$$

and set

$$f_\gamma(k) = 1^k 0^{q-k} + \gamma.$$

Notice that in $E(Q_q)$,

$$f_\gamma(\hat{k}) = \hat{k} \cdot (1^k 0^{q-k} + \gamma) = 1^k 0^{q-k} + \hat{k} \cdot \gamma = 1^{k-1} 0^{q-k+1} + \hat{k} \cdot \gamma.$$

The family $\{f_\gamma\}$ provides $|\, B_q \,| = 2^{q-1}$ ways of embedding P_q in Q_q, and P_q has q edges, so altogether the family $\{f_\gamma\}$ sends $q \cdot 2^{q-1}$ edges to Q_q while $|\, E(Q_q) \,| = q \cdot 2^{q-1}$. Therefore if the family $\{f_\gamma\}$ cover $E(Q_q)$ then they cover each edge just once, i.e.

the path images of $\{f_\gamma\}$ are pairwise edge-disjoint. To see that this is the case, let $e = \{\alpha, \alpha'\}$ denote any edge of Q_q; then $e = \hat{k} \cdot \alpha$ where the unique coordinate that differs between α and α' is the kth. Put $\gamma = k^0 \cdot (\alpha + 1^k 0^{q-k})$ and observe that $f_\gamma(\hat{k}) = \hat{k} \cdot \alpha = e$. We have proved

Lemma 1 *The family of graph embeddings $\{f_\gamma : P_q \longrightarrow Q_q \,|\, \gamma \in B_q\}$ defines a partition of $E(Q_q)$ into edge-disjoint paths indexed by B_q.*

(As mentioned in the Introduction, a more general result, for *all* trees on q edges, appears in [6] and in [14].)

The results in the next lemma are also in [14] but we include short proofs here so this article can be self-contained.

Lemma 2 (a) $P_2 <_D Q_3$.
(b) *If $P_{2^m} <_D Q_q$, where q is odd, then $q \geq 2^m$.*

Proof. (a) Q_3 may be viewed as an inner Q_2 joined to an outer Q_2 via a perfect matching. Decompose the inner Q_2 into 2 edge-disjoint P_2's. Each of the remaining 8 edges decompose into 4 P_2's, with one edge of the outer Q_2 joined to an incident matching edge.

(b) Every vertex of Q_q has odd degree, so at every vertex at least one embedded path must start or end there. So there must be at least $|\,V(Q_q)\,|/2$ paths, i.e. $q \cdot 2^{q-1}/2^m \geq 2^q/2$, which implies that $q \geq 2^m$. □

3 Stretched Graphs

Definition 2 Let G be a graph and let m be a positive integer. The *m-stretch* of G, denoted $m * G$, is the graph obtained by replacing each edge of G by a path of length m.

Remark The m-stretch of G is a special case of a subdivision of G, in which exactly $m-1$ new vertices are placed along each edge. One might therefore call this an $(m-1)$-subdivision of G, although we shall continue to use the term 'm-stretch of G'.

Lemma 3 (a) $1 * G \simeq G$ *for any graph* G.
(b) $|E(m * G)| = m|E(G)|$.
(c) $|V(m * G)| = (m-1)|E(G)| + |V(G)|$.
(d) $m_1 * (m_2 * G) \simeq (m_1 m_2) * G$.
(e) *If $H <_D G$, then $m * H <_D m * G$.*
(f) $m * P_q \simeq P_{mq}$.

The proofs are trivial.

The importance for hypercubes of stretched graphs comes from the next theorem.

Theorem 2 *$m * Q_q <_D Q_{mq}$ for any $m \geq 1, q \geq 1$.*

For example, from this and Lemmas 2(a) and 3(e,f) it follows easily that $P_6 = 3 * P_2 <_D 3 * Q_3$, which divides Q_9. By transitivity of divisibility, one obtains $P_6 <_D Q_9$, which is already a new result. To prove Theorem 2, the cases of m odd and $m = 2$ are considered separately. It should be clear from Lemma 3(d,e) that if $m_1 * Q_q <_D Q_{m_1 q}$ for any q and if $m_2 * Q_q <_D Q_{m_2 q}$ for any q, then $m_1 m_2 * Q_q <_D Q_{m_1 m_2 q}$ for any q, so the cases of m odd and $m = 2$ suffice.

Proposition 1 *$m * Q_q <_D Q_{mq}$ for m odd, $q \geq 1$.*

Before jumping into the proof, let us establish some notation for vertices and edges of Q_{mq} and $m * Q_q$. We consider a vertex of Q_{mq} to consist of q vectors of length m, (view Q_{mq} as Q_m^q) i.e. $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(q)}), \alpha^{(k)} \in V(Q_m) = \mathbb{Z}_2^m$. As before, 0^m is $(0, \ldots, 0) \in \mathbb{Z}_2^m$ and 1^m is $(1, \ldots, 1) \in \mathbb{Z}_2^m$.

Notation for $m * Q_q$ is as follows. First, each vertex of Q_q is carried over as a vertex into $m * Q_q$, so if $\alpha = (a_1, \ldots, a_q) \in V(Q_q)$, we also view α as a vertex of $m * Q_q$. In addition, for each edge $\hat{j} \cdot \alpha \in E(Q_q)$, let $j^k : \alpha$ denote the kth vertex on the path that replaced $\hat{j} \cdot \alpha$, where $0 \leq k \leq m$. We also identify $j^0 : \alpha$ with α, and $j^m : \alpha$ with $\overline{j} \cdot \alpha$ (which is the other endpoint of $\hat{j} \cdot \alpha$). Note that the edges of $m * Q_q$ connect $j^{k-1} : \alpha$ with $j^k : \alpha, k = 1, \ldots, m$. i The vertices and edges can be counted coming from either end of the path, hence

$$j^k : \alpha = j^{m-k} : (\overline{j} \cdot \alpha).$$

So one must be careful that any definition involving $j^k : \alpha$ is independent of choice of notation. One way to make the above notation unique for the vertices not inherited from $V(Q_q)$ is to apply it only to $\alpha \in B_q$. Then

$$V(m * Q_q) = V(Q_q) \cup \{j^k : \alpha \,|\, 1 \leq k < m, 1 \leq j \leq q, \alpha \in B_q\}.$$

Proof of Proposition 1.
Let $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(q)}) \in B_m^q \subseteq \mathbb{Z}_2^{mq}$, i.e. a vector where each length-m subvector $\gamma^{(i)}$ has parity 0. Define embeddings $F_\gamma : m * Q_q \to Q_{mq}$ as follows. If $\alpha = (a_1, \ldots, a_q) \in V(Q_q)$, put

$$F_\gamma(\alpha) = (a_1^m, a_2^m, \ldots, a_q^m) + \gamma.$$

Otherwise, for a vertex of $m * Q_q$ of the form $j^k : \alpha$, with $\alpha \in B_q$, put

$$F_\gamma(j^k : \alpha) = (c^{(1)}, c^{(2)}, \ldots, c^{(q)}) + \gamma,$$

where

$$c^{(s)} = \begin{cases} a_s^m & \text{if } s \neq j \\ a_j^m + 1^k 0^{m-k} & \text{for } s = j. \end{cases}$$

Note that $F_\gamma(j^0 : \alpha) = F_\gamma(\alpha)$ by this definition and likewise $F_\gamma(j^m : \alpha) = F_\gamma(\bar{j} \cdot \alpha)$, as needed for notational consistency and for F_γ to send edges to edges.

We will show that the $\{F_\gamma\}$ comprise an edge partition of Q_{mq} into copies of $m * Q_q$. Now $|\, E(m * Q_q) \,| = m \cdot (q \cdot 2^{q-1}) = mq \cdot 2^{q-1}$, and with $|\, B_m^q \,| = (2^{m-1})^q = 2^{mq-q}$ embeddings, at most $(2^{mq-q})(mq \cdot 2^{q-1}) = mq \cdot 2^{mq-1}$ edges will be covered by the union of their images. But this is exactly $|\, E(Q_{mq}) \,|$, so the $\{F_\gamma\}$ comprise an edge partition if and only if

$$\bigcup_\gamma F_\gamma(E(m * Q_q)) \supseteq E(Q_{mq}),$$

i.e. it suffices to show that every edge of Q_{mq} is in the image of some F_γ.

Let an edge of Q_{mq} be written as $(\alpha^{(1)}, \ldots, \hat{k} \cdot \zeta, \ldots, \alpha^{(q)})$, where $\alpha^{(s)} \in Q_m$ and $\zeta = k^0 \cdot \alpha^{(j)} \in B_m$. The idea here is that the unique coordinate that changes over the edge is at some position (call it k) of some length-m segment (call it the jth). Put

$$\gamma^{(s)} = (\rho(\alpha^{(s)}))^m + \alpha^{(s)} \text{ for } s \neq j.$$

Then $\gamma^{(s)} \in B_m$ because the parity of m copies of either 0 or 1 is (respectively) either 0 or 1. (Note: This is the *only* place in the proof where the premise that m is odd is used.) Put

$$c = \rho(\alpha^{(1)}) + \ldots + \rho(\alpha^{(j-1)}) + \rho(\alpha^{(j+1)}) + \ldots + \rho(\alpha^{(q)})$$

and set

$$\gamma^{(j)} = k^0 \cdot (c^m + 1^k 0^{m-k} + \zeta).$$

Putting $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(q)})$, we have $\gamma \in B_m^q$. Let

$$b_s = \begin{cases} c & \text{for } s = j \\ \rho(\alpha^{(s)}) & \text{for } s \neq j, \end{cases}$$

and let $\beta = (b_1, \ldots, b_q)$. Then $\beta \in B_q$ and

$$F_\gamma(j^k : \beta) = (b_1^m + \gamma^{(1)}, \ldots, c^m + 1^k 0^{m-k} + \gamma^{(j)}, \ldots, b_q^m + \gamma^{(q)})$$
$$= (\alpha^{(1)}, \ldots, k^{\rho(c+k)} \cdot \zeta, \ldots, \alpha^{(q)}).$$

and likewise

$$F_\gamma(j^{k-1} : \beta) = (b_1^m + \gamma^{(1)}, \ldots, c^m + 1^{k-1} 0^{m-k+1} + \gamma^{(j)}, \ldots, b_q^m + \gamma^{(q)})$$
$$= (\alpha^{(1)}, \ldots, k^{\rho(c+k-1)} \cdot \zeta, \ldots, \alpha^{(q)}).$$

This shows that the edge of $m * Q_q$ that links $j^k : \beta$ and $j^{k-1} : \beta$ is sent via F_γ to the edge $(\alpha^{(1)}, \ldots, \hat{k} \cdot \zeta, \ldots, \alpha^{(q)})$ of Q_{mq}. This completes the proof of Proposition 1.

$\square$

Proposition 2 *We have* $2 * Q_q <_D Q_{2q}$*, for any* $q \geq 1$.

Proof. Again, begin with notation for vertices and edges of Q_{2q} and $2*Q_q$. This time we set up Q_{2q} slightly differently, namely we identify Q_{2q} as $\mathbb{Z}_2^q \times \mathbb{Z}_2^q$ so (α, β) would be the notation for a vertex of Q_{2q}, where $\alpha, \beta \in V(Q_q)$. The notation for $V(2 * Q_q)$ is similar to before, but there is no need for the superscript 'k' because $k = 1$, and we will simply write $j : \alpha$ for the midpoint of $\hat{j} \cdot \alpha$. Notice that $j : \alpha = j : (\overline{j} \cdot \alpha)$. A unique notation for $V(2 * Q_q)$ is implicit in

$$V(2 * Q_q) = V(Q_q) \cup \{j : \alpha \mid \alpha \in B_q, 1 \leq j \leq q\}.$$

Let $j\#\alpha$ denote the unique edge of $2 * Q_q$ connecting α and $j : (j^0 \cdot \alpha)$. For any $\gamma \in B_q$, define the embeddings $F_\gamma^0, F_\gamma^1 : 2 * Q_q \to Q_{2q}$ by

$$F_\gamma^0(\alpha) = F_\gamma^1(\alpha) = (\alpha, \alpha + \gamma);$$

$$F_\gamma^0(j : \alpha) = (j^0 \cdot \alpha, (j^1 \cdot \alpha) + \gamma);$$

$$F_\gamma^1(j : \alpha) = (j^1 \cdot \alpha, (j^0 \cdot \alpha) + \gamma).$$

Note that $|\,E(2 * Q_q)\,| = 2|\,E(Q_q)\,| = 2q \cdot 2^{q-1} = q \cdot 2^q$, and there are 2^{q-1} elements in B_q and so $2 \cdot 2^{q-1} = 2^q$ embeddings. Their combined images cover at most $(2^q)(q \cdot 2^q) = q \cdot 2^{2q}$ edges, and again $|\,E(Q_{2q})\,| = (2q) \cdot 2^{2q-1} = q \cdot 2^{2q}$ also, so the family $\{F_\gamma^\epsilon \mid \epsilon = 0 \text{ or } 1, \gamma \in B_q\}$ provides an edge partition of Q_{2q} into copies of $2*Q_q$ if and only if

$$E(Q_{2q}) \subseteq \bigcup_{\gamma,\epsilon} F_\gamma^\epsilon(E(2 * Q_q)).$$

To prove this we consider two cases. An edge e of Q_{2q} is either $(\hat{j} \cdot \alpha, \beta)$, where $\alpha \in B_q$ and $\beta \in \mathbb{Z}_2^q$, or $(\alpha, \hat{j} \cdot \beta)$, where $\alpha \in \mathbb{Z}_2^q, \beta \in B_q$. Define γ by $\gamma = j^0 \cdot (\alpha + \beta) \in B_q$, and let $\tilde{\alpha} = \beta + \gamma$.

Suppose the edge e is $(\hat{j} \cdot \alpha, \beta)$, with $\alpha \in B_q$ and $\beta \in \mathbb{Z}_2^q$. We split this case further into two subcases. If $\rho(\beta) = 0$, then $\gamma = \alpha + \beta$ and $\tilde{\alpha} = \alpha \in B_q$. We have

$$F_\gamma^1(j : \alpha) = (j^1 \cdot \alpha, (j^0 \cdot \alpha) + \gamma) = (\overline{j} \cdot \alpha, \beta)$$

while

$$F_\gamma^1(\alpha) = (\alpha, \alpha + \gamma) = (\alpha, \beta).$$

Hence F_γ^1 maps $j\#\alpha$ to $(\hat{j} \cdot \alpha, \beta)$.

Otherwise, if $\rho(\beta) = 1$, then $\gamma = \overline{j} \cdot \alpha + \beta$ and $\tilde{\alpha} = \overline{j} \cdot \alpha = j^1 \cdot \alpha$. We find

$$F_\gamma^0(\tilde{\alpha}) = (\tilde{\alpha}, \tilde{\alpha} + \gamma) = (\tilde{\alpha}, \beta) = (\overline{j} \cdot \alpha, \beta)$$

while

$$F_\gamma^0(j : \tilde{\alpha}) = (j^0 \cdot \tilde{\alpha}, (j^1 \cdot \tilde{\alpha}) + \gamma) = (\alpha, \tilde{\alpha} + \gamma) = (\alpha, \beta).$$

So F_γ^0 carries the edge $j\#\tilde{\alpha}$ to $(\hat{j} \cdot \alpha, \beta)$.

Now consider the alternate situation where $e = (\alpha, \hat{j} \cdot \beta) \in E(Q_{2q})$, with $\alpha \in \mathbb{Z}_2^q, \beta \in B_q, j \in \{1, 2, \ldots, q\}$. Then $\gamma = (j^0 \cdot \alpha) + \beta \in B_q$. We consider separately subcases where $\rho(\alpha) = 0$ versus where $\rho(\alpha) = 1$. If $\rho(\alpha) = 0$, note that $j^0 \cdot \alpha = \alpha$ and $F_\gamma^0(\alpha) = (\alpha, \alpha + \gamma) = (\alpha, \beta)$ while $F_\gamma^0(j : \alpha) = (j^0 \cdot \alpha, (j^1 \cdot \alpha) + \gamma) = (\alpha, \overline{j} \cdot \beta)$. So F_γ^0 carries the edge $j\#\alpha$ to $(\alpha, \hat{j} \cdot \beta)$. If instead $\rho(\alpha) = 1$, then $\alpha = j^1 \cdot \alpha$ and

$$F_\gamma^1(j : \alpha) = (j^1 \cdot \alpha, (j^0 \cdot \alpha) + \gamma) = (\alpha, \beta)$$

while

$$F_\gamma^1(\alpha) = (\alpha, \alpha + \gamma) = (\alpha, \overline{j} \cdot \beta)$$

so again, the edge $(\alpha, \hat{j} \cdot \beta)$ is covered. This completes the proof of Proposition 2, and with it the proof of Theorem 2. □

4 For q odd, when does P_m divide Q_q?

This section is devoted to answering the question above. We will show that for q odd, $P_m <_D Q_q$ if and only if $m \leq q$ and $m | q \cdot 2^{q-1}$. The proof has three parts. Part one is to reduce to the case where m is a power of 2, and this is essentially done by Proposition 1. Part two constructs the edge decomposition for m a power of 2, for a small range of odd values of q just above m. Part three extends the result from this small range to all q. The second part is the hardest. We will settle part three in Proposition 3 below, then focus on part two, and then pull the pieces together.

We begin by citing from [16], [1] and [4], the fact that Q_{2n} has an edge-decomposition into Hamiltonian cycles. In this article we will use this fact only in the special case where $2n$ is a power of 2, and for that special case there is a simple and elegant construction which we offer in the Appendix.

Theorem 3 *For every $n \geq 1, Q_{2n}$ has an edge decomposition into n copies of $C_{2^{2n}}$, i.e., $C_{2^{2n}} <_D Q_{2n}$.*

Because $P_{2^t} <_D C_{2^{2n}}$ when $t < 2n$, an immediate corollary is

Corollary 1 *For any $t < 2n, P_{2^t} <_D Q_{2n}$.*

Recall that the Cartesian product of two graphs G and G', denoted $G \Box G'$, is the graph whose vertex set is $V(G) \times V(G')$ and whose edge set consists of pairs that are either $\{(x_1, y), (x_2, y)\}$, where $\{x_1, x_2\}$ is an edge of G and $y \in V(G')$, or $\{(x, y_1), (x, y_2)\}$ where $x \in V(G)$ and $\{y_1, y_2\}$ is an edge of G'. It should be clear that $Q_q \Box Q_{q'} \simeq Q_{q+q'}$. To begin to relate Cartesian products and edge decompositions, we have

Lemma 4 (a) *Let H, G, G' be graphs. If $H <_D G$ and $H <_D G'$ then $H <_D G \Box G'$.* (b) *If $P_m <_D Q_q$ and $P_m <_D Q_{q'}$ then $P_m <_D Q_{q+pq'}$ for any $p \geq 1$.*

Proof. Part (a) is obvious because $E(G\Box G')$ consists of $|V(G)|$ copies of $E(G')$ and $|V(G')|$ copies of $E(G)$. Part (b) is a specialization making use of $Q_q \Box Q_{q'} \simeq Q_{q+q'}$ and induction on p. □

Proposition 3 *Let $t < 2n$ and suppose that $P_{2^t} <_D Q_q$ for all odd integers q in the range $2^t + 1$ to $2^t + 2n - 1$. Then $P_{2^t} <_D Q_q$ for all $q \geq 2^t$ (q odd or even).*

Proof. If q is even and $2^t \leq q$, put $n = q/2$. It follows from Corollary 1 that $P_{2^t} <_D Q_q$ since $t < 2^t \leq q = 2n$. If q is odd, write $q = 2^t + s + p \cdot 2n$ where $0 < s < 2n$, and apply Lemma 4(b) and Corollary 1. □

Proposition 3 shows that for each t only a finite number of Q_q's need to have path decompositions constructed, to infer that $P_{2^t} <_D Q_q$ for all $q \geq 2^t$.

The key idea for the construction is to extend paths shorter than length 2^t to paths of length 2^t. The object we utilize for doing this is defined next.

Definition 3 Let G be a graph. A *disjoint collection of vertex-originating paths of length k*, henceforth DVOP[k], is a collection of paths of length k $\{p_v : P_k \longrightarrow G\}$ indexed by $V(G)$, satisfying
(a) disjointness, i.e. $p_v(\hat{j}) = p_{v'}(\hat{j}') \Longrightarrow v = v'$ and $j = j'$, and
(b) $p_v(0) = v$, i.e. each vertex originates one path.
Here, as above, the vertices of P_k are taken to be $\{0, 1, \ldots, k\}$ and $\hat{j}$ denotes the edge joining $j - 1$ and j. Clearly there are $k|V(G)|$ edges in the combined images of all the paths in a DVOP[k].

Proposition 4 *For $0 \leq k \leq 3$ and $n \geq k$, there is an edge decomposition of Q_{2n} into $n - k$ Hamiltonian cycles and a DVOP[k].*

Proof. The case $k = 0$ merely reiterates Theorem 3. For $k > 0$ start with Theorem 3 giving an edge decomposition of Q_{2n} into n copies of $C_{2^{2n}}$. Call three of the cycles $C^{(1)}, C^{(2)}, C^{(3)}$ (or stop at $C^{(k)}$ if $k < 3$), and choose a direction on each cycle. Define set bijections $h_i : V(Q_{2n}) \longrightarrow V(Q_{2n})$, $1 \leq i \leq k$, by letting $h_i(v)$ be the vertex reached by traveling one edge along $C^{(i)}$ from v, in the chosen direction. Let $p_v : P_k \to Q_{2n}$ be the path defined by: $p_v(0) = v, p_v(1) = h_1(v), p_v(2) = h_2(h_1(v)), p_v(3) = h_3(h_2(h_1(v)))$ (stop sooner if $k < 3$). This is clearly a graph map because $p_v(i-1)$ and $p_v(i)$ are connected by an edge in $C^{(i)}$, and it obviously originates on v. It is a path (i.e. an embedded copy of P_k) if the $k+1$ vertex images are all distinct. Because adjacent vertices have opposite parity on a hypercube, $p_v(0) \neq p_v(1) \neq p_v(2) \neq p_v(3) \neq p_v(0)$. To see that $p_v(0) \neq p_v(2)$ note that these two vertices are connected by distinct edges to $p_v(1)$ so must be distinct in Q_{2n}. Similarly $p_v(1) \neq p_v(3)$ because they are connected by distinct edges to $p_v(2)$. Finally, if two edges coincide, say $p_v(\hat{i}) = p_w(\hat{j})$, then because $p_v(\hat{i}) \in C^{(i)}$ while $p_w(\hat{j}) \in C^{(j)}$ we have $C^{(i)} \cap C^{(j)} \neq \emptyset$ forcing $i = j$. The fact that p_v and p_w follow the same chosen orientations of the $C^{(j)}$'s means that $p_v(i) = p_w(i)$, and then bijectivity of the h_j's leads to $v = w$. □

For generating path decompositions of length 2^t for $t \leq 7$ we rely on

Definition 4 Let G be a graph and $m \geq 1$. Let

$$m\#G$$

denote the graph obtained by drawing two copies of G, (call them G' and G''), and connecting each vertex $v' \in V(G')$ to the corresponding vertex $v'' \in V(G'')$ by a path of length m.

Lemma 5 (a) $1\#G \simeq P_1 \Box G$.
(b) $|\,V(m\#G)\,| = (m+1)|\,V(G)\,|$.
(c) $|\,E(m\#G)\,| = 2|\,E(G)\,| + m|\,V(G)\,|$.

Proof. Trivial. □

Lemma 6 *If m is odd, $m\#Q_q <_D Q_{m+q}$.*

Proof. Denote a vertex of Q_{m+q} as (α, β), where $\alpha \in V(Q_m)$ and $\beta \in V(Q_q)$. Utilize the edge decomposition of Q_m into copies of P_m:

$$\{f_\gamma : P_m \longrightarrow Q_m |\, \gamma \in B_m\}$$

defined in Lemma 1. Denote the embedded copies of Q_q in $m\#Q_q$ as Q'_q and Q''_q. Denote the j^{th} point on the edge of $m\#Q_q$ joining β' to β'' as $\beta_{\langle j\rangle}$. Thus $\beta_{\langle 0\rangle} = \beta'$ and $\beta_{\langle m\rangle} = \beta''$. Define for $\gamma \in B_m$

$$F_\gamma : m\#Q_q \longrightarrow Q_{m+q}$$

by $F_\gamma(\beta_{\langle j\rangle}) = (f_\gamma(j), \beta)$. This is a collection of 2^{m-1} embeddings of $m\#Q_q$ into Q_{m+q}. Since $|\,E(m\#Q_q)\,| = (m+q)2^q$, there are altogether $2^{m-1}(m+q)2^q = (m+q)2^{m+q-1}$ edge images of all the $\{F_\gamma\}$. Since $|E(Q_{m+q})| = (m+q)2^{m+q-1}$, the family $\{F_\gamma\}$ is a decomposition into disjoint copies if their collective images are onto $E(Q_{m+q})$. But this is easy, because an edge of Q_{m+q} is either $(\hat{j}\cdot\alpha, \beta)$, where $\hat{j}\cdot\alpha$ is an edge of Q_m, or $(\alpha, \hat{k}\cdot\beta)$, where $\hat{k}\cdot\beta$ is an edge of Q_q. Clearly $(\hat{j}\cdot\alpha, \beta) \in im(F_\gamma)$ if $\hat{j}\cdot\alpha \in im(f_\gamma)$. The edge $(\alpha, \hat{k}\cdot\beta)$ equals $F_\alpha(\hat{k}\cdot\beta')$ if α has even parity, and it equals $F_\gamma(\hat{k}\cdot\beta'')$ for $\gamma = \alpha + 1^m$ if α has odd parity (note that m must be odd for this to work). □

Lemma 7 *Suppose G has an edge decomposition into a DVOP$[k']$ and a complementary edge set E', as well as an edge decomposition into a DVOP$[k'']$ and a complementary edge set E''. Then $m\#G$ has an edge decomposition into $|\,V(G)\,|$ copies of $P_{k'+m+k''}$ and one copy each of E' and E''.*

Proof. Let $\{p_{v'} : P_{k'} \longrightarrow G\}$ and $\{p_{v''} : P_{k''} \longrightarrow G\}$ be the DVOP's. Simply concatenate the paths $p_{v'} \in G'$, the path from v' to v'' in $m\#G$, and the path $p_{v''}$ in G'', to make the path $\tilde{p_v} : P_{k'+m+k''} \longrightarrow m\#G$. Then $\{E(\tilde{p_v})\,|\, v \in V(G)\,\} \cup E' \cup E''$ is an edge decomposition of $m\#G$. □

Proposition 5 (a) $P_4 <_D Q_5$.
(b) $P_4 <_D Q_7$.
(c) $P_8 <_D Q_9$.
(d) $P_8 <_D Q_{11}$.

Proof. (a). Viewing Q_5 as $1\#Q_4$, apply Proposition 4 to obtain $E(Q_4)$ as a DVOP[2] (with an empty complementary set) and also write $E(Q_4)$ as a DVOP[1] with a single C_{16} as the complementary set. We have an application of Lemma 7 with $k' = 2, k'' = 1, m = 1$. Thus $E(Q_5)$ decomposes into 16 copies of P_4 and one copy of C_{16}. Since C_{16} is 4 copies of P_4, we have shown that $P_4 <_D Q_5$.
(b) Apply Proposition 4 for a DVOP[1] and a complementary set C_{16} in Q_4, as well as an empty DVOP[0] and a complementary set consisting of 2 copies of C_{16}. Then $3\#Q_4$ has an edge decomposition into 16 copies of $P_{1+3+0} = P_4$ and 3 copies of C_{16}, i.e. $P_4 <_D 3\#Q_4$. By Lemma 6, $P_4 <_D Q_7$.
(c) The proof that $P_8 <_D 5\#Q_4$ likewise applies Lemma 7 with $k' = 2$ and $k'' = 1$, but now with $m = 5$ so that paths have length $k' + m + k'' = 8$. It follows that $5\#Q_4$ has an edge decomposition into 16 copies of P_8 and one copy of C_{16}, hence a decomposition into 18 copies of P_8. We have $P_8 <_D 5\#Q_4 <_D Q_9$, so $P_8 <_D Q_9$.
(d) Similarly, $P_8 <_D 7\#Q_4$ (Lemma 7 with $k' = 1, k'' = 0, m = 7$), and $7\#Q_4 <_D Q_{11}$. □

Corollary 2 *$P_4 <_D Q_q$ for all $q \geq 4$ and $P_8 <_D Q_q$ for all $q \geq 8$.*

Proof. This follows immediately from Propositions 3 (put $n = 2$) and 5. □

Note: The first half of Corollary 2 was proved by an ad hoc method in [11].

Lemma 8 *Q_{2n} has a DVOP[n] (with an empty complementary set).*

Proof. Let $f_\gamma : P_n \longrightarrow Q_n$ be defined as before, but without the restriction that $\gamma \in B_n$. Write a vertex of Q_{2n} as (α, β), where $\alpha \in V(Q_n), \beta \in V(Q_n)$. Let

$$p_{(\alpha,\beta)}(j) = \begin{cases} (f_\alpha(j), \beta) & \text{if } \alpha + \beta \text{ has even parity} \\ (\alpha, f_\beta(j)) & \text{if } \alpha + \beta \text{ has odd parity.} \end{cases}$$

Then $p_{(\alpha,\beta)} : P_n \longrightarrow Q_{2n}$ is a path and $p_{(\alpha,\beta)}(0) = (\alpha, \beta)$. The family $\{p_{(\alpha,\beta)}\}$ provides 2^{2n} paths of length n, and $|E(Q_{2n})| = n2^{2n}$, so the family is edge disjoint if and only if their images together cover $E(Q_{2n})$. An edge of $E(Q_{2n})$ is either $(\hat{k} \cdot \alpha, \beta)$ for some k and for some $\alpha \in B_q$, or $(\alpha, \hat{k} \cdot \beta)$ where $\beta \in B_q$. For an edge that is $(\hat{k} \cdot \alpha, \beta)$, use the edge surjectivity of the $\{f_\gamma\}$ to choose $\gamma \in B_q$ for which $f_\gamma(\hat{k}) = \hat{k} \cdot \alpha$ and put $\tilde{\alpha} = k^{\rho(\beta)} \cdot \gamma$. Then $\rho(\tilde{\alpha} + \beta) = 0$ so $p_{(\tilde{\alpha},\beta)}(\hat{k}) = (\hat{k} \cdot \alpha, \beta)$. Likewise for an edge that is $(\alpha, \hat{k} \cdot \beta)$, choose γ so that $f_\gamma(\hat{k}) = \hat{k} \cdot \beta$ and put $\tilde{\beta} = k^{1-\rho(\alpha)} \cdot \gamma$. Then $p_{(\alpha,\tilde{\beta})}(\hat{k}) = (\alpha, \hat{k} \cdot \beta)$. □

To get results like Corollary 2 for P_{16}, we need to make use of Q_8.

Lemma 9 *For $0 \leq k \leq 4, Q_8$ has a DVOP$[k]$ and a complementary set that consists of $4-k$ copies of C_{256}.*

Proof. Use Lemma 8 (for $k = 4$) and Proposition 4 (for $k < 4$). $\square$

Lemma 10 *For $t = 4, 5, 6, 7$, and for $s = 1, 3, 5, 7$, we have $P_{2^t} <_D Q_{2^t+s}$.*

Proof. Let $m = 2^t + s - 8$. Put $k' = 0$ if $s > 4$ and put $k' = 4$ if $s < 4$. Put $k'' = 8 - s - k'$, which will equal either 1 or 3 depending on s. Apply Lemmas 9, 7, and 6 to deduce that $P_{2^t} <_D m\#Q_8 <_D Q_{m+8} = Q_{2^t+s}$. We are using the premise that $t < 8$ to infer that P_{2^t} divides the complementary set consisting of copies of C_{256}. $\square$

Corollary 3 *For q odd and for $t = 4, 5, 6, 7$, we have $P_{2^t} <_D Q_q$ if and only if $q \geq 2^t$.*

Proof. This follows from Proposition 3 (put $n = 4$) and Lemma 10. $\square$

To extend beyond P_{128}, two approaches were considered. Lemma 7 will do it if we can construct DVOP$[k]$'s in Q_{2^r} for k up to 2^{r-1}. The Hamiltonian cycle method as used above can be extended, but the hard part is demonstrating that one gets true paths rather than path images, i.e. that no vertex is repeated. As mentioned at the end of the Introduction, an earlier version of this article did that, for $r = 4$ and $r = 5$, but we were unable to go beyond $r = 5$. The second approach, developed next, is to generalize Definition 4.

Definition 5 is illustrated in Figure 1. Like a cubist painting we will break up the "faces" of $m\#G$, i.e. the embedded copies of G, and move the pieces around while preserving all of them. Consider an edge partition $E(G) = S_1 \cup \ldots \cup S_p$ and let $s' = (s'_1, \ldots, s'_p)$ and $s'' = (s''_1, \ldots, s''_p)$ be two lists of positions along the path $P_m = [0, \ldots, m]$. Referring to Figure 1, components of G' are translated to the levels specified by s', and components of G'' are translated to the levels specified by s''.

Definition 5 Let $m \geq 1$ and $p \geq 1$. Let $s' = (s'_1, \ldots, s'_p)$ and $s'' = (s''_1, \ldots, s''_p)$ be lists of integers where $s'_i \in [0, \ldots, m]$ and $s''_i \in [0, \ldots, m]$. Let G be a graph and let $E(G) = S_1 \cup \ldots \cup S_p$ be an edge partition of G into p parts. Define a graph

$$m\#\#G$$

as follows. It is a subgraph of $P_m \square G$. Its vertex set is $[0, \ldots, m] \times V(G)$. Its edge set consists of

$$(E(P_m) \times V(G)) \cup \bigcup_{i=1}^{p} (s'_i \times S_i) \cup \bigcup_{i=1}^{p} (s''_i \times S_i).$$

Remark It will be convenient to refer to the induced subgraph of $m\#\#G$ on $i \times V(G)$ as "level i" (cf. Figure 1). Note that s', s'', and the partition are utilized by the definition but are suppressed from the notation. Note as well that $m\#G$ is the special case where $p = 1, s'_1 = 0$, and $s''_1 = m$.

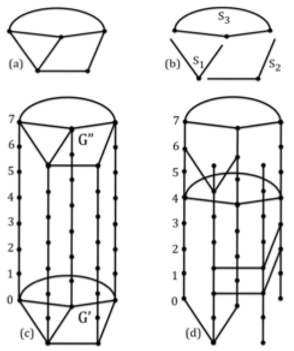

Figure 1: (a) Graph G. (b) Edge partition of G. (c) $m\#G$ (for $m = 7$). (d) $m\#\#G$ construction for $m = 7, s' = (0, 2, 4)$ and $s'' = (6, 3, 7)$.

Lemma 11 *Let $E(Q_q) = S_1 \cup \ldots \cup S_p$ denote any edge decomposition of Q_q and let m be odd. Suppose $s' = (s'_1, \ldots, s'_p)$ is a list of even integers in $[0, \ldots, m]$ and suppose $s'' = (s''_1, \ldots, s''_p)$ is a list of odd integers in $[0, \ldots, m]$. Then $m\#\#Q_q <_D Q_{m+q}$.*

Proof. The proof of Lemma 6 can be copied practically verbatim, until the very end when one is verifying that an edge of Q_{m+q} of the type $(\alpha, \hat{k} \cdot \beta)$ lies in the image of some F_γ. Locate $\hat{k} \cdot \beta$ in a part S_i. If $\rho(\alpha) = 0$ put $\gamma = 1^{s'_i} 0^{m-s'_i} + \alpha$. If $\rho(\alpha) = 1$ put $\gamma = 1^{s''_i} 0^{m-s''_i} + \alpha$. Then $(\alpha, \hat{k} \cdot \beta) \in im(F_\gamma)$. □

Lemma 12 *Let $E(G) = S_1 \cup \ldots \cup S_p$ be an edge decomposition of a graph G into p parts and suppose each S_i is a DVOP[1]. Let $m \geq 1$ be odd, and suppose $s' = (s'_1, \ldots, s'_p) \in [0, \ldots, m]^p$ and $s'' = (s''_1, \ldots, s''_p) \in [0, \ldots, m]^p$ have the property that $|\{s'_1, \ldots, s'_p, s''_1, \ldots, s''_p\}| = 2p$, i.e. the $\{s'_i\}_{1 \leq i \leq p}$ and $\{s''_j\}_{1 \leq j \leq p}$ are all distinct. (This forces m to be at least $2p-1$.) Then for any k', k'' satisfying $0 \leq k', k'' \leq p, m\#\#G$ has an edge decomposition into $|\, V(G)\, |$ copies of $P_{m+k'+k''}$ and a complementary set consisting of $\{s'_j \times S_j\}_{j>k'} \cup \{s''_j \times S_j\}_{j>k''}$.*

Proof. Because of $\{s'_i\} \cup \{s''_j\}$ being distinct, at each level $i \in [0, \ldots, m]$ there is either one DVOP[1] or no edges at all. First let us consider the case where $k' = k'' = p$. Given any vertex $v \in V(G)$, define a path originating at v as follows. If there is a DVOP at level 0, move along it for one edge; if no DVOP at level 0 skip that step. Then move "vertically", i.e. along the embedded P_m, for one edge. Repeat at level 1: if there is a DVOP traverse it for one edge, then take the unique edge to level 2. Continue to alternate DVOP's (when present) and vertical steps. The result is a path of length $m + 2p$. No vertices repeat because the path spends at most one step within any level. The paths are pairwise edge-disjoint because there is a unique way to go forward or backward at each stage, and they cover $E(m\#\#G)$ with empty complementary set.

If $k' < p$ or $k'' < p$, perform the DVOP steps only at levels $\{s'_1, \ldots, s'_{k'}\}$ and $\{s''_1, \ldots, s''_{k''}\}$. The complementary set consists of the unused parts, and each path's length is $m + k' + k''$. $\square$

The generalization of Proposition 5 and Lemma 10 is

Proposition 6 *Let $t \geq 8$, and choose r so that $2^{r-1} \leq t < 2^r$. Then*
(a) *For s an odd integer in the range $1 \leq s \leq 2^r - 1$, $P_{2^t} <_D Q_{2^t+s}$; and*
(b) *A positive odd integer q satisfies $P_{2^t} <_D Q_q$ if and only if $q > 2^t$.*

Proof. (b) follows from (a) and Proposition 3 .
For (a), as in Lemma 10 put $m = 2^t + s - 2^r$ and let k' be either 0 if $s > 2^{r-1}$, or $k' = 2^{r-1}$ if $s < 2^{r-1}$. Put $k'' = 2^r - s - k'$. Then k'' is an odd number between 0 and 2^{r-1}. Note that $m + k' + k'' = 2^t$. Also, because $t \geq 8$ we have $r \geq 4$ and so $2^{r-1} > r + 1$, from which we deduce $2^t \geq 2^{2^{r-1}} > 2^{r+1} = 2^r + 2^r$. Hence

$$m = 2^t + s - 2^r > 2^t - 2^r > 2^r = 2p,$$

where $p = 2^{r-1}$.

Use the partition of Q_{2^r} into $p = 2^{r-1}$ copies of $C_{2^{2^r}}$ each of which is oriented to make it a DVOP[1]. Put $s' = (0, 2, 4, \ldots, 2^r - 2)$ and $s'' = (1, 3, 5, \ldots, 2^r - 1)$, These lists have length p and we verified above that $m > 2p$, so s' and s'' are in $[0, \ldots, m]^p$ and consist of all-distinct even and odd entries respectively. Apply Lemmas 11 and 12 and the fact that P_t divides the complementary parts that are $C_{2^{2^r}}$ to infer $P_{2^t} <_D m\#\#Q_{2^r} <_D Q_{m+2^r} = Q_{2^t+s}$. $\square$

Combining Proposition 6 with Corollaries 2 and 3 and the trivial $P_2 <_D Q_q$ for $q \geq 2$, we have shown

Theorem 4 *For any $t \geq 1$ and for q odd, $P_{2^t} <_D Q_q$ if and only if $q > 2^t$.*

$\square$

Theorem 5 *Let q be odd. A necessary and sufficient condition for P_m to divide Q_q is that $m \leq q$ and $m \mid q \cdot 2^{q-1}$.*

Proof. For necessity, that $m \mid q \cdot 2^{q-1}$ is obvious since $|\,E(Q_q)\,|$ must be a multiple of $|\,E(P_m)\,|$. Because every vertex of Q_q has odd degree, at least one path must start or end there. Each path provides just two "starts" or "ends", and there are $q \cdot 2^{q-1}/m$ paths, hence $2(q \cdot 2^{q-1}/m) \geq |\,V(Q_q)\,| = 2^q$. This reduces to $q \geq m$.

For sufficiency, let $d = gcd(m, q)$. Because q is odd, d is odd. Consider the cases $d = 1$ and $d > 1$ separately. If $d = 1$, $m \mid 2^{q-1}$ so m is a power of 2. Let 2^t be the largest power of 2 that is smaller than q. Since $m < q$, $m \mid 2^t$. So $P_m <_D P_{2^t}$ and we only have to show that $P_{2^t} <_D Q_q$. This is covered by Theorem 4.

Now suppose $d > 1$. We reduce to the case $d = 1$. Let $m' = m/d$ and let $q' = q/d$. Then $m' \mid q' \cdot 2^{q-1}$. But m' and q' are relatively prime so $m' \mid 2^{q-1}$, making m' a power of 2. Since $m' \leq q' \leq 2^{q'-1}$, we see that $m' \,|\, 2^{q'-1} \,|\, q' \cdot 2^{q'-1}$. Then m' and q' are relatively prime so fall under the previous case, hence $P_{m'} <_D Q_{q'}$, i.e. $P_{m/d} <_D Q_{q/d}$. Apply Theorem 2 and Lemma 3(e) to see that $P_m = d * P_{m/d} <_D d * Q_{q/d} <_D Q_q$. $\square$

Appendix: Simple construction of Hamiltonian cycle decomposition of Q_{2^r}

For $r \geq 1$, we define a set of 2^{r-1} Hamiltonian cycles in Q_{2^r} indexed by $\delta = (d_1, \ldots, d_{r-1}) \in \mathbb{Z}_2^{r-1}$, denoted $g_\delta : C_{2^{2^r}} \longrightarrow Q_{2^r}$, as follows. For $r = 1$ there is just one cycle, denoted $g : C_4 \longrightarrow Q_2$, which traces the unique cycle starting at 0^2, i.e. $g(0) = (0,0); g(1) = (1,0); g(2) = (1,1); g(3) = (0,1)$. The vertices of C_n are identified with the integer range $[0, n-1]$ viewed modulo n. For $r = 2$ define $g_0, g_1 : C_{16} \longrightarrow Q_4$ by

$$g_0(4u + v) = (g(v-u), g(u)); g_1(4u+v) = (g(u), g(v-u)); \text{ where } 0 \leq u, v \leq 3.$$

Then $\{g_0, g_1\}$ is a set of two maps from $[0, 15]$ to Q_4 indexed by $\mathbb{Z}_2$. They are illustrated in Figure 2 and their images are edge-disjoint cycles that partition $E(Q_4)$.

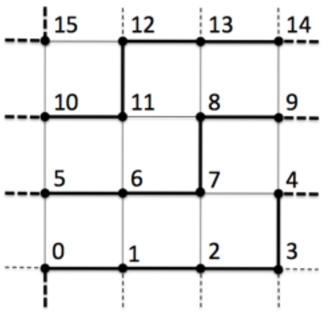

Figure 2: Illustration of Q_4 with the cycle g_0 in thicker lines and the cycle g_1 in thinner lines. For $0 \leq w \leq 15$, vertex labeled w is $g_0(w)$. Dashed lines are edges that wrap around to connect with the vertex on the opposite side of the diagram, *e.g.* "4" is joined by an edge of Q_4 to "5", and "14" has eges joining it to "15" and to "3".

Now let $r \geq 2$ and suppose the $\{g_\delta : C_{2^{2^r}} \longrightarrow Q_{2^r} \,|\, \delta \in \mathbb{Z}_2^{r-1}\}$ have been defined. The vertices of $Q_{2^{r+1}}$ will be identified with $Q_{2^r} \square Q_{2^r}$ and may be written as (α, β), where $\alpha, \beta \in V(Q_{2^r})$. For $\delta = (d_1, \ldots, d_{r-1}) \in \mathbb{Z}_2^{r-1}$, let $\delta 0$ (respectively $\delta 1$) denote $(d_1, \ldots, d_{r-1}, 0)$ (respectively $(d_1, \ldots, d_{r-1}, 1)) \in \mathbb{Z}_2^r$. Define the cycles $\{g_{\delta 0} : C_{2^{2^{r+1}}} \longrightarrow Q_{2^{r+1}}\}$ and $\{g_{\delta 1} : C_{2^{2^{r+1}}} \longrightarrow Q_{2^{r+1}}\}$ by these formulas:

$$g_{\delta 0}(2^{2^r} u + v) = (g_\delta(v-u), g_\delta(u)),$$

$$g_{\delta 1}(2^{2^r} u + v) = (g_\delta(u), g_\delta(v-u)),$$

for $u, v \in [0, 2^{2^r} - 1]$. Taken together, $\{g_{\delta 0}\} \cup \{g_{\delta 1}\}$ is a set of 2^r cycles in $Q_{2^{r+1}}$, indexed by $\mathbb{Z}_2^r$, that comprises the construction for $r + 1$.

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