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Eternal domination on $3 \times n$ grid graphs

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Abstract

In the eternal dominating set problem, guards form a dominating set on a graph and at each step, a vertex is attacked. After each attack, if the guards can "move" to form a dominating set that contains the attacked vertex, then the guards have successfully defended against the attack. We wish to determine the minimum number of guards required to successfully defend against any possible sequence of attacks, the eternal domination number.

Since the domination number for grid graphs has been recently determined [Gonçalves et al., *SIAM J. Discrete Math.* 25 (2011), 1443–1453] grid graphs are a natural class of graphs to consider for the eternal dominating set problem. Though the eternal domination number has been determined for $2 \times n$ grids and $4 \times n$ grids, it has remained only bounded for the $3 \times n$ grid. The results in this paper provide major improvements to both the upper and lower bounds of the eternal domination number for $3 \times n$ grid graphs. In particular, we show the conjectured value in [Goldwasser et al., *Util. Math.* 91 (2013), 47–64] is too small for certain values of n.

1 Introduction and Definitions

In *graph protection*, mobile agents or guards are placed on vertices in order to defend against a sequence of attacks on a network. Interest began on the topic with a series of papers appearing in *Scientific American*, *John Hopkins Magazine*, *American Mathematical Monthly*, and *Military Operations Research* in the late twentieth century. These papers considered Emperor Constantine's strategies for defending the vast Roman Empire against enemy attacks. See the survey [10] for more background and the state of the art of *graph protection*.

In this paper, we consider the "all guards move" model of the eternal dominating set problem. Informally, a set of guards initially form a dominating set on a graph, where each vertex either contains a guard or is adjacent to one containing a guard. At each step, a vertex is attacked. After each attack, if the guards can "move" so that a guard is located on the attacked vertex and the set of guards again forms a dominating set on the graph, then we say the guards have *successfully defended against the attack*. We note, however, that the guards' movements are restricted: after an attack, each guard may remain where it is or move to a neighbouring vertex. We wish to find the minimum number of guards to defend against *any possible* sequence of attacks on a particular graph G. We denote this parameter $\gamma_{\rm all}^{\infty}(G)$ and refer to it as the "eternal domination number" of G. This "all guards move model" or "multiple guards move version" was introduced by Goddard et al.[3], where it was called the "eternal m-security number" and $\gamma_{\rm all}^{\infty}(G)$ was denoted $\sigma_m(G)$.

General bounds of $\gamma(G) \leq \gamma_{\rm all}^{\infty}(G) \leq \alpha(G)$ were determined in [3], where $\gamma(G)$ denotes the minimum cardinality among all dominating sets of G and $\alpha(G)$ the maximum cardinality among all independent sets of G (sets of vertices where no two vertices in the set are adjacent). The exact eternal domination number for various classes of graphs was also given in [3]. Though $\gamma_{\rm all}^{\infty}(G) = \gamma(G)$ for all Cayley graphs G (see [3]), a characterization of those graphs G for which $\gamma_{\rm all}^{\infty}(G) = \gamma(G)$ remains unknown. The eternal domination number of trees was determined in [9], and shows the upper bound of the inequality given by [3] is tight. We consider previous results on the grid graphs below, but for additional results, see [1, 2, 3, 4, 5, 8, 9], or the survey [10].

The $m \times n$ grid graph is often denoted $P_m \,\square\, P_n$ as it is the Cartesian product of two paths P_m and P_n. The domination number of $P_m \,\square\, P_n$ has been well-studied over the past thirty years and is known for all values of m and n. Indeed, the first results appeared thirty years ago in [7] and the final results appeared two years ago in [6]. Consequently, grid graphs are a natural class of graphs to consider for the eternal dominating set problem, with the domination number providing a lower bound for $\gamma_{\rm all}^{\infty}(G)$.

Although the values for all n are known for $\gamma_{\rm all}^{\infty}(P_2 \,\square\, P_n)$ (see [5]) and $\gamma_{\rm all}^{\infty}(P_4 \,\square\, P_n)$ (see [2]), the values for $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n)$ are largely unknown. After determining

$$\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n) = n \quad \text{for } 2 \leq n \leq 8,$$

Goldwasser, Klostermeyer and Mynhardt [5] found the surprising result of

$$\gamma_{\text{all}}^{\infty}(P_3 \,\square\, P_9) = 8,$$

which provides the general upper bound

$$\gamma_{\text{all}}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{8n}{9} \right\rceil \text{ for } n \geq 9. \tag{1}$$

Ultimately, they made the following conjecture.

Conjecture 1 *[5] For $n > 9$, $\gamma_{\text{all}}^{\infty}(P_3 \,\square\, P_n) = \lceil \frac{4n}{5} \rceil + 1$.*

The main contribution of this paper is a significant improvement to both the upper and lower bound on the eternal domination number for $P_3 \,\square\, P_n$. We improve the lower bound from $\lfloor \frac{3n+4}{4} \rfloor$ (the domination number, see [7]) to $\lceil \frac{4n+1}{5} \rceil + 1$ and we improve the upper bound from (1) to $\lceil \frac{6n+2}{7} \rceil$ for $n \geq 2$.

The organization of the paper is as follows. In Section 2, we determine various bounds on $\gamma_{\text{all}}^{\infty}(P_3 \,\square\, P_n)$ to be used in subsequent sections and prove the lower bound of $\lceil \frac{4n}{5} \rceil + 1$ in Theorem 6. In Section 3, we determine the eternal domination number of $P_3 \,\square\, P_{15}$ and a lower bound for $P_3 \,\square\, P_n$ when $n \geq 20$ is a multiple of 5. This gives a new lower bound of $\lceil \frac{4n+1}{5} \rceil + 1$ (Theorem 14). These results provide counterexamples to Conjecture 1. Finally, in Section 4, we state the eternal domination number for some n values and derive a new upper bound of $\lceil \frac{6n+2}{7} \rceil$ for $n \geq 2$ in Theorem 16.

We conclude this section with formal definitions. Let $G = (V, E)$ be a graph. A *dominating set* of G is a subset of V whose closed neighbourhood is V. The smallest cardinality of a dominating set is denoted $\gamma(G)$ and is called the *domination number* of G. Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. Let $D, D' \in \mathbb{D}_q(G)$. We will say D can be *transformed* to D' (or D *transforms* to D') if $D = \{v_1, v_2, \ldots, v_q\}$, $D' = \{u_1, u_2, \ldots, u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \ldots, q$.

In the "eternal dominating set problem", a defender is given q guards to protect the graph from a series of attacks on vertices made by an attacker. An *eternal dominating family of G* is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and D transforms to D'. When the value of q in the above definition is known we will refer to this family as an eternal dominating family with q guards. A set $D \in \mathbb{D}_q(G)$ is an *eternal dominating set* if it is a member of some eternal dominating family. Note that the set of all eternal dominating sets of a particular cardinality is an eternal dominating family, provided the family is non-empty.

The *Cartesian product* of graphs G and H is denoted by $G \,\square\, H$. The vertex set of $G \,\square\, H$ is $V(G \,\square\, H) = \{(u, v) | u \in V(G), v \in V(H)\}$, and two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. When $G = P_m$ and $H = P_n$, these graphs are also known as *grids* or *grid graphs* of dimensions $m \times n$. Label the vertices of P_m (respectively P_n) in their usual ordering $u_1, u_2, \ldots, u_m$ (resp. $v_1, v_2, \ldots, v_n$). In this paper, we discuss the eternal domination numbers of grid graphs with $m = 3$. Each copy of P_3, corresponding to a vertex of P_n is referred to as a column. We refer to each of the columns as the first column, second column, etc. and as column 1, column 2, etc. starting from one the columns corresponding to a leaf of P_n and proceeding consecutively.

In constructing eternal dominating families we make use of the symmetries of the $3 \times n$ grid graph. Given a dominating set $D \in \mathbb{D}_q(P_3 \square P_n)$, a vertical reflection of D (about the horizontal line of symmetry) is denoted D_v, while a horizontal reflection (about the vertical line of symmetry) is denoted D_h. A rotation of a dominating set D by 180° (which is the same as both the vertical reflection of D_h and the horizontal reflection of D_v) is denoted D_r. When we wish to discuss an arbitrary symmetry of a dominating set D, we denote it D_s.

For example, in the eternal dominating family for $P_3 \square P_4$ illustrated in Table 1, there are four dominating sets, or one set and its three symmetries. In each set, a vertex containing a guard is denoted by a bullet (•). Each vertex not containing a guard is labeled with the name of the set to which to transform in order to protect against an attack on that vertex. In the table, any of the sets can transform to any of the other sets; thus the protection against an attack on the middle vertex of the first column of set D by transformation to D_h could also be accomplished by a transformation to D_r.

	1	2	3	4
	•	D_v	D_r	D_h
D	D_h	D_h	•	•
	D_v	•	D_h	D_r

	1	2	3	4
	D	D_v	D_r	•
$\mathbf{D_h}$	•	•	D	D
	D_v	D	•	D_r

	D	•	D_r	D_h
$\mathbf{D_v}$	D_r	D_r	•	•
	•	D	D_h	D_r

	D	D_v	•	D_h
$\mathbf{D_r}$	•	•	D_v	D_v
	D_v	D	D_h	•

Table 1: Eternal dominating family for $P_3 \square P_4$ with 4 guards.

Theorem 2 *[2] Given dominating sets $D, E \in \mathbb{D}_q(P_m \square P_n)$ and any arbitrary symmetry s resulting from a reflection or rotation, D transforms to E if and only if D_s transforms to E_s.*

2 A Lower Bound on the Number of Guards

We begin with an observation of Goldwasser et al. [5]. We note that, by symmetry, statements and arguments referring to the first i columns also apply to the last i columns, for any i.

Lemma 3 *[5] The $3 \times n$ grid graph, $n > 5$, cannot be defended if at any time the first six columns have at most four guards.*

Lemma 3 implies that in an eternal dominating set of $P_3 \square P_n$ there are at least five guards in the first six columns. To obtain a lower bound we extend this result.

Lemma 4 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$. If there are at least k guards in the first i columns for each dominating set $D \in \mathcal{E}$, then for any set $D' \in \mathcal{E}$ all of the following hold.*

1. *If there are at most k guards in the first $i+1$ columns, then there are k guards in the first i columns, no guards in column $i+1$ and three guards in column $i+2$.*
2. *If there are at most $k+1$ guards in the first $i+2$ columns, then there are $k+1$ guards in the first $i+1$ columns, no guards in column $i+2$ and at least two guards in column $i+3$.*
3. *If there are at most $k+2$ guards in the first $i+3$ columns, then there are $k+2$ guards in the first $i+2$ columns, no guards in column $i+3$ and at least two guards in column $i+4$.*
4. *If there are at most $k+3$ guards in the first $i+4$ columns, then there are $k+3$ guards in the first $i+3$ columns, no guards in column $i+4$ and at least one guard in column $i+5$.*
5. *There are at least $k+4$ guards in the first $i+5$ columns.*

Proof: Assume the guards are positioned on the vertices of $D' \in \mathcal{E}$. As $\mathcal{E}$ is an eternal dominating family, we can insist that after an attack the defender moves the guards in such a way that D' is transformed to a set in $\mathcal{E}$.

1. There are k guards in the first i columns and hence there are no guards in column $i+1$. If an attack occurs in column $i+1$ and a guard from column i responds, then there remains at most $k-1$ guards in the first i columns, a contradiction. Thus, the response to any attack in column $i+1$ must be made by a guard in column $i+2$, requiring three guards in column $i+2$.

2. If there are at most k guards in the first $i+1$ columns, then by 1., there are k guards in the first $i+1$ columns and 3 guards in column $i+2$, which yields a contradiction (of $k+3$ guards in the first $i+2$ columns). Thus, there are $k+1$ guards in the first $i+1$ columns and no guards in column $i+2$. As there are at least k guards in the first i columns, there is at most one guard in column $i+1$ (which dominates at most one vertex in column $i+2$). Consequently, there must be at least 2 guards in column $i+3$ (to dominate the remaining two vertices in column $i+2$).

3. If there are at most $k+1$ guards in the first $i+2$ columns, then by 2., there are $k+1$ guards in the first $i+1$ columns and at least two guards in column $i+3$, which yields a contradiction in the number of guards. Thus, there are $k+2$ guards in the first $i+2$ columns and no guards in column $i+3$. If there are two or more guards in column $i+2$ there are at most k guards in the first $i+1$ columns and hence

by 1., k guards in the first i columns, no guards in column $i+1$ and three guards in column $i+2$. This yields a contradiction (of $k+3$ guards in the first $i+2$ columns); therefore, there is at most one guard in column $i+2$ (which dominates at most one vertex in column $i+3$). Consequently, there must be at least 2 guards in column $i+4$ (to dominate the remaining two vertices in column $i+3$).

4. If there are at most $k+2$ guards in the first $i+3$ columns, then by 3., there are $k+2$ guards in the first $i+2$ columns and at least two guards in column $i+4$, which yields a contradiction in the number of guards. Thus, there are exactly $k+3$ guards in the first $i+3$ columns and no guards in column $i+4$. If there are three guards in column $i+3$, then there are at most k guards in the first $i+1$ columns and hence by 1., k guards in the first i columns and three guards in column $i+2$. Thus, there are $k+6$ guards in the first $i+3$ columns, a contradiction. Consequently, there are at most two guards in column $i+3$ (each dominates at most one vertex in column $i+4$), and there is at least one guard in column $i+5$ (to dominate the remaining vertex in column $i+4$).

5. If there are more than $k+3$ guards in the first $i+4$ columns, then the result follows. If there are at most $k+3$ guards in the first $i+4$ columns, then 4. implies there are $k+3$ guards in the first $i+3$ columns and at least one guard in column $i+5$; the result follows. ■

Corollary 5 *In any eternal dominating set of $P_3 \,\square\, P_n$, for any $\ell \geq 2$, the first ℓ columns contain at least $\left\lceil \frac{4\ell-3}{5} \right\rceil$ guards.*

Proof: We proceed with strong induction on ℓ. For each $\ell = 2, \dots, 7$, the result is either implied by Lemma 3 or is easy to check. Suppose the result holds for all $\ell = 2, 3, \dots, s-1$, for some $s > 7$. Recall that the non-empty set of all eternal dominating sets of a particular cardinality forms an eternal dominating family. Then all eternal dominating sets have at least $\left\lceil \frac{4(s-5)-3}{5} \right\rceil$ guards in the first $s-5$ columns. By Lemma 4 (5.), all eternal dominating sets have at least $\left\lceil \frac{4(s-5)-3}{5} \right\rceil + 4 = \left\lceil \frac{4s-3}{5} \right\rceil$ guards in the first s columns. ■

We now show the conjectured value for $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n)$ given by Goldwasser et al. [5] in Conjecture 1 is, in fact, a lower bound.

Theorem 6 *For $n \geq 10$, $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n) \geq \left\lceil \frac{4n}{5} \right\rceil + 1$.*

Proof:

Claim: Let $\mathcal{E}$ be an eternal dominating family of $P_3 \,\square\, P_n$ with fewer than $\left\lceil \frac{4n}{5} \right\rceil + 1$ guards and $n \geq 10$. Then for any $1 \leq \ell \leq n-3$, there are at least $\ell - 1$ guards in the first ℓ columns in every dominating set in $\mathcal{E}$.

Proof of Claim: By Lemmas 3 and 5, the claim is true for $1 \leq \ell \leq 6$. Therefore, let $\ell \geq 7$ be the smallest counterexample. That is, there is a set $D \in \mathcal{E}$ such that there are $\ell-2$ guards in the first ℓ columns of D, but in every set in $\mathcal{E}$, there are at least $\ell-2$ guards in the first $\ell-1$ columns. By apply Lemma 4 (1.) (applied with $i=\ell-1$ and $k=\ell-2$), D has $\ell-2$ guards in the first $\ell-1$ columns, no guards in column ℓ and three guards in column $\ell+1$. Hence D has $\ell+1$ guards in the first $\ell+1$ columns. Since $n-(\ell+1)\geq 2$, by Corollary 5, D has at least $\left\lceil\frac{4(n-(\ell+1))-3}{5}\right\rceil$ guards in the last $n-(\ell+1)$ columns. Therefore,

$$|D| \geq \ell+1+\left\lceil\frac{4(n-(\ell+1))-3}{5}\right\rceil = \left\lceil\frac{4n+\ell-2}{5}\right\rceil.$$

As $\ell \geq 7$, this implies $|D| \geq \left\lceil\frac{4n}{5}\right\rceil+1$ guards, which contradicts the assumption that $\mathcal{E}$ is an eternal dominating family with fewer than $\left\lceil\frac{4n}{5}\right\rceil+1$ guards, thus proving the Claim. □

If $\gamma_{\text{all}}^{\infty}(P_3\,\square\,P_n) < \left\lceil\frac{4n}{5}\right\rceil+1$, then there is an eternal dominating family, $\mathcal{E}$, with fewer than $\left\lceil\frac{4n}{5}\right\rceil+1$ guards. By the Claim, for any $1\leq\ell\leq n-3$, there are at least $\ell-1$ guards in the first ℓ columns in every dominating set in $\mathcal{E}$. Let $\ell=n-3$ and observe there are at least $\ell-1=n-4$ guards in the first $\ell-3$ columns in every dominating set in $\mathcal{E}$. By Corollary 5, there are at least 2 guards in the last 3 columns, giving a total of at least $n-2$ guards. However, if $n\geq 15$, then $n-2\geq\left\lceil\frac{4n}{5}\right\rceil+1$, which contradicts the assumption that a dominating set in $\mathcal{E}$ has fewer than $\left\lceil\frac{4n}{5}\right\rceil+1$ guards and the result follows. The result also holds for $10\leq n\leq 14$ (the values for $\gamma_{\text{all}}^{\infty}(P_3\,\square\,P_n)$ were determined in [5]). ■

3 Improving the Lower Bound

In this section, we improve the bound given in Theorem 6. The following lemma provides a tool to identify cases in which the number of guards used is larger than the conjectured bound. This lemma is then used to show Conjecture 1 fails when $n=15$ (and subsequently, when $n\equiv 0 \bmod 5$ for $n\geq 15$).

Lemma 7 *For $n\geq 12$, suppose that in each set in some eternal dominating family $\mathcal{E}$ of $P_3\,\square\,P_n$ there are at least six guards in the first seven columns. Then $\mathcal{E}$ is an eternal dominating family with at least $\left\lceil\frac{4n+7}{5}\right\rceil$ guards.*

Proof: The proof is similar to Theorem 6. Suppose for some $n\geq 12$, there is an eternal dominating family $\mathcal{E}$ with fewer than $\left\lceil\frac{4n+7}{5}\right\rceil$ guards.

<u>Claim 1:</u> For any $\ell\geq 3$ and any set $D\in\mathcal{E}$, there are at least $\left\lceil\frac{4\ell-2}{5}\right\rceil$ guards in the first ℓ columns.

<u>Proof of Claim 1:</u> We proceed with strong induction on ℓ. For each $\ell=3,\ldots,7$, the result is assumed by the premise of the Lemma or was proven in Corollary 5. Suppose the result holds for all $\ell=3,\ldots,s-1$, for some $s>7$. Then for each dominating

set in $\mathcal{E}$, there are at least $\left\lceil \frac{4(s-5)-2}{5} \right\rceil$ guards in the first $s-5$ columns. By Lemma 4 (5.), for each dominating set in $\mathcal{E}$ there are at least $\left\lceil \frac{4(s-5)-2}{5} \right\rceil + 4 = \lceil \frac{4s-2}{5} \rceil$ guards in the first s columns, proving Claim 1. □

Claim 2: For any $1 \leq \ell \leq n-4$ and any set $D \in \mathcal{E}$, there are at least $\ell - 1$ guards in the first ℓ columns.

Proof of Claim 2: By Lemma 3 and Corollary 5, the claim holds for $1 \leq \ell \leq 6$. Therefore, let $\ell \geq 7$ be the smallest counterexample. That is, there is a $D \in \mathcal{E}$ such that there are at least $\ell - 2$ guards in the first ℓ columns, but for every set in $\mathcal{E}$, there are $\ell - 2$ guards in the first $\ell - 1$ columns. By the assumption given in the statement of the Lemma, there are at least 6 guards in the first 7 columns, so $\ell \geq 8$. By Lemma 4 (1.) (applied with $i = \ell - 1$ and $k = \ell - 2$), D has $\ell - 2$ guards in the first $\ell - 1$ columns, no guards in column ℓ and three guards in column $\ell + 1$. Hence D has $\ell + 1$ guards in the first $\ell + 1$ columns. By Claim 1 and symmetry, D has at least $\left\lceil \frac{4(n-(\ell+1))-2}{5} \right\rceil$ guards in the last $n - (\ell + 1)$ columns. Therefore,

$$|D| \geq \ell + 1 + \left\lceil \frac{4(n-(\ell+1))-2}{5} \right\rceil = \left\lceil \frac{4n+\ell-1}{5} \right\rceil.$$

Since $\ell \geq 8$, this implies $\mathcal{E}$ has at least $\lceil \frac{4n+7}{5} \rceil$ guards, a contradiction that proves Claim 2. □

Recall the assumption that $\mathcal{E}$ is an eternal dominating family with fewer than $\lceil \frac{4n+7}{5} \rceil$ guards. By Claim 2 (with $\ell = n-4$), every dominating set in $\mathcal{E}$ has at least $n-5$ guards in the first $n-4$ columns. As a consequence, there can be at most 3 guards in the last 4 columns; otherwise, at least $n - 1 \geq \lceil \frac{4n+7}{5} \rceil$ guards are used (as $n \geq 12$). By Corollary 5 and symmetry, there are exactly 3 guards in the last 4 columns. By Lemma 4 (3.) (applied with $k = n-5$ and $i = n-4$), for any $D \in \mathcal{E}$, if there are at most $n-3$ in the first $n-1$ columns, then there are $n-3$ in the first $n-1$ columns and two guards in column n, hence $n-1$ guards in total, a contradiction. Otherwise for every dominating set in $\mathcal{E}$, there are $n-2$ guards in the first $n-1$ columns and there are no guards in the last column. This contradicts the assumption that $\mathcal{E}$ is an eternal dominating family and can respond to attacks in the last column, showing that $\mathcal{E}$ can not exist. ■

The next two results consider the domination number of a graph G, denoted $\gamma(G)$, and a minimum dominating set (as opposed to an eternal dominating set). The first is due to Jacobson and Kinch [7].

Theorem 8 *[7]* $\gamma(P_3 \square P_n) = \lfloor \frac{3n+4}{4} \rfloor$.

Observation 9 *The unique minimum dominating set (up to reflection) on* $P_3 \square P_6$ *is the following.*

1	2	3	4	5	6
		•			
•				•	•
		•			

Proof: By Theorem 8, a minimum dominating set on $P_3 \Box P_6$ is of size 5.

No column contains 3 guards since this would leave three columns with only 2 guards, which is insufficient. With 5 guards in six columns, at least one column contains no guards. Having one guard in each of the other five columns leaves at least one vertex in the column with no guards that are not dominated. Thus at least one column must contain 2 guards.

Suppose column 1 (or by symmetry column 6) contains 2 guards. By Theorem 8, 4 guards must be located on the remaining 3×4 grid formed by the last four columns. Such a dominating set, containing 6 guards, is not minimum.

Suppose column 2 (or by symmetry 5) contains 2 guards. There remains one vertex in column 1 that is not dominated, thus one vertex of column 1 must contain a guard. By Theorem 8, 3 guards must be located on the 3×3 grid formed by the last three columns. Such a dominating set, containing 6 guards, is not minimum.

Therefore, only column 3 (or by symmetry 4) contains 2 guards. There is one vertex in column 2 and three vertices in column 1 that are not dominated. Thus, at minimum, the middle vertex of column 1 must contain a guard. This forces the guards in column 3 to be located at the top and bottom vertices. Note that the middle vertex in column 4 and all vertices in columns 5 and 6 are not yet dominated. Consequently, the last two guards must be placed at the middle vertices of columns 5 and 6. ∎

Lemma 10 *No eternal dominating set of $P_3 \Box P_{15}$ with at most 13 guards will have exactly two guards in the eighth column.*

Proof: Suppose D is an eternal dominating set of $P_3 \Box P_{15}$ with exactly two guards in the eighth column and $|D| \leq 13$. Of the remaining (at most eleven) guards, without loss of generality, there are at most five guards in the first seven columns. However, Lemma 3 and Lemma 4 (1.) imply that D has three guards in the eighth column. ∎

In the following lemma, we show that every eternal dominating set of $P_3 \Box P_{15}$ with 13 guards does not have three guards in the eighth column. If one did, Lemma 3 and Observation 9 shows there are exactly three possible configurations of guards, up to symmetry, which we label **A**, **B**, **C** as shown below. We show these configurations are not eternal dominating sets.

Lemma 11 *No eternal dominating set of $P_3 \Box P_{15}$ with at most 13 guards will have exactly three guards in the eighth column.*

Proof: Suppose D is an eternal dominating set of $P_3 \square P_{15}$ with exactly three guards in the eighth column and $|D| \leq 13$. By Lemma 3, there are at least five guards in the first six columns and five guards in the last six columns. It follows that $|D| = 13$ and there are no guards in the seventh column and no guards in the ninth column. It now follows that the five guards in the first six columns must form a dominating set in the first six columns. By Observation 9, there are exactly two possible configurations of guards in the first six columns. Analogously, there are exactly two possible configurations of guards in the last six columns and hence, up to symmetry, there are exactly three possible configurations of guards, which we label **A**, **B**, **C** as shown in Figure 1.

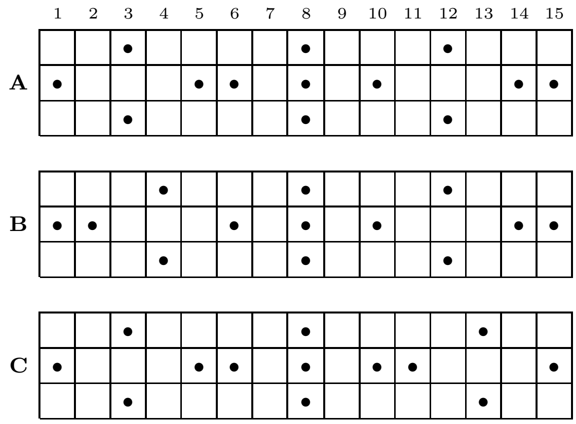

Figure 1: The three possible eternal dominating sets with three guards in column 8.

We wish to show that **A** is not an eternal dominating set. To establish this we first consider a sequence of two attacks. This sequence and the corresponding responses by the defender (the guards) are depicted in Figure 2.

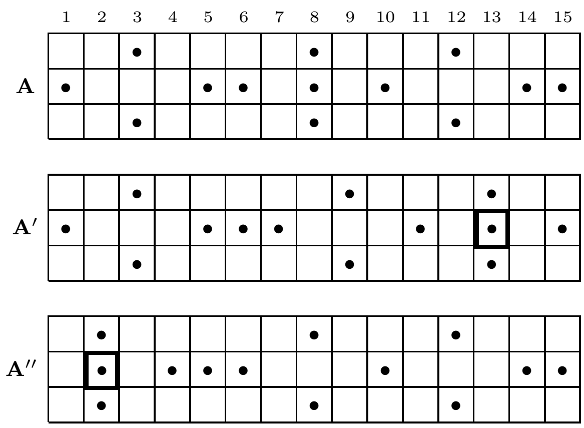

Figure 2: Two attacks and defender responses, starting from **A**.

The first attack is in the middle vertex of column 13. To successfully defend against the attack, the defender must move the guards to transform **A** to a set

$\mathbf{A}' \in \mathbb{D}_{13}(P_3 \square P_{15})$ which contains the attacked vertex. In particular, the defender must:

- move the guard in column 14 to the attacked vertex.
- leave the guard in the last column stationary so that $\mathbf{A}'$ has a vertex in the neighbourhood of each of the vertices of the last column.
- move the two guards in column 12 to column 13 so that $\mathbf{A}'$ has guards in the neighbourhood of all the vertices in column 14.
- move the guard in column 10 to column 11 so that $\mathbf{A}'$ has guards in the neighbourhood of all the vertices in column 11.
- move the guards in the top and bottom vertices in column 8 to column 9 so that $\mathbf{A}'$ has guards in the neighbourhood of all the vertices in column 9 and column 10.
- move the guard in the middle vertex of column 8 to the middle vertex of column 7 so that $\mathbf{A}'$ has guards in the neighbourhood of all the vertices in column 8 and column 7. Note: not moving one of the guards in column 8 of $\mathbf{A}$ to column 7 would result in $\mathbf{A}'$ containing only five guards in the first seven columns and, by Lemma 3 and Lemma 4 (1.), three guards in column 8, which is not possible.

As each guard must remain stationary or move to an adjacent vertex, the guard on the middle vertex of column 6 in $\mathbf{A}$ must be in the closed neighbourhood of the middle vertex of column 6 in $\mathbf{A}'$. Thus having that vertex dominated by the guard in the middle vertex of column 7 does not affect that the remaining five guards dominate each vertex in the first six columns in $\mathbf{A}'$. By Observation 9, it can be seen that the defender does not move these five guards in the transformation from $\mathbf{A}$ to $\mathbf{A}'$. (Also, one can easily see that $\mathbf{A}'$ will not form a dominating set if any of these last five guards in $\mathbf{A}$ is moved to a different vertex in $\mathbf{A}'$.)

The guards are now located on the vertices of $\mathbf{A}'$. The middle vertex of column 2 is now attacked. To successfully defend against the attack, the defender must transform $\mathbf{A}'$ to a dominating set $\mathbf{A}'' \in \mathbb{D}_{13}(P_3 \square P_{15})$ which contains the attacked vertex. That is, the defender must:

- move the guard in column 1 to the attacked vertex.
- move the guards at the top and bottom of column 3 to the top and bottom of column 2 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in the first column.
- move the guard in column 5 to the adjacent vertex in column 4 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 4.
- move the guard in column 6 to the adjacent vertex in column 5 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 5.

- move the guard in column 7 to the adjacent vertex in column 6 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 6.
- move the guards at the top and bottom of column 9 to the top and bottom of column 8 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 7.
- move the guard in column 11 to the adjacent vertex in column 10 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 9.
- move the guards at the top and bottom of column 13 to the top and bottom of column 12 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 11.
- move the guard on the middle vertex of column 13 to the adjacent vertex in column 14 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 14.
- leave the guard on the middle vertex of column 15 so that $\mathbf{A}''$ has guards in the neighbourhood of all the vertices in column 15.

Starting at dominating set $\mathbf{A}$, the attacker can force the guards to be positioned at the vertices of $\mathbf{A}''$. Therefore, any eternal dominating family which contains $\mathbf{A}$ must also contain $\mathbf{A}''$. By Lemma 10, $\mathbf{A}''$ is in no eternal dominating family of $P_3 \,\square\, P_{15}$ and hence $\mathbf{A}$ is in no eternal dominating family. By definition, $\mathbf{A}$ is not an eternal dominating set for $P_3 \,\square\, P_{15}$.

An almost identical method can be used to show $\mathbf{B}$ and $\mathbf{C}$ are not eternal dominating sets. We omit most of the details. To see $\mathbf{B}$ is not an eternal dominating set, one considers the defender's response to a sequence of two attacks: first the middle vertex of column 3 and then the middle vertex of column 13. Then Lemma 10 can be used to show the guards form a dominating set which is not an eternal dominating set. In the case of $\mathbf{C}$, two attacks which force the defender to move the guards to such a set are: first the middle vertex of column 2 and then the middle vertex of column 14. ∎

We are now ready to disprove Conjecture 1.

Theorem 12 $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_{15}) = 14.$

Proof: Let $\mathcal{E}$ be an eternal dominating family of $P_3 \,\square\, P_{15}$ which uses 13 guards. Recall from Lemma 3, for all $D \in \mathcal{E}$ there are at least 5 guards in the first 7 columns. If there exists $D \in \mathcal{E}$ such that D contains exactly 5 guards in the first 7 columns, then by Lemma 4 (1.), D has three guards in column 8, which contradicts Lemma 11. Consequently, for all $D \in \mathcal{E}$ there are at least 6 guards in the first 7 columns. By Lemma 7, $\mathcal{E}$ has at least $\left\lceil \frac{4n+7}{5} \right\rceil = 14$ guards, which is a contradiction. Thus, $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_{15}) \geq 14$. The upper bound of 14 is proved in [5]. ∎

In fact, the Conjecture fails for all $n \equiv 0 \bmod 5$ when $n \geq 15$.

Theorem 13 *If $n \equiv 0$ mod 5 and $n \geq 20$, then $\gamma^{\infty}_{\text{all}}(P_3 \Box P_n) \geq \frac{4n}{5} + 2$.*

Proof: Suppose $\mathcal{E}$ is the eternal dominating family of all eternal dominating sets of $P_3 \Box P_n$ with $\frac{4n}{5}+1$ guards. It follows from (the contrapositive of) Lemma 7 that there must be a $D \in \mathcal{E}$ with exactly five guards in the first seven columns. By Lemma 3 and Lemma 4 (1.), D has no guards in the seventh columns and three guards in the eighth column. Consequently, D has $\frac{4n}{5} - 7$ guards in the last $n - 8$ columns. But by Corollary 5 and symmetry, for every dominating set in $\mathcal{E}$, there are at least:

(1) $\left\lceil \frac{4(n-9)-3}{5} \right\rceil = \frac{4n}{5} - 7$ guards in the last $n - 9$ columns

(2) $\left\lceil \frac{4(n-14)-3}{5} \right\rceil = \frac{4n}{5} - 11$ guards in the last $n - 14$ columns

(3) $\left\lceil \frac{4(n-15)-3}{5} \right\rceil = \frac{4n}{5} - 12$ guards in the last $n - 15$ columns

From (1), it follows that D has no guards in the ninth column and hence there are $\frac{4n}{5} - 7$ guards in the last $n - 9$ columns. Therefore, by (2), there are at most four guards in column 10 through column 14. As $\gamma(P_3 \Box P_4) = 4$, (in order to dominate column 10 through column 13) D has exactly four guards in column 10 through column 14. By (2) and (3) we conclude, that D has at most one guard in column 15.

We now argue there are four possible configurations of the four guards in column 10 through column 14 in D. It is easily seen that none of these columns can contain three guards (as there is at most one guard in column 15) and at least one of column 10 through column 14 must contain no guards. Any column containing no guards must be adjacent to a column containing at least two guards and hence two of column 10 through column 14 contain no guards. It follows that either column 11 and column 13 contain no guards (with column 12 containing two guards) or column 12 and column 14 contain no guards (with column 13 containing two guards). This leads to three cases (up to symmetry) shown in Figure 3.

In Case 1, consider an attack at the top vertex of column 14 and in Case 2, consider an attack at the middle vertex of column 14. The defender responds by transforming D to a set in $\mathcal{E}$. In both cases, the defender is forced to move the guard in column 15 to the attacked vertex. The result is an eternal dominating set with 13 guards in the first 14 columns. By (2), any set in $\mathcal{E}$ has $\frac{4n}{5} - 11$ guards in the last $n - 14$ columns and hence there are at least $\frac{4n}{5} - 11 + 13 = \frac{4n}{5} + 2$ guards in total.

In Case 3, we must consider the five guards in D in the first seven columns. As there is no guard in the seventh column, the five guards in the first six columns must form a dominating set in the first six columns. By Observation 9, there are exactly two possible configurations of guards in the first six columns. Consider an attack at the middle vertex of column 13. The defender responds by transforming D to a set $D' \in \mathcal{E}$. The defender is forced to move the guard in column 14 to the attacked vertex. Thus D' is a set with 12 guards in the first 13 columns and exactly $\frac{4n}{5} - 11$ guards in the last $n - 13$ columns. By (2), any set in $\mathcal{E}$ has at least $\frac{4n}{5} - 11$ guards in the last $n - 14$ columns and consequently, Lemma 4 (1.) (with $k = \frac{4n}{5} - 11$ and

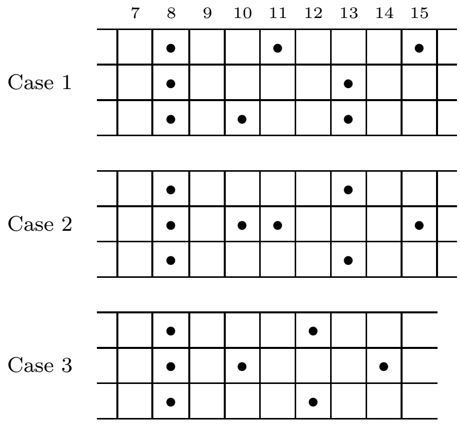

Figure 3: The three possible configurations (up to symmetry) of guards in column 7 through column 14 in D. In Case 1 and Case 2 the guard in column 15 is included.

$i = n - 14$) implies D' has no guards in column 14 (which is the $(n-13)^{\text{th}}$ column from last) and three guards in column 13 (which is the $(n-12)^{\text{th}}$ column from last).

Therefore, to transform D to D' the defender must:

- move the two guards in column 12 to the adjacent vertices in column 13,
- move the guard on column 10 to the adjacent vertex in column 11 (to dominate the top and bottom vertices of column 11),
- move the top and bottom guards in column 8 to the adjacent vertices in column 9 (to dominate the top and bottom vertices of column 10),
- move the middle guard in column 8 to the adjacent vertex in column 7 (else there are 5 guards in the first 7 columns, which requires there be 3 guards in column 8 by Lemma 3 and Lemma 4 (1.)).

The guard which was on the middle vertex of column six in D must be on a vertex in the closed neighbourhood of the middle vertex of the sixth column in D'. It follows that the five guards in the first six columns of D' must form a dominating set in the first six columns and hence, by Observation 9, remain stationary. The two possible configurations of D' are shown in Figure 4.

The remaining logic of the guard movements is similar to the logic presented in the proof of Lemma 11 and hence the details are omitted. Consider an attack in the middle row of the second column or third column (as appropriate), followed by an attack in the middle of column 9. This forces the defender to move the guards into a dominating set with 9 guards in the first 9 columns. By (1), there are at least $\frac{4n}{5} - 7$ guards in the last $n - 9$ columns of this dominating set. Hence there are at least $\frac{4n}{5} + 2$ guards.

Each case leads to a contradiction, completing the proof. ■

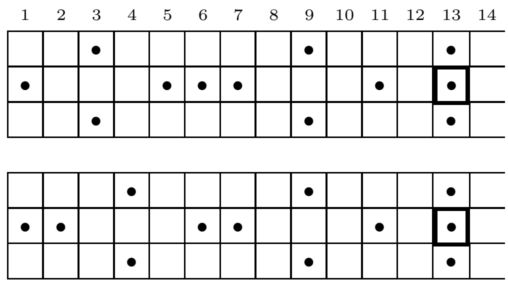

Figure 4: The two possible configurations for guard in the first fourteen columns of D'.

We end this section by summarizing our lower bounds in one theorem.

Theorem 14 *For any $n \geq 11$, $\gamma_{\text{all}}^{\infty}(P_3 \square P_n) \geq \left\lceil \frac{4n+1}{5} \right\rceil + 1$.*

Proof: For $n = 11, 12, 13, 14$, the result was found in [5]. If $n \equiv 0 \mod 5$, the result follows from Theorem 12 and Theorem 13. If $n \not\equiv 0 \mod 5$, $\left\lceil \frac{4n+1}{5} \right\rceil + 1 = \left\lceil \frac{4n}{5} \right\rceil + 1$ and hence the result follows from Theorem 6. ∎

4 A New Upper Bound

The previously known values of $\gamma_{\text{all}}^{\infty}(P_3 \square P_n)$ from [5] are given in Table 2. In this section, we add new values to the Table and include a new upper bound.

n	$\gamma_{\text{all}}^{\infty}(P_3 \square P_n)$	n	$\gamma_{\text{all}}^{\infty}(P_3 \square P_n)$	n	$\gamma_{\text{all}}^{\infty}(P_3 \square P_n)$
1	2	8	8	15	$\in \{13, 14\}$
2	2	9	8	16	$\in \{14, 15\}$
3	3	10	9	17	$\in \{15, 16\}$
4	4	11	10	18	
5	5	12	11	19	17
6	6	13	12	20	
7	7	14	13	21	

Table 2: Values for $P_3 \square P_n$ determined in [5].

By Theorem 12, $\gamma_{\text{all}}^{\infty}(P_3 \square P_{15}) = 14$. Using the new lower bounds obtained by Theorem 14, we obtain the following:

$$\begin{aligned} \gamma_{\text{all}}^{\infty}(P_3 \square P_{18}) &\geq 16; \\ \gamma_{\text{all}}^{\infty}(P_3 \square P_{20}) &\geq 18. \end{aligned}$$

Lemma 15 *[5] If t guards can defend the $3 \times n$ grid graph and r guards can defend the $3 \times s$ grid graph, then $t + r$ guards can defend the $3 \times (n + s)$ grid graph.*

Using two copies of a 3×9 eternal dominating family, where $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_9) = 8$ [5], Lemma 15 defines an eternal dominating family for 3×18, thus $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_{18}) = 16$. Adding a copy of a 3×2 family gives $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_{20}) = 18$. With the assistance of a computer program, eternal dominating families were generated which achieve the lower bounds for $n = 16, 17, 21, 22$ defined by Theorem 14, as shown in Table 3. We provide such a family for $P_3 \,\square\, P_{16}$ in Table 4 and for $P_3 \,\square\, P_{21}$ in Table 5, using the notation introduced in Table 1.

Using a copy of a 3×21 eternal dominating family, along with copies of a 3×2, a 3×3, and a 3×4 family, Lemma 15 provides the values for $n = 23, 24, 25$ matching those lower bounds defined by Theorem 14. The remaining ranges (and value) in Table 3 are obtained from the lower bounds obtained by Theorem 14, along with combinations of copies of smaller eternal dominating families for known values of n.

n	$\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n)$	n	$\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n)$	n	$\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n)$
15	$\boxed{\mathbf{14}}$	22	$\boxed{\mathbf{19}}$	29	$\in \{25, 26\}$
16	$\boxed{\mathbf{14}}$	23	$\boxed{\mathbf{20}}$	30	$\boxed{\mathbf{26}}$
17	$\boxed{\mathbf{15}}$	24	$\boxed{\mathbf{21}}$	31	$\in \{26, 27\}$
18	$\boxed{\mathbf{16}}$	25	$\boxed{\mathbf{22}}$	32	$\in \{27, 28\}$
19	17	26	$\in \{22, 23\}$	33	$\in \{28, 29\}$
20	$\boxed{\mathbf{18}}$	27	$\in \{23, 24\}$	34	$\in \{29, 30\}$
21	$\boxed{\mathbf{18}}$	28	$\in \{24, 25\}$	35	$\in \{30, 31\}$

Table 3: Known values for $P_3 \,\square\, P_n$.

The values of $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n)$ for n up to 25 given in Table 3 and, in particular, the result $\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_{20}) = \gamma_{\rm all}^{\infty}(P_3 \,\square\, P_{21}) = 18$, improves the upper bound to the following.

Theorem 16 *For $n \geq 2$,*

$$\gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{6n}{7} \right\rceil + \begin{cases} 1 & \textit{if } n \equiv 7, 8, 14, \textit{ or } 15 \pmod{21} \\ 0 & \textit{otherwise.} \end{cases}$$

5 Conclusion

We have closed the gap between the upper and lower bounds considerably to

$$\left\lceil \frac{4n+1}{5} \right\rceil + 1 \leq \gamma_{\rm all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{6n+2}{7} \right\rceil$$

for $n \geq 11$ (note this version of the upper bound is a bit relaxed from that in Theorem 16).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	$\bullet$	A_v	B_r	B_v	B	E_v	$\bullet$	B_h	B	B_r	$\bullet$	B	B_r	B	$\bullet$	B_h
A	B_r	G_r	$\bullet$	$\bullet$	$\bullet$	E	B	J	$\bullet$	B_r	E	B	$\bullet$	B_r	$\bullet$	B
	A_v	$\bullet$	B_r	B	B_r	E	$\bullet$	B_r	B_v	B	$\bullet$	B_r	B_h	B	$\bullet$	B_r
	$\bullet$	B_v	B_r	B_v	$\bullet$	A_r	B_r	F	$\bullet$	B_r	A	$\bullet$	B_r	$\bullet$	B_r	A_h
B	B_r	A_r	$\bullet$	A	B_r	D_r	$\bullet$	A_r	A	B_r	E_v	$\bullet$	A	B_r	A	$\bullet$
	B_v	$\bullet$	B_r	$\bullet$	B_r	A_r	$\bullet$	B_r	B_v	$\bullet$	A	B_r	D_r	$\bullet$	A	B_r
	$\bullet$	C_r	B_r	C_h	$\bullet$	C_r	C_h	C_h	$\bullet$	$\bullet$	C_v	C_r	$\bullet$	D_v	C_r	$\bullet$
C	B_r	D_r	$\bullet$	C_r	B_r	D_r	$\bullet$	K	F_r	C_r	D_v	I	$\bullet$	C_r	D_v	D_v
	C_r	$\bullet$	B_r	C_r	$\bullet$	C_h	C_r	$\bullet$	C_h	C_v	$\bullet$	C_r	C_v	E	$\bullet$	C_r
	$\bullet$	D_v	D_r	$\bullet$	C_v	D_r	$\bullet$	D_v	$\bullet$	B_r	D_r	C_r	$\bullet$	D_v	D_r	C_h
D	D_r	D_r	C_v	$\bullet$	B_r	D_r	C_v	F	H_v	B_r	$\bullet$	G	D_r	B_r	$\bullet$	$\bullet$
	D_v	$\bullet$	B_r	D_r	B_r	$\bullet$	D_v	$\bullet$	D_v	D_r	$\bullet$	B_r	D_r	$\bullet$	C_h	D_r
	A	A_r	$\bullet$	$\bullet$	B_h	A_r	A	$\bullet$	C	A_r	$\bullet$	B_h	C	$\bullet$	A	A_h
E	$\bullet$	A_r	A	A	A	$\bullet$	B_v	A_r	A	B_h	$\bullet$	A_r	A	A_r	A	$\bullet$
	A_v	A	$\bullet$	C_r	B_v	$\bullet$	A	C	$\bullet$	A_r	A	B_v	$\bullet$	$\bullet$	A	A_r
	B	F_h	$\bullet$	D	B	$\bullet$	D	$\bullet$	F_h	F_v	$\bullet$	B	F_v	F_h	$\bullet$	F_v
F	$\bullet$	H_h	B	$\bullet$	I_v	E_v	B	$\bullet$	F_h	C_r	D	B	$\bullet$	C_r	D	F_h
	F_h	B	$\bullet$	F_h	I_v	$\bullet$	F_h	F_v	J_r	$\bullet$	F_h	C_r	$\bullet$	F_h	F_v	$\bullet$
	B	A_r	$\bullet$	B_v	B	$\bullet$	B_r	B_h	B	$\bullet$	D_v	B	B_r	B	$\bullet$	A_h
G	$\bullet$	A_r	B	$\bullet$	B_r	E	B	$\bullet$	J_r	B_r	D	$\bullet$	$\bullet$	A_r	$\bullet$	B
	B_v	A_r	$\bullet$	B	B_r	$\bullet$	B	B_r	B_v	$\bullet$	D	B_r	B_h	B	$\bullet$	A_r
	A	H_r	$\bullet$	B_v	$\bullet$	H_r	H_v	$\bullet$	H_h	H_r	$\bullet$	H_h	B_r	H_r	$\bullet$	B_h
H	$\bullet$	H_r	A	H_r	$\bullet$	E	B_v	H_r	$\bullet$	B_r	D_v	H_r	$\bullet$	B_r	$\bullet$	H_r
	A_v	H_r	$\bullet$	D_v	H_v	H_r	$\bullet$	H_v	H_r	H_h	$\bullet$	H_r	B_h	H_r	$\bullet$	B_r
	C	A_r	$\bullet$	B_v	$\bullet$	A_r	B_v	$\bullet$	I_h	I_r	C_v	I_h	C	$\bullet$	A_r	A_h
I	$\bullet$	A_r	B_v	A_r	$\bullet$	I_r	I_r	A_r	J_r	$\bullet$	$\bullet$	$\bullet$	A_r	A_r	G	$\bullet$
	B_v	A_r	$\bullet$	K_h	B_v	A_r	$\bullet$	I_h	$\bullet$	I_h	C	I_r	C_v	$\bullet$	A_h	A_r
	A	A_v	$\bullet$	C_h	H	C_r	J_h	$\bullet$	J_h	$\bullet$	J_v	C_r	K_v	$\bullet$	A	C_h
J	$\bullet$	G_h	A	A	$\bullet$	$\bullet$	I_r	$\bullet$	J_h	C_r	J_h	$\bullet$	A	C_r	A	$\bullet$
	A_v	A	$\bullet$	C_r	H_v	J_h	A	$\bullet$	J_h	J_v	$\bullet$	C_r	K	$\bullet$	A	C_r
	C	K_h	$\bullet$	D	C	K_h	$\bullet$	$\bullet$	K_h	K_h	$\bullet$	C_r	K_v	K_h	$\bullet$	K_v
K	$\bullet$	H_h	C	K_h	$\bullet$	I_h	C	$\bullet$	K_h	C_r	D	K_h	$\bullet$	C_r	D	K_h
	K_h	C	$\bullet$	K_h	C	$\bullet$	K_h	K_v	J_r	$\bullet$	K_h	C_r	$\bullet$	K_h	K_v	$\bullet$

Table 4: Eternal dominating family for $P_3 \square P_{16}$ with 14 guards.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	C_r	B	•	C_r	F_r	C	B	•	C	F_h	C_r	•	C	C_v	C_r	C	•	C_r	F_r	C	C_r
A	•	B	D_r	B	•	•	G_r	•	B	•	C	F_r	C_r	•	•	I	F_r	C	•	•	•
	C	B	•	C	F_h	C_r	B	•	C_r	F_r	C	•	C_r	C	C	C_r	•	C	F_h	C_r	C
	D_r	•	A	E_r	E_v	E_h	•	A	E_h	E	H_h	•	E	E_r	E_h	E	•	E_v	E	D_h	D_r
B	A	•	D_r	•	•	A	G_r	A	•	•	H_r	E_r	I_r	•	•	E	E_r	G_r	•	•	•
	D_h	•	A	E_h	E	E_r	•	A	E_r	E_h	H_r	•	E_v	E	E_r	E_v	•	E	E_r	D_r	D_h
	C_r	•	A	C_r	A_h	•	C_r	C_r	•	A_h	C_r	A	•	C_v	C_r	•	A	C_r	A_h	•	C_r
C	A	A_h	A_h	•	A	A	A_h	A	•	A	•	A_h	C_r	A	A	A_h	A_h	•	A	A	A
	•	C_r	A	•	A_h	C_r	•	A	C_r	A_h	•	A	C_r	•	•	C_r	A	•	A_h	C_r	•
	D_v	•	F_h	E_r	•	G	F_h	E_r	F_r	•	H	E_v	F_v	A_h	•	H_v	E_r	E_v	A_h	•	D_v
D	A_h	A_h	•	G	E_v	E_r	•	•	I	E_v	H	•	•	A_h	G	A_h	•	•	•	B_h	A_h
	•	D_v	E_r	G	•	E_r	E_v	F_h	E_r	•	H_r	E_r	E_v	A_h	•	E_v	F	H_v	A_h	D_v	•
	D_r	B	•	K_v	B_h	E_h	•	F	E_h	•	I_v	E_h	•	L_v	E_h	•	B	K_r	•	B_h	D_r
E	•	B	B_h	B	•	E_h	B_h	B_h	B	•	H_r	E_h	B_h	B	B	•	E_h	B_h	B	B	•
	E_h	•	F	E_h	•	K_v	B	•	K_v	E_h	H_r	•	F	•	B_h	K_r	E_h	•	L_v	E_h	•
	D_r	•	E	E_r	•	K	D_r	•	K	A_h	I_r	•	E	A_h	•	E	D_r	G_r	•	D	•
F	A_h	A_h	A_h	D_r	•	E_r	A_h	A_h	D_r	•	I_r	A_h	D	A_h	D_r	A_h	•	D	D	I_v	A_h
	•	D_r	•	K	A_h	E_r	•	E	E_r	•	I_v	D_r	•	•	D	G_r	•	E	A_h	•	D
	G_v	•	G_r	G_v	A_h	•	G_v	G_r	G_h	•	C_r	G_r	•	G_v	G_r	G_h	H_h	G_r	•	G_h	G_r
G	G_r	A_h	A_h	•	G_r	G_r	G_r	•	C	G_r	C	•	B_h	G_r	•	•	•	G_r	D	B_h	•
	•	G_v	G_r	•	A_h	G_v	•	G_h	G_r	•	C	G_r	G_v	•	G_h	G_r	F_r	G_h	•	G_r	G_h
	D_r	•	H_h	H_h	•	C	D_r	•	C	B_h	•	D_r	C	•	B_h	C	H_h	•	•	H_h	D_r
H	H_h	B_h	B_h	C	•	H_h	B_h	B_h	C	D_r	•	B_h	B_h	D_r	D_r	•	H_h	B_h	D	B_h	•
	•	D_r	•	C	B_h	H_h	•	C_h	•	B_h	C	D_r	•	C	H_h	•	D_r	C	•	B_h	H_h
	I_v	•	A	I_r	B_h	•	I_v	I_r	•	B_h	I_r	A	•	I_v	I_r	•	A	I_r	•	B_h	I_r
I	I_r	I_r	B_h	•	A	I_r	B_h	A	•	A	•	B_h	I_r	A	A	•	B_h	I_r	A	•	•
	•	I_v	I_r	•	B_h	I_r	•	A	I_r	B_h	•	A	I_r	•	B_h	I_r	A	•	I_v	I_r	D
	C_v	J_v	•	C_v	J_r	E_h	•	J_v	E_h	E	•	E_h	E	J_r	J_h	E	•	C_v	J_h	J_r	C_v
J	•	J_r	J_r	C_v	•	E_h	J_r	J_r	•	E	C_v	E_h	•	•	•	E	J_r	C_v	•	•	•
	E_h	•	J_v	E_h	•	C_v	J_v	•	C_v	E_h	•	E	C_v	J_h	J_r	C_v	•	E	J_r	J_h	E
	K_r	•	A	K_v	K_r	•	K_v	A	•	K_r	C_r	•	K_h	C_v	K_r	K_h	•	K_r	F	•	K_r
K	A	I_r	K_r	•	A	A	K_r	K_r	•	•	C	K_r	K_r	•	•	E_v	E_r	K_r	•	A	A
	•	K_r	A	•	K_r	K_v	•	A	K_v	K_r	C	•	K_r	C	K_h	K_r	•	K_h	E_r	K_r	•
	L_r	•	H_r	L_r	•	H_r	C_r	L_r	F_r	•	C_r	L_r	•	L_v	B_h	•	L_r	L_v	•	B_h	L_r
L	L_r	B_h	•	C_r	E_v	L_r	•	•	C_v	L_r	C_r	•	B_h	L_r	L_r	•	B_h	B_h	L_r	B_h	•
	•	L_v	L_r	H_r	•	L_r	E_v	H_r	L_r	•	H_r	L_r	L_v	•	B_h	L_v	L_r	•	L_v	L_r	•

Table 5: Eternal dominating family for $P_3 \square P_{21}$ with 18 guards.

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