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Cubic leaves

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Abstract

We determine those cubic graphs with at most 22 vertices which are leaves of partial triple systems. As a first step towards a complete characterization of cubic leaves, we also show several classes of cubic graphs to be leaves.

1 Introduction

A *partial triple system* of order v, or PTS(v), is a set V of v elements, together with a collection $\mathcal{B}$ of 3-subsets of V called *triples* or *blocks* such that every 2-subset of V is contained in at most one triple of $\mathcal{B}$. The *leave* of a partial triple system is a graph (V, E) in which E is the set of unordered pairs *not* appearing in a triple of $\mathcal{B}$; thus the leave is a simple undirected graph. A partial triple system is *maximal* if its leave is triangle-free. If the leave is empty, the PTS is a *Steiner triple system.*

It was shown in [4] that it is an NP-complete problem to decide whether a graph is a leave of a partial triple system. On the other hand, the complexity of determining whether a graph is a leave of a *maximal* partial triple system remains unknown. In [5], maximal partial triple systems with quadratic (=2-regular) leaves are characterized. However, our knowledge of partial triple systems with *cubic* leaves is very scant: we are aware of a single paper [3] addressing this question. In this article, we make first steps towards characterizing cubic leaves of partial triple systems (not necessarily maximal).

For a cubic graph of order n to be a leave it is necessary that $n \equiv 0$ or $4 \pmod 6$. Indeed, when G is a cubic graph with n vertices, $n \equiv 2 \pmod 6$, the number of edges of the complement of G is not a multiple of 3.

For a Steiner triple system $(V, \mathcal{B})$ of order v and a triple $B = \{x, y, z,\} \in \mathcal{B}$, consider a graph with vertex set $V \setminus \{x, y, z\}$ and edge-set $\mathcal{E} = \{\{a, b\} : \{x, a, b\} \in \mathcal{B}\} \cup \{\{a, b\} : \{y, a, b\} \in \mathcal{B}\} \cup \{\{a, b\} : \{z, a, b\} \in \mathcal{B}\}$. Then $(V \setminus \{x, y, z\}, \mathcal{E})$ is a cubic leave with $v-3$ vertices. This shows that for each $n = v-3 \equiv 0$ or $4 \pmod 6$ there exist cubic leaves with n vertices (provided n is sufficiently large, actually a multitude of them, albeit all of Class 1).

The two nonisomorphic cubic graphs of order 6 are the complete bipartite graph $K_{3,3}$ and the triangular prism $Pr(3)$. The former is a leave while the latter is not.

Those cubic graphs which are leaves of partial triple systems of order 10 have been determined in [3]. In Section 2, we extend that result slightly by enumerating all such PTSs, up to an isomorphism. In Section 3 those cubic graphs of order 12 that are leaves are determined, together with a census of the corresponding PTSs. Section 4 deals briefly with cubic graphs of order $16, 18$ and 22.

A conjecture of Nash-Williams (cf. [10]) states that every even graph (a graph whose all vertices are of even degree) with n vertices having minimum degree $\delta \geq \frac{3}{4}n$ and number of edges divisible by 3 can be partitioned into triangles. Thus our computer verification in Section 4 is in agreement with this conjecture. It "should" be then the case that all cubic graphs with n vertices, $n \geq 16$ are leaves; however, we appear to be far from proving this, nor is it clear how should such a proof proceed. Instead, all we are able to do is to show in subsequent sections that several classes of cubic graphs are indeed leaves (and establish that the conjecture holds for $n = 16, 18$ and 22).

Given a partial triple system $(V, \mathcal{B})$ with a cubic leave $\mathcal{L}$, it is immediately clear that $\mathcal{L}$ as a subgraph has to contain all vertices of V, i.e. must be a factor (otherwise the complement of $\mathcal{L}$ would contain vertices whose degrees are of different parity, a clear impossibility). The corresponding PTS $(V, \mathcal{B})$ is maximal if and only if $\mathcal{L}$ is triangle-free. However, in what follows we allow $\mathcal{L}$ to possibly contain triangles.

2 Cubic leaves on 10 vertices

There exist in the literature several sources containing the census of 10-vertex cubic graphs ([1], [2], [3], [7], [11], [12]). There are 19 connected and 2 disconnected, for a total of 21 ten-vertex cubic graphs. In [11] and [12] only the diagrams of the 19 connected cubic graphs are given while [3] adopts the same numbering as [7] (but also lists the diagrams). The following table lists for convenience the correspondence in the above sources of the 19 connected 10-vertex cubic graphs. Unfortunately, we could not avail ourselves of the source [1].

The connected 10-vertex cubic graphs:

[2]	[11]	[7]	leave	[2]	[11]	[7]	leave
No.1	C25	G14	no	No.11	C16	G7	yes
No.2	C21	G15	no	No.12	C15	G16	yes
No.3	C14	G18	no	No.13	C10	G8	yes
No.4	C17	G19	no	No.14	C26	G3	no
No.5	C22	G17	no	No.15	C23	G5	yes
No.6	C9	G11	no	No.16	C11	G6	no
No.7	C12	G12	no	No.17	C24	G2	no
No.8	C13	G9	no	No.18	C19	G4	no
No.9	C18	G13	yes	No.19	C27	G1	yes
No.10	C20	G10	yes				

The seven connected graphs that are in fact leaves were determined in [3] (see Fig. 1). Here we extend slightly this result by determining, for each of the 7 leaves, all nonisomorphic partial triple systems with this leave.

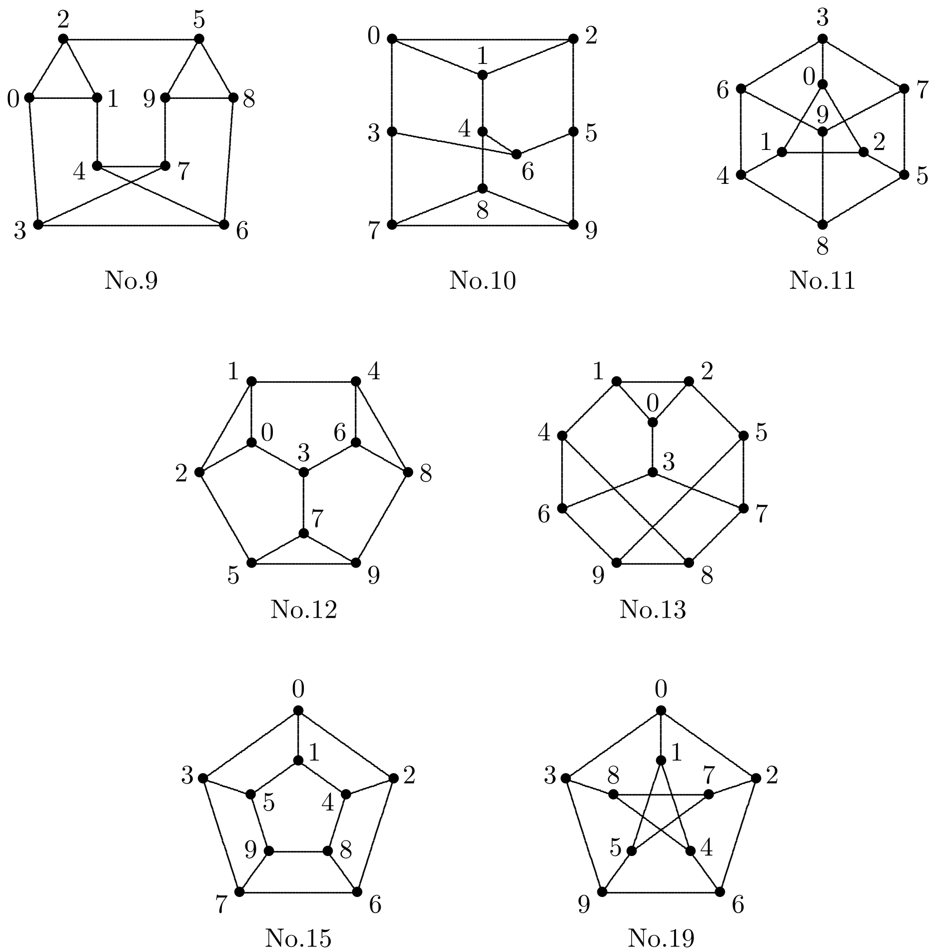

Figure 1

Graph No.9:
The leave: $0\colon 1,2,3$; $1\colon 2,4$; $2\colon 5$; $3\colon 6,7$; $4\colon 6,7$; $5\colon 8,9$; $6\colon 8$; $7\colon 9$; $8\colon 9$.
Unique solution: $\{0,4,8\},\{0,5,7\},\{0,6,9\},\{1,3,9\},\{1,5,6\},\{1,7,8\},\{2,3,8\},\{2,4,9\},\{2,6,7\},\{3,4,5\}$.

Graph No.10:
The leave: $0\colon 1,2,3$; $1\colon 2,4$; $2\colon 5$; $3\colon 6,7$; $4\colon 6,8$; $5\colon 6,9$; $7\colon 8,9$; $8\colon 9$.
Two nonisomorphic solutions:
(1) $\{0,4,7\},\{0,5,8\},\{0,6,9\},\{1,3,9\},\{1,5,7\},\{1,6,8\},\{2,3,8\},\{2,4,9\},\{2,6,7\},\{3,4,5\}$.
(2) $\{0,4,9\},\{0,5,8\},\{0,6,7\},\{1,3,9\},\{1,5,7\},\{1,6,8\},\{2,3,8\},\{2,4,7\},\{2,6,9\},\{3,4,5\}$.

Graph No.11:
The leave: $0\colon 1,2,3$; $1\colon 2,4$; $2\colon 5$; $3\colon 6,7$; $4\colon 6,8$; $5\colon 7,8$; $6\colon 9$; $7\colon 9$; $8\colon 9$.
Unique solution: $\{0,4,7\},\{0,5,9\},\{0,6,8\},\{1,3,9\},\{1,5,6\},\{1,7,8\},\{2,3,8\},\{2,4,9\},\{2,6,7\},\{3,4,5\}$.

Graph No.12:
The leave: $0\colon 1,2,3$; $1\colon 2,4$; $2\colon 5$; $3\colon 6,7$; $4\colon 6,8$; $5\colon 7,9$; $6\colon 8$; $7\colon 9$; $8\colon 9$.
Two nonisomorphic solutions:
(1) $\{0,4,7\},\{0,5,8\},\{0,6,9\},\{1,3,9\},\{1,5,6\},\{1,7,8\},\{2,3,8\},\{2,4,9\},\{2,6,7\},\{3,4,5\}$.
(2) $\{0,4,9\},\{0,5,8\},\{0,6,7\},\{1,3,9\},\{1,5,6\},\{1,7,8\},\{2,3,8\},\{2,4,7\},\{2,6,9\},\{3,4,5\}$.

Graph No.13:
The leave: $0\colon 1,2,3$; $1\colon 2,4$; $2\colon 5$; $3\colon 6,7$; $4\colon 6,8$; $5\colon 7,9$; $6\colon 9$; $7\colon 8$; $8\colon 9$.
Unique solution: $\{0,4,9\},\{0,5,8\},\{0,6,7\},\{1,3,8\},\{1,5,6\},\{1,7,9\},\{2,3,9\},\{2,4,7\},\{2,6,8\},\{3,4,5\}$.

Graph No.15 (prism):
The leave: $0\colon 1,2,3$; $1\colon 4,5$; $2\colon 4,6$; $3\colon 5,7$; $4\colon 8$; $5\colon 9$; $6\colon 7,8$; $7\colon 9$; $8\colon 9$.
Unique solution: $\{0,4,7\},\{0,5,8\},\{0,6,9\},\{1,2,9\},\{1,3,6\},\{1,7,8\},\{2,3,8\},\{2,5,7\},\{3,4,9\},\{4,5,6\}$.

Graph No.19 (Petersen graph):
The leave: $0\colon 1,2,3$; $1\colon 4,5$; $2\colon 6,7$; $3\colon 8,9$; $4\colon 6,8$; $5\colon 7,9$; $6\colon 9$; $7\colon 8$.
Two nonisomorphic solutions:
(1) $\{0,4,5\},\{0,6,7\},\{0,8,9\},\{1,2,3\},\{1,6,8\},\{1,7,9\},\{2,4,9\},\{2,5,8\},\{3,4,7\},\{3,5,6\}$.
(2) $\{0,4,5\},\{0,6,7\},\{0,8,9\},\{1,2,8\},\{1,3,6\},\{1,7,9\},\{2,3,5\},\{2,4,9\},\{3,4,7\},\{5,6,8\}$.

In the following table, *nonis* stands for the number of nonisomorphic partial triple systems with a given cubic leave, while *dist* stands for the number of *distinct* partial triple systems with a given fixed cubic leave; g is the order of the automorphism group of the graph, and G is the order of the automorphism group(s) of the corresponding PTSs.

Graph No.	*nonis*	*dist*	g	G
9	1	2	8	4
10	2	4	12	4,12
11	1	2	6	3
12	2	3	6	3,6
13	1	1	2	2
15	1	2	20	10
19	2	6	120	120,24
total	10	20		

Neither of the two disconnected 10-vertex cubic graphs is a leave. Indeed, any PTS with a cubic leave $\mathcal{L}$ must contain exactly 10 triples. There are 24 edges joining vertices in different components of the complement of $\mathcal{L}$ which requires 12 triples (alternatively, the complement of $\mathcal{L}$ can contain at most 6 pairwise disjoint triples).

3 Cubic leaves on 12 vertices

There are 85 connected and 9 disconnected cubic graphs on 12 vertices (see [2] for an explicit listing of edges of the connected graphs, and [11] or [12] for their diagrams). Of these 94 cubic graphs, exactly 18 are *not* leaves. We show this first.

Lemma 3.1 *Let G be a disconnected 12-vertex cubic graph with two 6-vertex components. Then G is not a leave of a PTS.*

PROOF: There are 36 edges in $\bar{G}$ joining vertices of the two components of G. Thus 18 triples are needed to cover these pairs of vertices but any PTS of order 12 with a cubic leave must contain exactly 16 triples. □

Corollary 3.2 *There are exactly three disconnected 12-vertex cubic graphs which are not leaves: $K_{3,3} + K_{3,3}$, $K_{3,3} + \bar{C}_6$, and $\bar{C}_6 + \bar{C}_6$.*

("Exactly" because the remaining 6 disconnected 12-vertex cubic graphs are all leaves, see below.)

Lemma 3.3 *Let G be a 12-vertex cubic graph with a bridge e such that the two components of $G - e$ have 5 and 7 vertices, respectively. Then G is not a leave.*

PROOF: There are 34 edges in $\bar{G}$ joining vertices of the two components of $G - e$ requiring 17 triples to cover these pairs of vertices, a contradiction. □

Corollary 3.4 *Neither of the following four 12-vertex cubic graphs is a leave: No.1 ($C101$), No.2 ($C89$), No.3 ($C100$), and No.4 ($C108$). (No.x refers to the numbering in [2], Cy refers to the diagram in [11]).*

Lemma 3.5 *Let G be a 12-vertex cubic graph with a 2-edge cut F such that the two components of $G \setminus F$ have 6 vertices each. Then G is not a leave.*

Proof: It is the same as that of Lemma 3.3. □

Corollary 3.6 *Neither of the following ten 12-vertex cubic graphs is a leave: No.5 (C97), No.6 (C71), No.7 (C85), No.8 (C98), No.23 (C73), No.24 (C74), No.25 (C87), No.26 (C99), No.27 (C102), No.64 (C112).*

There is only one other 12-vertex cubic graph which is not a leave.

Lemma 3.7 *The 12-vertex cubic graph No.76 [2] (C95 [11]) is not a leave.*

Proof: The graph whose edges are $1: 2,3,4$; $2: 5,6$; $3: 5,7$; $4: 8,9$; $5: 8$; $6: 7,10$; $7: 11$; $8: 12$; $9: 10,11$; $10: 12$; $11: 12$ has been shown first by an exhaustive computer search not to be a leave. A (less than satisfactory) non-computer proof goes as follows:

Suppose this graph is a leave, and consider the corresponding partial triple system, i.e a partition of the edges of the complement into triangles. The edge $\{6,8\}$ can occur only in one of the triples $\{1,6,8\}, \{3,6,8\}, \{6,8,9\}, \{6,8,11\}$ so w.l.o.g we may assume that the pts contains the triple $\{1,6,8\}$ (as there exists an automorphism of the graph preserving $\{6,8\}$ and mapping 1 to 3, 9, or 11, respectively). Then there are four possibilities for the four triples containing 1:
(1) $\{1,6,8\}, \{1,5,11\}, \{1,7,10\}, \{1,9,12\}$;
(2) $\{1,6,8\}, \{1,10,11\}, \{1,5,7\}, \{1,9,12\}$;
(3) $\{1,6,8\}, \{1,10,11\}, \{1,5,9\}, \{1,7,12\}$;
(4) $\{1,6,8\}, \{1,10,11\}, \{1,5,12\}, \{1,7,9\}$.

Case (1). There are three possibilities for triples containing 6:
(a) $\{3,6,9\}, \{4,6,11\}, \{5,6,12\}$;
(b) $\{3,6,11\}, \{4,6,12\}, \{5,6,9\}$;
(c) $\{3,6,12\}, \{4,6,11\}, \{5,6,9\}$,
and two possibilities for triples containing 5:
(d) $\{4,5,10\}, \{5,6,9\}, \{5,7,12\}$;
(e) $\{4,5,10\}, \{5,6,12\}, \{5,7,9\}$.
Combining (a), (b), (c) with (d),(e) leads to three possibilities:
A. $\{3,6,9\}, \{4,6,11\}, \{5,6,12\}, \{4,5,10\}, \{5,7,9\}$;
B. $\{4,5,10\}, \{5,6,9\}, \{5,7,12\}, \{3,6,11\}, \{4,6,12\}$;
C. $\{4,5,10\}, \{5,6,9\}, \{5,7,12\}, \{3,6,12\}, \{4,6,11\}$.
There must be two more triples containing 7, and these are forced to be $\{2,7,8\}$, $\{4,7,12\}$ in case A, and $\{2,4,7\}, \{7,8,9\}$ in case B or C. There is now only more more triple containing 4 to be added, but this is impossible.

Case (2). Each triple containing 5 must also contain 4 or 6 thus no three further disjoint triples containing 5 can be found.

Case (3). There is only one possibility for further three triples containing 5, namely $\{4,5,12\},\{5,6,11\},\{5,7,10\}$. But then there is only one possibility for further triples containing 6, namely $\{3,4,6\},\{6,9,12\}$. There are now two further disjoint triples containing 4 to be added but this is impossible.

Case (4). Same as Case (2). □

All the remaining 76 twelve-vertex cubic graphs, 70 connected and 6 disconnected, are leaves. The number of nonisomorphic partial triple systems with a given leave ranges from 1 (twice) to 24 (also twice), for a total of 574 systems, while the number of distinct PTSs totals 3172. The number of PTSs is too large for all of them to be presented here. The details of all these systems can be found in [9]. Here we show, as an example, only some "prominent" 12-vertex cubic graphs and their corresponding PTSs, including the four vertex-transitive cubic graphs on 12 vertices.

A *cubic circulant* $C(2m;s,m)$ is a cubic graph whose vertices may be labelled with Z_{2m} and whose edges are $\{x,y\}$ if and only if $\min(|x-y|, 2m-|x-y|) = s$ or m, where $s \in \{1,2,\ldots,m-1\}$. The circulant $C(2m;s,m)$ is connected if and only if $gcd(s,m)=1$.

A *generalized Petersen graph* $\mathrm{GP}(n,k)$ is a cubic graph whose vertex set is $Z_n \times \{1,2\}$ and whose edges are $\{i_1,(i+1)_1\}$, $\{i_1,i_2\}$, and $\{i_2,(i+k)_2\}$.

The *prism* $\mathrm{Pr}(n)$ is the generalized Petersen graph $\mathrm{GP}(n,1)$.

The prism $\mathrm{Pr}(6)$ (No.71 in [2], C106 in [11]):
The leave: $0\colon 1,2,3$; $1\colon 4,5$; $2\colon 4,6$; $3\colon 5,7$; $4\colon 8$; $5\colon 9$; $6\colon 8,10$; $7\colon 9,10$; $8\colon 11$; $9\colon 11$; $10\colon 11$.
Two nonisomorphic solutions:
(1) $\{0,4,7\},\{0,5,8\},\{0,6,11\},\{0,9,10\},\{1,2,9\},\{1,3,6\},\{1,7,11\},\{1,8,10\},$ $\{2,3,10\},\{2,5,11\},\{2,7,8\},\{3,4,11\},\{3,8,9\},\{4,5,10\},\{4,6,9\},\{5,6,7\}$.
(2) $\{0,4,7\},\{0,5,8\},\{0,6,11\},\{0,9,10\},\{1,2,10\},\{1,3,6\},\{1,7,11\},\{1,8,9\},$ $\{2,3,9\},\{2,5,11\},\{2,7,8\},\{3,4,11\},\{3,8,10\},\{4,5,10\},\{4,6,9\},\{5,6,7\}$.
The automorphism group of each system has order 3. The total number of distinct systems is 16.

The 12-vertex circulants:

Consider the cubic circulants $C(12;s,6)$ where $s \in \{1,2,3,4,5\}$. The circulant $C(12;2,6) \simeq C(12;4,6)$ is disconnected and is not a leave (cf. Corollary 3.2). The circulant $C(12;3,6)$ is disconnected as well, and is isomorphic to $3K_4$; any decomposition of its complement into triangles is, in effect, a 3-GDD of type 4^3, or a latin square of order 4, thus there are exactly two nonisomorphic solutions (and 576 distinct solutions).

Finally, there is the circulant $C(12;1,6) \simeq C(12;5,6)$ which is isomorphic to the graph No.72 in [2] (C104 in [11]). The automorphism group of this graph has order 24; it is a leave, and its complement admits two nonisomorphic partial triple systems (and 8 distinct systems):
The leave: $0\colon 1,2,3$; $1\colon 4,5$; $2\colon 4,6$; $3\colon 5,7$; $4\colon 8$; $5\colon 9$; $6\colon 8,10$; $7\colon 9,11$; $8\colon 11$; $9\colon 10$;

10: 11.
Two nonisomorphic solutions:
(1) $\{0,4,7\},\{0,5,10\},\{0,6,11\},\{0,8,9\},\{1,2,9\},\{1,3,11\},\{1,6,7\},\{1,8,10\}$, $\{2,3,8\},\{2,5,11\},\{2,7,10\},\{3,4,10\},\{3,6,9\},\{4,5,6\},\{4,9,11\},\{5,7,8\}$.
(2) $\{0,4,7\},\{0,5,11\},\{0,6,9\},\{0,8,10\},\{1,2,9\},\{1,3,10\},\{1,6,11\},\{1,7,8\}$, $\{2,3,11\},\{2,5,8\},\{2,7,10\},\{3,4,6\},\{3,8,9\},\{4,5,10\},\{4,9,11\},\{5,6,7\}$.
The automorphism group of the systems has order 12 and 4, respectively.

The generalized Petersen graph GP(6, 2):

Note that the generalized Petersen graph GP(6, 1) is isomorphic to the prism Pr(6). The generalized Petersen graph GP(6, 2) (No.44 in [2], graph C92 in [11]) is neither vertex- nor edge-transitive; its automorphism group has order 12. It is a leave, and its complement admits 2 nonisomorphic partial triple systems (and 16 distinct systems).
The leave: 0: 1, 2, 3; 1: 2, 4; 2: 5; 3: 6, 7; 4: 6, 8; 5: 7, 8; 6: 9; 7: 10; 8: 11; 9: 10, 11; 10: 11.
Two nonisomorphic solutions:
(1) $\{0,4,7,\},\{0,5,9\},\{0,6,11\},\{0,8,10\},\{1,3,10\},\{1,5,6\},\{1,7,11\},\{1,8,9\}$, $\{2,3,8\},\{2,4,11\},\{2,6,10\},\{2,7,9\},\{3,4,9\},\{3,5,11\},\{4,5,10\},\{6,7,8\}$.
(2) $\{0,4,7\},\{0,5,10\},\{0,6,11\},\{0,8,9\},\{1,3,9\},\{1,5,6\},\{1,7,11\},\{1,8,10\}$, $\{2,3,8\},\{2,4,11\},\{2,6,10\},\{2,7,9\},\{3,4,10\},\{3,5,11\},\{4,5,9\},\{6,7,8\}$.
The automorphism group of the systems has order 1 and 3, respectively.

Finally, here are examples of the other two (of four) vertex-transitive cubic graphs on 12 vertices. First, there is No.59 [2], C105 [11] with an automorphism group of order 24. It is a leave, its complement admits 5 nonisomorphic partial triple systems (and 40 distinct systems).
The leave: 0: 1, 2, 3; 1: 2, 4; 2: 5; 3: 6, 7; 4: 8, 9; 5: 10, 11; 6: 7, 8: 7: 10; 8: 9; 9: 11; 10: 11.
Five nonisomorphic solutions:
(1) $\{0,4,5\},\{0,6,9\},\{0,7,11\},\{0,8,10\},\{1,3,10\},\{1,5,6\},\{1,7,9\},\{1,8,11\}$, $\{2,3,8\},\{2,4,7\},\{2,6,11\},\{2,9,10\},\{3,4,11\},\{3,5,9\},\{4,6,10\},\{5,7,8\}$.
(2) $\{0,4,5\},\{0,6,9\},\{0,7,11\},\{0,8,10\},\{1,3,11\},\{1,5,6\},\{1,7,8\},\{1,9,10\}$, $\{2,3,9\},\{2,4,7\},\{2,6,10\},\{2,8,11\},\{3,4,10\},\{3,5,8\},\{4,6,11\},\{5,7,9\}$.
(3) $\{0,4,5\},\{0,6,10\},\{0,7,9\},\{0,8,11\},\{1,3,8\},\{1,5,6\},\{1,7,11\},\{1,9,10\}$, $\{2,3,11\},\{2,4,7\},\{2,6,9\},\{2,8,10\},\{3,4,10\},\{3,5,9\},\{4,6,11\},\{5,7,8\}$.
(4) $\{0,4,5\},\{0,6,11\},\{0,7,9\},\{0,8,10\},\{1,3,8\},\{1,5,6\},\{1,7,11\},\{1,9,10\}$, $\{2,3,10\},\{2,4,7\},\{2,6,9\},\{2,8,11\},\{3,4,11\},\{3,5,9\},\{4,6,10\},\{5,7,8\}$.
(5) $\{0,4,5\},\{0,6,11\},\{0,7,9\},\{0,8,10\},\{1,3,9\},\{1,5,7\},\{1,6,10\},\{1,8,11\}$, $\{2,3,11\},\{2,4,6\},\{2,7,8\},\{2,9,10\},\{3,4,10\},\{3,5,8\},\{4,7,11\},\{5,6,9\}$.
The automorphism groups of these systems have orders 6, 2, 3, 2, and 6, respectively.

Second, there is No.83 in [2] (C110 in [11]) with automorphism group of order 48. It also is a leave, its complement admits 4 nonisomorphic partial triple systems, and 64 distinct systems.
The leave: 0: 1, 2, 3; 1: 4, 5; 2: 4, 6; 3: 7, 8; 4: 9; 5: 10, 11; 6: 10, 11: 7: 9, 10; 8: 9, 11.

Four nonisomorphic solutions:
(1) $\{0,4,5\},\{0,6,7\},\{0,8,10\},\{0,9,11\},\{1,2,8\},\{1,3,10\},\{1,6,9\},\{1,7,11\},$ $\{2,3,11\},\{2,5,7\},\{2,9,10\},\{3,4,6\},\{3,5,9\},\{4,7,8\},\{4,10,11\},\{5,6,8\}$.
(2) $\{0,4,5\},\{0,6,9\},\{0,7,8\},\{0,10,11\},\{1,2,7\},\{1,3,10\},\{1,6,8\},\{1,9,11\},$ $\{2,3,11\},\{2,5,8\},\{2,9,10\},\{3,4,6\},\{3,5,9\},\{4,7,11\},\{4,8,10\},\{5,6,7\}$.
(3) $\{0,4,5\},\{0,6,9\},\{0,7,8\},\{0,10,11\},\{1,2,7\},\{1,3,11\},\{1,6,8\},\{1,9,10\},$ $\{2,3,10\},\{2,5,8\},\{2,9,11\},\{3,4,6\},\{3,5,9\},\{4,7,11\},\{4,8,10\},\{5,6,7\}$.
(4) $\{0,4,7\},\{0,5,6\},\{0,8,10\},\{0,9,11\},\{1,2,8\},\{1,3,10\},\{1,6,9\},\{1,7,11\},$ $\{2,3,11\},\{2,5,7\},\{2,9,10\},\{3,4,6\},\{3,5,9\},\{4,5,8\},\{4,10,11\},\{6,7,8\}$.
The automorphism group of each of these systems has order 3.

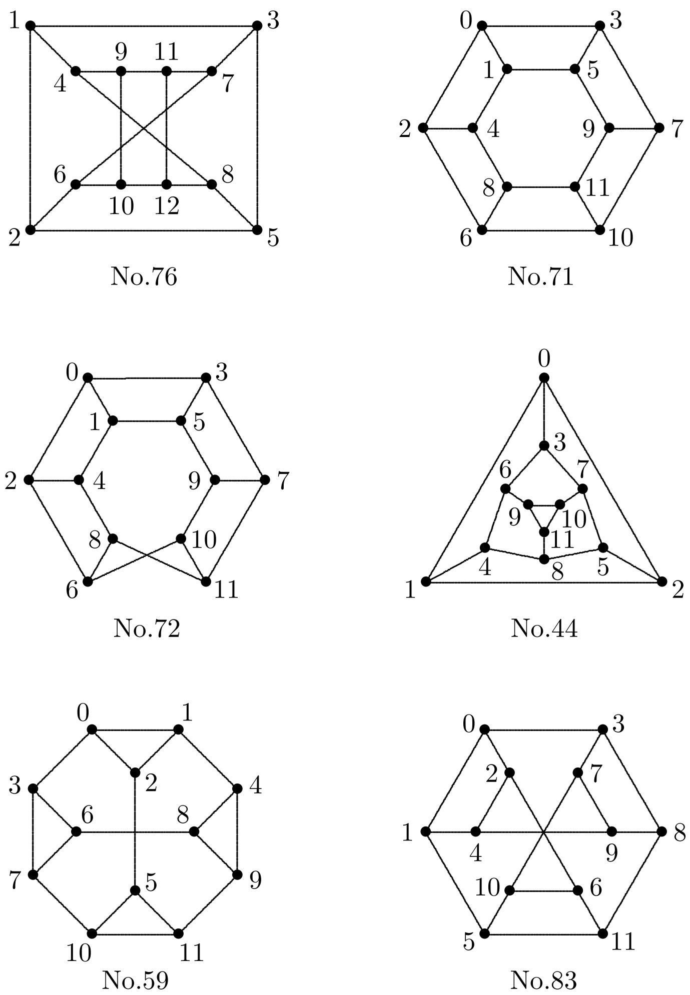

Figure 2

4 Cubic leaves on 16, 18 and 22 vertices

As mentioned in the introduction, there are 4060 connected and 147 disconnected cubic graphs on 16 vertices, and each of these is a leave. Due to the large number of these graphs and the corresponding partial triple systems, we refer the reader to [9] where a representative partial triple system for each particular leave can be found. Here we mention only some examples.

So, for instance, the number of nonisomorphic partial triple systems into which the complement of each of the three nonisomorphic circulants $C(16;1,8)$, $C(16;2,8)$, $C(16;4,8)$ can be decomposed is 1477064, 54964, and 23, respectively, a truly wide (and almost unbelievable) span. Of course, the last of these numbers has been known before (see [6]) since $C(16;4,8)$ is nothing else as $4K_4$.

For the complement of the unique 16-vertex cubic graph without a 1-factor, the number of noinsomorphic PTSs is 98721. For generalized Petersen graphs, the numbers are 1485912 for GP$(8,1)$, 3386464 for GP$(8,2)$, and 563776 for GP$(8,3)$.

We are thankful to an anonymous referee for his suggestion to extend the computational work to larger orders. We were able to do so for orders 18 and 22. There are 41301 connected and 809 disconnected cubic graphs on 18 vertices (cf. [12]); we were able to verify, with the help of a computer, that each of these is a leave. Similarly, each of the 7319447 connected and 54477 disconnected cubic graphs on 22 vertices is a leave.

In all the above cases, hillclimbing has been used to find, for any cubic graph G (connected or disconnected), a single partial triple system whose leave is G.

5 Cubic circulants as leaves

For the definition of a cubic circulant, see Section 3 above.

An *extended Skolem sequence* of order n is a sequence ES$=(s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ integers such that
1. for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in$ ES such that $s_i = s_j = k$,
2. if $s_i = s_j = k$ with $i < j$ then $j - i = k$, and
3. there is exactly one $s_i \in$ ES such that $s_i = 0$.

A *near-Skolem sequence* of order n and defect $m, n \geq m$, is a sequence NS$=(s_1, s_2, \ldots, s_{2n-2})$ of $2n-2$ integers such that
1. for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots n\}$ there exist exactly two elements $s_i, s_j \in$ NS such that $s_i = s_j = k$, and
2. if $s_i = s_j = k$ with $i < j$ then $j - i = k$ (cf. [12], [13]).

A *near-Rosa sequence* of order n and defect m is a sequence $R(s_1, s_2, \ldots, s_{2n-1})$ such that
1. for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ there are exactly two elements $s_i, s_j \in R$ such that $s_i = s_j = k$,
2. if $s_i = s_j = k$ with $i < j$ then $j - i = k$, and
3. $s_n = 0$, (cf. [14]).

A (p, q)-*extended Rosa sequence* of order n, $R_n(p, q) = (s_1, s_2, \ldots, s_{2n+2})$, is a sequence of $2n+2$ integers such that

1. for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in R_n(p, q)$, $i, j \neq p, q$ such that $s_i = s_j = k$,
2. $s_p = s_q = 0$, $p \neq q$, and
3. if $s_i = s_j > 0$ with $i < j$ then $j - i = k$ (cf. [8]).

It is known [8] that an $R_n(p, q)$ exists if and only if
(i) $p \not\equiv q \pmod 2$ and $n \equiv 0, 1 \pmod 4$, except when $(p, q) \in \{(2, 3), (5, 6)\}$ ($n = 1$ and $n = 4$, respectively), or
(ii) $p \equiv q \pmod 2$ and $n \equiv 2, 3 \pmod 4$.

Theorem 5.1 *Let $G = C(n; s, \frac{n}{2})$, and let*
(a) $n \equiv 4$ or $22 \pmod{24}$ and $s \equiv 1 \pmod 2$;
(b) $n \equiv 10$ or $16 \pmod{24}$ and $s \equiv 0 \pmod 2$;
(c) $n \equiv 6$ or $12 \pmod{24}$, $s \equiv 1 \pmod 2$;
(d) $n \equiv 0$ or $18 \pmod{24}$, $s \equiv 0 \pmod 2$, $s \neq \frac{n}{3}$.
Then G is a leave.

PROOF: We show this directly by arranging the edges of the complement $\bar{G}$ of G into triples, using extended Skolem sequences ([13]), near-Skolem sequences (see [14]), near-Rosa sequences ([15]) as well as (p, q)-extended Rosa sequences (see [8]).

Let $n \equiv 22 \pmod{24}, n = 24t + 22, t \geq 0$. By [13], an extended Skolem sequence ES $= (s_1, s_2, \ldots, s_{8t+7})$ of order $4t + 3$ exists for all even positions of 0. For each $k \in \{1, \ldots, 4t + 3\}$, form the difference triple $(k, 4t + 3 + i, 4t + 3 + j)$ provided $s_j = s_i = k$. Then the union of orbits of triples $\{0, k, 4t + 3 + i + k\}$ mod n, $k = 1, 2, \ldots, 4t + 3$, is a decomposition of the complement $\bar{C}(n; s, \frac{n}{2})$ of $C(n; s, \frac{n}{2})$ into triples, for each odd $s \in \{4t+5, \ldots, 12t+9\}$. For $s \in \{1, 3, \ldots, 4t+3\}$, we utilize in a similar manner a near-Skolem sequence NS $= (s_1, s_2, \ldots, s_{8t+6})$ of order $4t+4$ and defect s (which exists for all odd $s, 1 \leq s \leq 4t+3$). For each $k \in \{1, 2, \ldots, 4t+4\} \setminus \{s\}$, form the difference triple $(k, 4t + 4 + i, 4t + 4 + j)$ provided $s_i = s_j = k$. Then the union of orbits of triples $\{0, k, 4t + 4 + i + k\}$ mod n, $k \in \{1, 2, \ldots, 4t + 4\} \setminus \{s\}$, is a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triples for each odd $s \in \{1, 3, \ldots, 4t + 3\}$.

Let now $n \equiv 4 \pmod{24}, n = 24t + 4, t \geq 1$. This case is completely analogous to the previous one, except that now we use an extended Skolem sequence of order $4t$ to form difference triples $(k, 4t+i, 4t+j)$, and a near-Skolem sequence of order $4t+1$ and defect s to form difference triples $(k, 4t + 1 + i, 4t + 1 + j)$, respectively. This completes the proof of case (a).

The proof for case (b) is very similar except that the parity of the "hook" changes. Let $n \equiv 10 \pmod{24}$, $n = 24t + 10, t \geq 0$. Use an extended Skolem sequence ES $= (s_1, s_2, \ldots, s_{8t+3})$ of order $4t+1$ and hook s which exists for all odd $s, 1 \leq s \leq 8t+3$. For each $k \in \{1, 2, \ldots, 4t + 1\}$, form the difference triple $(k, 4t + 1 + i, 4t + 1 + j)$ provided $s_j = s_i = k$. Then the union of orbits of triples $\{0, k, 4t+1+i+k\}$ mod n, $k = 1, 2, \ldots, 4t + 1$, is a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triples, for even $s \in \{4t + 2, \ldots, 12t + 4\}$. For $s \in \{2, 4, \ldots, 4t\}$, use a near-Skolem sequence $(s_1, s_2, \ldots, s_{8t+2})$ of order $4t + 2$ and defect s; the latter exists for all even s, $2 \leq s \leq 4t + 2$. For each $k \in \{1, 2, \ldots, 4t+2\} \setminus \{s\}$, form the difference triple $(k, 4t+2+i, 4t+2+j)$ provided $s_i = s_j = k$. Then the union of orbits of triples $\{0, k, 4t+2+i+k\}$ mod n, $k \in \{1, 2, \ldots, 4t+2\} \setminus \{s\}$, is a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triples, for each even $s \in \{2, \ldots, 4t+2\}$. The proof in the case of $n \equiv 16 \pmod{24}$, $n = 24t+16, t \geq 0$ proceeds completely analogously, and is therefore omitted.

To deal with the case (c), let first $n \equiv 6 \pmod{24}$, $n = 24t+6$; since for $t = 0$ the graph $C(6; 1, 3)$ is a leave, while $C(6; 2, 3)$ is not, we may assume $t \geq 1$. We use a $(4t+2, u)$-extended Rosa sequence $R_{4t}(s_1, s_2, \ldots, s_{8t+2})$ of order $4t$ which by [8] exists for all $t \geq 1$ and u odd, $1 \leq u \leq 8t+1$, except when $t = 1$ and $u = 5$. For each $k \in \{1, 2, \ldots, 4t\}$ such that $s_j = s_i = k$, form the difference triple $(k, 4t+i, 4t+j)$; then take, for each k, the corresponding orbit of triples $\{0, k, 4t+i+k\}$. The union of these orbits, together with the short orbit of triples $\{0, \frac{n}{3}, \frac{2n}{3}\}$ mod n will provide a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triangles for each odd $s = u + 4t$, $s \in \{4t+1, \ldots, 12t+1\}$ with the exception noted above. This exception is eliminated by observing that the union of orbits of triples mod 30 corresponding to the difference triples $(1, 13, 14), (2, 4, 6), (3, 8, 11), (5, 7, 12)$ together with the short orbit of triples $\{0, 10, 20\}$ mod 30 yields a decomposition of $\bar{C}(30; 9, 15)$ into triangles. For $s \leq 4t+1$, we utilize in a similar manner a near-Rosa sequence $\mathrm{R} = (s_1, s_2, \ldots, s_{8t+1})$ of order $4t+1$ and defect s which by [14] exists for all odd $s \in \{1, \ldots, 4t+1\}$. For each $k \in \{1, 2, \ldots, 4t+1\} \setminus \{s\}$, form the difference triple $(k, 4t+1+i, 4t+1+j)$ provided $s_i = s_j = k$. Then the union of orbits of triples $\{0, k, 4t+1+i+k\}$ mod n, $k \in \{1, 2, \ldots, 4t+1\} \setminus \{s\}$, together with the short orbit $\{0, \frac{n}{3}, \frac{2n}{3}\}$ mod n is a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triples for each odd $s \in \{1, \ldots, 4t+1\}$.

Let now $n \equiv 12 \pmod{24}$, $n = 24t+12, t \geq 0$. We use a $(4t+3, u)$-extended Rosa sequence $R_{4t+1}(4t+3, u) = (s_1, s_2, \ldots, s_{8t+4})$ of order $4t+1$ which by [8] exists for all $t \geq 0$ and u even, $2 \leq u \leq 8t+4$, except when $t = 0$ and $u = 2$. For each $k \in \{1, 2, \ldots, 4t+1\}$ such that $s_j = s_i = k$, form the difference triple $(k, 4t+1+i, 4t+1+j)$, and then form for each k the corresponding orbit of triples $\{0, k, 4t+1+i+k\}$. The union of these orbits together with the short orbit of triples $\{0, \frac{n}{3}, \frac{2n}{3}\}$ gives a decomposition of $\bar{C}(n; s, \frac{n}{2})$ into triangles for each odd $s = u+4t+1$, $s \in \{4t+3, \ldots, 12t+5\}$ with the above noted exception. However, $C(12; 3, 6)$ is clearly a leave since it is isomorphic to $3K_4$. For $s \leq 4t+1$ we proceed analogously, except that now we use a near-Rosa sequence $\mathrm{R} = (s_1, s_2, \ldots, s_{8t+3})$ of order $4t+2$ and defect s which by [14] exists for all odd $s \in \{1, \ldots, 4t+1\}$.

The proof for the case (d) is very similar, with the change of the parity of the "hook". Let $n \equiv 0 \pmod{24}$, $n = 24t, t \geq 1$. We use a $(4t+1, u)$-extended Rosa sequence $R_{4t-1}(4t+1, u) = (s_1, s_2, \ldots, s_{8t})$ of order $4t-1$, $u \neq 4t+1$, to form difference triples $(k, 4t-1+i, 4t-1+j)$ for each $k \in \{1, 2, \ldots, 4t-1\}$ such that $s_j = s_i = k$. Then form for each k the corresponding orbit of triples $\{0, k, 4t-1+i+k\}$; their union together with the short orbit of triples $\{0, \frac{n}{3}, \frac{2n}{3}\}$ mod n gives a decomposition of $\bar{C}(n; s, \frac{n}{3})$ into triangles for each even $s = u+4t-1$, $s \in \{4t, \ldots, 12t-2\} \setminus \{8t\}$. For $s \leq 4t$, we use a near-Rosa sequence $\mathrm{R} = (s_1, s_2, \ldots, s_{8t-1})$ of order $4t$ and defect s which by [14] exists for all even $s \in \{2, \ldots, 4t\}$ except $t = 1$ and $s = 2$. The exception is eliminated by observing that the difference triples $(1, 6, 7)$, $(3, 10, 11)$,

$(4, 5, 9)$ form the union of three orbits which together with the short orbit yields a decomposition of $\bar{C}(24; 2, 12)$ into triangles.

The proof in the case when $n \equiv 18 \pmod{24}$, $n = 24t+18, t \geq 0$ proceeds completely analogously. For $s \geq 4t+4$ we use a $(4t+4, u)$-extended Rosa sequence $R_{4t+2}(4t+4, u) = (s_1, s_2, \dots, s_{8t+6})$ of order $4t+2$, $u \neq 4t+4$. For $s \leq 4t+2$ we use a near-Rosa sequence $\mathrm{R} = (s_1, s_2, \dots, s_{8t+5})$ of order $4t+3$ and defect s which by [14] exists for all even $s \in \{2, \dots, 4t+2\}$ except $t = 0$ and $s = 2$. The exception is eliminated by observing that the difference triples $(1, 4, 5)$, $(3, 7, 8)$ form the union of two orbits which together with the short orbit yields a decomposition of $\bar{C}(18; 2, 9)$ into triangles. □

Remark Note that for $n \equiv 1 \pmod 2$, the prism $Pr(n) \simeq C(2n; 2, n)$, thus we have the following corollary.

Corollary 5.2 *For $n \equiv 5 \pmod{12}$, the prism $Pr(n)$ is a leave.*

6 Generalized Petersen graphs as leaves

We consider generalized Petersen graphs $\mathrm{GP}(n, k)$ with n odd, $n = 2m+1$. A necessary condition for $\mathrm{GP}(2m+1, k)$ to be a leave is $m \equiv 1$ or $2 \pmod 3$.

Theorem 6.1 *Let $m \equiv 1, 2 \pmod 3$, $m \geq 5$ and $1 \leq k \leq m$, $k \neq \frac{2m+1}{3}$ when $m \equiv 1 \pmod 3$. Then $G = GP(2m+1, k)$ is a leave.*

Proof: Let $V(PTS) = V(G) = \{0, 1, \dots, 2m\} \times \{1, 2\}$. An edge $\{v_p, v_q\}$ is a *pure edge of type 1* [*pure edge of type 2*] when $p = q = 1$ [$p = q = 2$], otherwise it is a *mixed edge*. We consider two main cases separately.

Case I: $m \equiv 2 \pmod 3$. Let $n = \frac{m+1}{3}$ and $r = \lceil \frac{m}{2} \rceil$.
Subcase A: $m \equiv 2, 11 \pmod{12}$. Let $E(G) = \{\{i_1, (i+1)_1\}, \{i_2, (i+k)_2\}, \{i_1, i_2\} : i = 0, 1, \dots, 2m\}$. We take a near-Skolem sequence S of order n and defect 1; since $n \equiv 0, 1 \pmod 4$ such a sequence exists [12]. We construct $n-1$ base triples of a PTS in a usual way: for every $i = 2, 3 \dots, n$, let s_p and s_q be two elements of S such that $s_p = s_q = i$. Then $\{0_1, (n+p)_1, (n+q)_1\}$ is a base triple. In this way pure edges of type 1 and lengths $2, 3, \dots, m-1$ are used in triples. Let $g = k$.
Subcase B: $m \equiv 5, 8 \pmod{12}$. Let $E(G) = \{\{i_1, (i+2)_1\}, \{i_2, (i+2k)_2\}, \{i_1, i_2\} : i = 0, 1, \dots, 2m\}$. Since $n \equiv 2, 3 \pmod 4$, we take a near-Skolem sequence S of order n and defect 2 [12]. We construct $n-1$ base triples of a PTS in a similar way as above: for every $i = 1, 3, 4, \dots, n$, if s_p and s_q are two elements of S such that $s_p = s_q = i$, then we take $\{0_1, (n+p)_1, (n+q)_1\}$ as a base triple. Thus pure edges of type 1 and lengths $1, 3, 4, \dots, m-1$ are used in triples. Let $g = \min\{2k, 2m+1-2k\}$.

Now we are going to construct a near one-factor F on the set of vertices $\{0, 1, \dots, 2m\}$. If $g = m$ we require that all edges in F have pairwise distinct lengths. Let $F = \{\{m+1-i, m+1+i\} : 1 \leq i \leq m\}$. In the contrary case, every l: $1 \leq l \leq m$

except $l = g$ must appear as the length of an edge in F and moreover the length m must be repeated. If g is even, let $h = g/2$ and then $F = \{\{-i, i\} : \; h+1 \le i \le m-h\} \cup \{\{1-i, i\} : \; 1 \le i \le h$ or $\; m-h+2 \le i \le m\} \cup \{-h, m-h+1\}$. If g is odd, let $h = (2m+1-g)/2$ and $F = \{\{-i, i\} : \; m-h+1 \le i \le h-1\} \cup \{\{1-i, i\} : 1 \le i \le m-h$ or $h+1 \le i \le m\} \cup \{-m+h, h\}$. Notice that in all three cases $m+1$ is unmatched in F. Now we construct m base triples of a PTS in the following way. If $g = m$, for the edge $\{m+1-r, m+1+r\}$ of length m we take the triple $\{(m+1-r)_1, (m+1+r)_1, (m+1)_2\}$. Similarly, if $g \neq m$, for the edge $\{-r, r\}$ of length m we take the triple $\{-r_1, r_1, m_2\}$. In both cases, for every remaining edge $\{i, j\}$ in F we take the triple $\{(m+1)_1, i_2, j_2\}$. All those m triples use together edges of pairwise distinct lengths: pure length m of type 1, pure lengths of type 2: $1, 2, \ldots, m$ except g, and moreover all mixed lengths except 0.

Case II: $m \equiv 1 \pmod 3$. Let $n = \frac{m-1}{3}$ and $r = \lceil \frac{m}{2} \rceil$.
Subcase A: $m \equiv 1, 10 \pmod{12}$. Let $E(G) = \{\{i_1, (i+1)_1\}, \{i_2, (i+k)_2\}, \{i_1, i_2\} : i = 0, 1, \ldots, 2m\}$. We take an $(n+1)$-extended 1-near Skolem sequence S of order n; since $n \equiv 0, 3 \pmod 4$ such a sequence exists, see Lemma 5.2 in [8]. We construct $n-1$ base triples of a PTS: for every $i = 2, 3 \ldots, n$, if s_p and s_q are two elements of S such that $s_p = s_q = i$, then $\{0_1, (n+p)_1, (n+q)_1\}$ is a base triple. One more base triple $\{0_1, (2n+1)_1, (4n+2)_1\}$ generates a short orbit. In this way pure edges of type 1 and lengths $2, 3, \ldots, m-2$ are used in triples. Let $g = k$.
Subcase B: $m \equiv 4, 7 \pmod{12}$. Let $E(G) = \{\{i_1, (i+2)_1\}, \{i_2, (i+2k)_2\}, \{i_1, i_2\} : i = 0, 1, \ldots, 2m\}$. Since $n \equiv 1, 2 \pmod 4$, we take an $(n+1)$-extended 2-near Skolem sequence S of order n [8]. We construct $n-1$ base triples of a PTS: for every $i = 1, 3, 4 \ldots, n$, if s_p and s_q are two elements of S such that $s_p = s_q = i$, then $\{0_1, (n+p)_1, (n+q)_1\}$ is a base triple. Again, a base triple $\{0_1, (2n+1)_1, (4n+2)_1\}$ generates a short orbit. In this way pure edges of type 1 and lengths $1, 3, 4, \ldots, m-2$ are used in triples. Let $g = \min\{2k, 2m+1-2k\}$.

Analogously as in Case I, we need to construct a particular near one-factor F on the set of vertices $\{0, 1, \ldots, 2m\}$. Among all lengths of edges in F, $\frac{2m+1}{3}$ must be missing; it will be used for the base triple $\{0_2, (2n+1)_2, (4n+2)_2\}$ of a PTS which generates a short orbit. If $g = m-1$ then the length m is repeated and to construct F and next base triples of a PTS we apply the method presented in Case I. If $g = m$ then the length $m-1$ must be repeated; we take the following near-1-factor $F = \{\{-i, i\} : \; \frac{m+2}{3} \le i \le r-1\} \cup \{\{1-i, 1+i\} : \; r+1 \le i \le \frac{2m-2}{3}\} \cup \{1-i, i\} : \; 2 \le i \le \frac{m-1}{3}\} \cup \{\{2-i, 1+i\} : \; \frac{2m+4}{3} \le i \le m\} \cup \{r, r+1\} \cup \{-\frac{m-1}{3}, \frac{2m+4}{3}\} \cup \{0, m+2\}\}$, where the vertex 1 remains unmatched. For the edge $\{-\frac{m-1}{3}, \frac{2m+4}{3}\}$ of length m we take the base triple $\{(-\frac{m-1}{3})_1, (\frac{2m+4}{3})_1, (\frac{m+2}{3})_2\}$, for the edge $\{0, m+2\}$ of length $m-1$ the base triple $\{0_1, (m+2)_1, (m+1)_2\}$, and for every remaining edge $\{i, j\}$ in F we take the base triple $\{1_1, i_2, j_2\}$. In a general case, if $g \le m-2$ then the length g must be missing too, and the lengths $m-1$ and m must be repeated. Depending on the parity of g and comparison of g with $\frac{2m+1}{3}$ we apply one of the following constructions:

(i) $g = 1$. $F = \{\{-i-1, i\} : \; \frac{m+2}{3} \le i \le r-2\} \cup \{\{-i, 1+i\} : \; r+1 \le i \le \frac{2m-2}{3}\} \cup \{-i, i\} : \; 2 \le i \le \frac{m-1}{3}\} \cup \{\{1-i, 1+i\} : \; \frac{2m+4}{3} \le i \le m-1\} \cup \{-r, r\} \cup \{r-$

$1, r+1\}\cup\{-\frac{m+2}{3}, \frac{2m+4}{3}\}\cup\{1, m+2\}\cup\{2m, m+1\}\}$, where the vertex 0 is unmatched. For the edge $\{1, m+2\}$ of length m we take the triple $\{1_1, (m+2)_1, (m+3)_2\}$, for the edge $\{2m, m+1\}$ of length $m-1$ the triple $\{2m_1, (m+1)_1, m_2\}$, and for every remaining edge $\{i, j\}$ in F we take the triple $\{0_1, i_2, j_2\}$.

(ii) g is odd, $g > 1$. W.l.o.g assume that $g < \frac{2m+1}{3}$. Let $h = \frac{2m+3-g}{2}$. Then $F = \{\{-i, i\} : \frac{m+2}{3} \leq i \leq r-1\} \cup \{\{1-i, 1+i\} : r+1 \leq i \leq \frac{2m-2}{3}\} \cup \{\{1-i, i\} : m+3-h \leq i \leq \frac{m-1}{3}\} \cup \{\{2-i, 1+i\} : \frac{2m+4}{3} \leq i \leq h-1\} \cup \{\{-i, i\} : 1 \leq i \leq m+1-h\}\cup\{\{1-i, 1+i\} : h \leq i \leq m-1\}\cup\{r, r+1\}\cup\{-\frac{m-1}{3}, \frac{2m+4}{3}\}\cup\{-h+2, m-h+2\} \cup \{0, m+2\}$, where $m+1$ is unmatched. For the edge $\{-h+2, m-h+2\}$ of length m we take the triple $\{(-h+2)_1, (m-h+2)_1, (3-2h)_2\}$, for the edge $\{0, m+2\}$ of length $m-1$ the triple $\{0_1, (m+2)_1, 1_2\}$, and for every remaining edge $\{i, j\}$ in F we take the triple $\{(m+1)_1, i_2, j_2\}$.

(iii) g is even and $g < \frac{2m+1}{3}$. Let $h = \frac{g}{2}$. Then $F = \{\{-i, i\} : \frac{m+2}{3} \leq i \leq r-1\}\cup\{\{1-i, 1+i\} : r+1 \leq i \leq \frac{2m-2}{3}\}\cup\{\{1-i, i\} : h+1 \leq i \leq \frac{m-1}{3}\}\cup\{\{2-i, 1+i\} : \frac{2m+4}{3} \leq i \leq m-h\} \cup \{\{-i, i\} : 1 \leq i \leq h-1\} \cup \{\{1-i, 1+i\} : m+1-h \leq i \leq m-1\} \cup \{r, r+1\} \cup \{-\frac{m-1}{3}, \frac{2m+4}{3}\} \cup \{h+1-m, h\} \cup \{0, m+1\}$, where $m+2$ is unmatched. For the edge $\{0, m+1\}$ of length m we take the triple $\{0_1, (m+1)_1, 2m_2\}$, for the edge $\{h+1-m, h\}$ of length $m-1$ the triple $\{h_1, (h+1-m)_1, 2h_2\}$, and for every remaining edge $\{i, j\}$ in F we take the triple $\{(m+2)_1, i_2, j_2\}$.

(iv) g is even and $\frac{2m+1}{3} < g$. Let $h = \frac{g}{2}$. Then $F = \{\{-i, i\} : h+1 \leq i \leq r-1\}\cup\{\{1-i, 1+i\} : r+1 \leq i \leq m-h\}\cup\{\{1-i, i\} : \frac{m+5}{3} \leq i \leq h\}\cup\{\{2-i, 1+i\} : m+2-h \leq i \leq \frac{2m+1}{3}\} \cup \{\{-i, i\} : 1 \leq i \leq \frac{m-1}{3}\} \cup \{\{1-i, 1+i\} : \frac{2m+4}{3} \leq i \leq m-1\} \cup \{r, r+1\} \cup \{-h, m+2-h\} \cup \{-\frac{2m-2}{3}, \frac{m+2}{3}\} \cup \{0, m+1\}$, where $m+2$ is unmatched. For the edge $\{0, m+1\}$ of length m we take the triple $\{0_1, (m+1)_1, 2m_2\}$, for the edge $\{-h, m+2-h\}$ of length $m-1$ the triple $\{-h_1, (m+2-h)_1, -2h_2\}$, and for every remaining edge $\{i, j\}$ in F we take the triple $\{(m+2)_1, i_2, j_2\}$. □

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References

[1] A.T. Balaban, Valence-isomerism of cyclopolyenes, *Rev. Roum. Chim.* 11 (1996), 1097–1116.

[2] A.M. Baraev and N.A. Faradzev, Postroenie i issledovanie na EVM odnorodnykh i odnorodnykh dvudol'nykh grafov, *Algoritmicheskie issledovania v kombinatorike*, Nauka, Moskva, 1978, pp. 25–60.

[3] D. Bryant, B. Maenhaut, K. Quinn and B. Webb, Existence and embeddings of partial Steiner triple systems of order ten with cubic leaves, *Discrete Math.* 184 (2004), 83–95.

[4] C.J. Colbourn, Embedding partial Steiner triple systems is NP-complete, *J. Combin. Theory Ser. A* 35 (1983), 165–179.

[5] C.J. Colbourn and A. Rosa, Quadratic leaves of maximal partial triple systems, *Graphs Combin.* 2 (1986), 317–337.

[6] F. Franek, R. Mathon and A. Rosa, On a class of linear spaces with 16 points, *Ars Combin.* 31 (1991), 97–104.

[7] W. Imrich, Zehnpunktige kubische Graphen, *Aequationes Math.* 6 (1971), 6–10.

[8] V. Linek and N. Shalaby, The existence of (p, q)-extended Rosa sequences, *Discrete Math.* 308 (2008), 1583–1602.

[9] M. Meszka, `http://home.agh.edu.pl/~meszka/cubic_leaves.html`

[10] C. St.J. A. Nash-Williams, Problem, in *Combinatorial Mathematics and its Applications*, Proc. Colloq. Balatonfüred, 1969, pp. 1179–1181.

[11] R.C. Read and R.J. Wilson, *An Atlas of Graphs, 3. Regular Graphs*, Clarendon Press, Oxford, 1998.

[12] G. Royle, Graphs and multigraphs, in *Handbook of Combinatoral Designs, 2nd Ed., VII.4*, (Eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton, 2007, pp. 731–740.

[13] N. Shalaby, The existence of near-Skolem and hooked near-Skolem sequences, *Discrete Math.* 135 (1994), 309–319.

[14] N. Shalaby, Skolem and Langford sequences, in *Handbook of Combinatorial Designs 2nd Ed., VI.53*, (Eds. C.J. Colbourn, J.H. Dinitz), CRC Press, Boca Raton, 2007, pp. 612–616.

[15] N. Shalaby, The existence of near-Rosa and hooked near-Rosa sequences, *Discrete Math.* 261 (2003), 435–450.

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